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DIRAC INEQUALITY FOR HIGHEST WEIGHT HARISH–CHANDRA MODULES II

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Abstract. Let G be a connected simply connected noncompact exceptional simple Lie group of Hermitian type. In this paper, we work with the Dirac inequality which is a very useful tool for the classification of unitary highest weight modules.

1. Introduction

Let G be a connected simply connected noncompact exceptional simple Lie group of Hermitian type. That means that G is either of type E_6 or of type E_7. Let Θ be a Cartan involution of G and let K be the group of fixed points of Θ. Then K/Z is a maximal compact subgroup of G/Z, where Z denotes the center of G.

We will denote by $\mathfrak{g}_0$ the Lie algebra of G and by $\mathfrak{k}_0$ the Lie algebra of K. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition and let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. Our assumptions on G imply that $\mathfrak{t}_0$ is also a Cartan subalgebra of $\mathfrak{g}_0$. We delete the subscript 0 to denote complexifications.

Let $\Delta_{\mathfrak{g}}^+ \supset \Delta_{\mathfrak{k}}^+$ denote fixed sets of positive respectively positive compact roots. Since the pair (G,K) is Hermitian, we have a K-invariant decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ and $\mathfrak{p}^\pm$ are abelian subalgebras of $\mathfrak{p}$. Let ρ denote the half sum of positive roots for $\mathfrak{g}$.

We will consider $\lambda \in \mathfrak{t}^*$ which are $\Delta_{\mathfrak{k}}^+$-dominant integral ($\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{N} \cup \{0\}$, $\forall \alpha \in \Delta_{\mathfrak{k}}^+$). Let $N(\lambda)$ denote the generalized Verma module. From definition $N(\lambda) \simeq S(\mathfrak{p}^-) \otimes F_\lambda$, where F_λ is the irreducible $\mathfrak{k}$-module with highest weight λ. The generalized Verma module $N(\lambda)$ is a highest weight module. In case $N(\lambda)$ is not irreducible, we will consider the irreducible quotient $L(\lambda)$ of $N(\lambda)$. Our main goal is to determine those weights λ which correspond to unitarizable $L(\lambda)$ using the Dirac inequality. We consider only real highest weights λ since this is a necessary condition for unitarity.

To learn more about highest weight modules see [1], [2], [3], [4], [5], [10].

The K-types of $S(\mathfrak{p}^-)$ are called the Schmid modules. For each of the Lie algebras in Table 2, the general Schmid module s is a nonnegative integer combination of the

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so called basic Schmid modules. The basic Schmid modules for each exceptional Lie algebra $\mathfrak{g}_0$ for which (G,K) is a Hermitian symmetric pair are given in Table 2. To learn more about the Schmid modules see [13].

The Dirac operator is an element of $U(\mathfrak{g})\otimes C(\mathfrak{p})$ defined as $D=\sum_i b_i\otimes d_i$ where b_i is a basis of $\mathfrak{p}$ and d_i is the dual basis of $\mathfrak{p}$ with respect to the Killing form B. The Dirac operator acts on the tensor product $X\otimes S$ where X is a $(\mathfrak{g},K)$-module, and S is the spin module for $C(\mathfrak{p})$. The square of the Dirac operator is:

$$D^2=-(\mathrm{Cas}_{\mathfrak{g}}\otimes 1+\|\rho\|^2)+(\mathrm{Cas}_{\mathfrak{k}_\Delta}+\|\rho_{\mathfrak{k}}^2\|),$$

where $\rho_{\mathfrak{k}}$ is a half sum of the compact positive roots. To learn more about the Dirac operators in representation theory see [6], [8], [9], [7]).

If a $(\mathfrak{g},K)$-module is unitary, then D is self adjoint with respect to an inner product, so $D^2\geqslant 0$. By the formula for D^2 the Dirac inequality becomes explicit on any K-type F_τ of $L(\lambda)\otimes S$

$$\|\tau+\rho_{\mathfrak{k}}\|^2\geqslant\|\lambda+\rho\|^2.$$

In [3] it was proved that $L(\lambda)$ is unitary if and only if $D^2>0$ on $F_\mu\otimes\bigwedge^{\mathrm{top}}\mathfrak{p}^+$ for any K-type F_μ of $L(\lambda)$ other than F_λ, that is if and only if

$$\|\mu+\rho\|^2>\|\lambda+\rho\|^2.$$

The following theorem gives us motivation to study the Dirac inequality (see [11] for the case of classical Lie groups):

THEOREM 1.1. *Let us assume that $\mathfrak{g},\rho,\lambda,s$ are as in tables 1 and 2.*
(1) Let s_0 be a Schmid module such that the strict Dirac inequality

$$\|(\lambda-s)^++\rho\|^2>\|\lambda+\rho\|^2 \tag{1.1}$$

holds for any Schmid module s of strictly lower level than s_0, and such that

$$\|(\lambda-s_0)^++\rho\|^2<\|\lambda+\rho\|^2.$$

Then $L(\lambda)$ is not unitary.
(2) If

$$\|(\lambda-s)^++\rho\|^2>\|\lambda+\rho\|^2 \tag{1.2}$$

holds for all Schmid modules s, then $N(\lambda)$ is irreducible and unitary.

In Theorem 1.1, $(\lambda-s)^+$ is the unique $\mathfrak{k}$-dominant $W_{\mathfrak{k}}$-conjugate of $\lambda-s$, which means that $(\lambda-s)^+$ is as in the third column of Table 2.

The proof of the above theorem requires some tools from representation theory, so we will omit it in this paper and prove it in [12].

In Table 1, $s_\alpha(\lambda) = \lambda - \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$ is the reflection of λ with respect to the hyperplane orthogonal to a root α, $W_\mathfrak{k}$ is the Weyl group of $\mathfrak{k}$ generated by the s_α and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

Table 1: ρ *and* $W_\mathfrak{k}$

Lie algebra	ρ	generators of $W_\mathfrak{k}$
$\mathfrak{e}_6$	$(0,1,2,3,4,-4,-4,4)$	$s_{\varepsilon_i\pm\varepsilon_j}$, $5\geqslant i>j$
$\mathfrak{e}_7$	$\left(0,1,2,3,4,5,-\frac{17}{2},\frac{17}{2}\right)$	$s_{\varepsilon_i\pm\varepsilon_j}$, $5\geqslant i>j$, $s_{\frac{1}{2}(\varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_5-\varepsilon_4-\varepsilon_3-\varepsilon_2+\varepsilon_1)}$

Table 2: *The weights of basic Schmid modules and the condition for the* $\mathfrak{k}$*-highest weights* $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_n)$

Lie algebra	basic Schmid modules	highest weights
$\mathfrak{e}_6$	$s_1 = \frac{1}{2}(1,1,1,1,1,-1,-1,1)$, $s_2 = (0,0,0,0,1,-1,-1,1)$	$\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_6,-\lambda_6)$ $\lvert\lambda_1\rvert \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_5$ $\lambda_i-\lambda_j\in\mathbb{Z}$, $2\lambda_i\in\mathbb{Z}$, $i,j\leqslant 5$.
$\mathfrak{e}_7$	$s_1 = (0,0,0,0,0,0,-1,1)$, $s_2 = (0,0,0,0,1,1,-1,1)$, $s_3 = (0,0,0,0,0,2,-1,1)$	$\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,-\lambda_7)$ $\lvert\lambda_1\rvert \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_5$ $\lambda_i-\lambda_j\in\mathbb{Z}$, $2\lambda_i\in\mathbb{Z}$, $i,j\leqslant 5$ and $\frac{1}{2}\left(\lambda_8-\sum_{i=2}^{7}\lambda_i+\lambda_1\right)\in\mathbb{N}_0$

Here λ and ρ are elements of $\mathfrak{t}^*$ which is identified with $\mathbb{C}^n$, and ε_i denotes the projection to the i-th coordinate. The roots are certain functionals on $\mathfrak{t}^*$ and the relevant ones are those in the subscripts of the reflections s in Table 1, like $\varepsilon_i-\varepsilon_j$ or $\varepsilon_i+\varepsilon_j$.

We will frequently use the following lemma in our calculations (see [11]):

LEMMA 1.1. *Let* $\mathfrak{g}$ *be one of the Lie algebras listed in the above tables. Let* μ *and* ν *be weights as in Table 2. Let* $w_1,w_2\in W_\mathfrak{k}$. *Then*

$$\|(w_1\mu-w_2\nu)^+ +\rho\|^2 \geqslant \|(\mu-\nu)^+ +\rho\|^2.$$

In Lemma 1.1, $(w_1\mu-w_2\nu)^+$ is the unique dominant $W_\mathfrak{k}$-conjugate of $w_1\mu-w_2\nu$, which means $(w_1\mu-w_2\nu)^+$ is as in the third column of Table 2. The proof requires some representation theory and we leave it for [12].

2. Dirac inequalities

2.1. Dirac inequality for $\mathfrak{e}_6$

The basic Schmid $\mathfrak{k}$-modules in $S(\mathfrak{p}^-)$ have lowest weight $-s_i$, $i=1,2$, where

$$s_1=\beta_1=\frac{1}{2}(1,1,1,1,1,-1,-1,1),$$
$$s_2=\beta_1+\beta_2=(0,0,0,0,1,-1,-1,1).$$

The highest weight $(\mathfrak{g},K)$-modules have highest weights of the form

$$\lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_6,-\lambda_6), \quad |\lambda_1|\leqslant\lambda_2\leqslant\lambda_3\leqslant\lambda_4\leqslant\lambda_5,$$
$$\lambda_i-\lambda_j\in\mathbb{Z},\ 2\lambda_i\in\mathbb{Z}, \quad i,j\in\{1,2,3,4,5\}$$

In this case

$$\rho=(0,1,2,3,4,-4,-4,4).$$

The basic necessary condition for unitarity is the Dirac inequality

$$||(\lambda-s_1)^+ +\rho||^2\geqslant||\lambda+\rho||^2.$$

As before, we write $(\lambda-s_1)^+=\lambda-\gamma_1$. Then the Dirac inequality is equivalent to

$$2\langle\gamma_1\,|\,\lambda+\rho\rangle\leqslant\|\gamma_1\|^2.$$

We have

$$\lambda-s_1=\left(\lambda_1-\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6+\frac{1}{2},\lambda_6+\frac{1}{2},-\lambda_6-\frac{1}{2}\right)$$
$$\lambda+\rho=(\lambda_1,\lambda_2+1,\lambda_3+2,\lambda_4+3,\lambda_5+4,\lambda_6-4,\lambda_6-4,-\lambda_6+4)$$

There are eight cases.

Case 1.1: $\lambda_1+\lambda_2\geqslant 1$. In this case $\gamma_1=s_1$. The basic inequality is equivalent to

$$\sum_{i=1}^{5}\lambda_i+20\leqslant 3\lambda_6.$$

Case 1.2: $\lambda_2=-\lambda_1$, $\lambda_3-\lambda_2\geqslant 1$. In this case $\gamma_1=\frac{1}{2}(-1,-1,1,1,1,-1,-1,1)$. The basic inequality is equivalent to

$$\sum_{i=1}^{5}\lambda_i+18\leqslant 3\lambda_6.$$

Case 1.3: $\lambda_3=\lambda_2=-\lambda_1$, $\lambda_2>0$, $\lambda_4-\lambda_2\geqslant 1$. In this case $\gamma_1=\frac{1}{2}(-1,1,-1,1,1,-1,-1,1)$. The basic inequality is equivalent to

$$\sum_{i=1}^{5}\lambda_i+16\leqslant 3\lambda_6.$$

Case 1.4: $\lambda_3=\lambda_2=\lambda_1=0$, $\lambda_4\geqslant 1$. In this case $\gamma_1=\frac{1}{2}(1,-1,-1,1,1,-1,-1,1)$. The basic inequality is equivalent to

$$\sum_{i=1}^{5}\lambda_i+14\leqslant 3\lambda_6.$$

Case 1.5: $\lambda_4=\lambda_3=\lambda_2=-\lambda_1$, $\lambda_2>0$, $\lambda_5-\lambda_2\geqslant 1$. In this case $\gamma_1=\frac{1}{2}(-1,1,1,-1,1,-1,-1,1)$. The basic inequality is equivalent to

$$\sum_{i=1}^{5}\lambda_i+14\leqslant 3\lambda_6.$$

Case 1.6: $\lambda_4=\lambda_3=\lambda_2=\lambda_1=0$, $\lambda_5-\lambda_2\geqslant 1$. In this case $\gamma_1=\frac{1}{2}(-1,-1,-1,-1,1,-1,-1,1)$. The basic inequality is equivalent to

$$\sum_{i=1}^{5}\lambda_i+8\leqslant 3\lambda_6.$$

Case 1.7: $\lambda_5=\lambda_4=\lambda_3=\lambda_2=-\lambda_1$, $\lambda_2>0$. In this case $\gamma_1=\frac{1}{2}(-1,1,1,1,-1,-1,-1,1)$. The basic inequality is equivalent to

$$\sum_{i=1}^{5}\lambda_i+12\leqslant 3\lambda_6.$$

Case 1.8: $\lambda_5=\lambda_4=\lambda_3=\lambda_2=\lambda_1=0$. In this case $\gamma_1=\frac{1}{2}(1,-1,-1,-1,-1,-1,-1,1)$. The basic inequality is equivalent to

$$\sum_{i=1}^{5}\lambda_i\leqslant 3\lambda_6,$$

i.e. $\lambda_6\geqslant 0$.

Now we are going to see in which cases the Dirac inequality holds for s_2. We have

$$\lambda-s_2=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5-1,\lambda_6+1,\lambda_6+1,-\lambda_6-1).$$

We write $(\lambda-s_2)^+=\lambda-\gamma_2$. Then the Dirac inequality for s_2

$$\|(\lambda-s_2)^++\rho\|^2\geqslant\|\lambda+\rho\|^2$$

is equivalent to

$$2\langle\gamma_2,\lambda+\rho\rangle\leqslant\|\gamma_2\|^2$$

There are seven cases.

Case 2.1: $\lambda_5 \neq \lambda_4$. In this case $\gamma_2 = s_2$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + 14 \leqslant 3\lambda_6.$$

Case 2.2: $\lambda_5 = \lambda_4 > \lambda_3$. In this case $\gamma_2 = (0,0,0,1,0,-1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + 13 \leqslant 3\lambda_6.$$

Case 2.3: $\lambda_5 = \lambda_4 = \lambda_3 > \lambda_2$. In this case $\gamma_2 = (0,0,1,0,0,-1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + 12 \leqslant 3\lambda_6.$$

Case 2.4: $\lambda_5 = \lambda_4 = \lambda_3 = \lambda_2 > |\lambda_1|$. In this case $\gamma_2 = (0,1,0,0,0,-1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + 11 \leqslant 3\lambda_6.$$

Case 2.5: $\lambda_5 = \lambda_4 = \lambda_3 = \lambda_2 = \lambda_1 > 0$. In this case $\gamma_2 = (1,0,0,0,0,-1,-1,1)$. The basic inequality for s_2 is equivalent to

$$\lambda_5 + 10 \leqslant 3\lambda_6.$$

Case 2.6: $\lambda_5 = \lambda_4 = \lambda_3 = \lambda_2 = -\lambda_1 > 0$. In this case $\gamma_2 = (-1,0,0,0,0,-1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + 10 \leqslant 3\lambda_6.$$

Case 2.7: $\lambda_5 = \lambda_4 = \lambda_3 = \lambda_2 = \lambda_1 = 0$. In this case $\gamma_2 = (0,0,0,0,-1,-1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + 6 \leqslant 3\lambda_6,$$

i.e. $\lambda_6 \geqslant 2$.

It is easy to see that in the cases $1.1, 1.2, 1.3, 1.4, 1.5$ and 1.7 if the Dirac inequality holds for s_1 then it also holds for s_2, since

$$\lambda_5 \leqslant \sum_{i=1}^{5} \lambda_i$$

Therefore we have three basic cases:

Case 1: $\lambda_i = 0$, $i \in \{1,2,3,4,5\}$.

In this case the basic Dirac inequality can be written as

$$\lambda_6 \geqslant 0.$$

The Dirac inequality for the second basic Schmid module is equivalent to

$$\lambda_6 \geqslant 2.$$

Case 2: $\lambda_i = 0,\ i \in \{1,2,3,4\},\ \lambda_5 \neq 0.$
In this case the basic Dirac inequality can be written as

$$\lambda_5 + 8 \leqslant 3\lambda_6.$$

The Dirac inequality for the second basic Schmid module is equivalent to

$$\lambda_5 + 14 \leqslant 3\lambda_6.$$

Case 3: λ is of type 1.1, 1.2, 1.3, 1.4, 1.5 or 1.7, i.e. $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \neq (0,0,0,0)$. The Dirac inequality for the second basic Schmid module is automatically satisfied if the basic Dirac inequality holds.

Let

$$s_{a,b} = as_1 + bs_2 = \left(\frac{a}{2},\frac{a}{2},\frac{a}{2},\frac{a}{2},\frac{a}{2}+b,-\frac{a}{2}-b,-\frac{a}{2}-b,\frac{a}{2}+b\right), \quad a,b \in \mathbb{N}_0, \quad a+b>0$$

be a general Schmid module.

THEOREM 2.1. (Case 1) *Let λ be the highest weight of the form $\lambda = (0,0,0,0,0,\lambda_6,\lambda_6,-\lambda_6)$.*

1. *If $\lambda_6 > 2$ then λ satisfies the strict Dirac inequality*

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 > \|\lambda+\rho\|^2 \quad \forall a,b \in \mathbb{N}_0, a+b \neq 0.$$

2. *If $0 < \lambda_6 < 2$ then*

$$\|(\lambda - s_2)^+ + \rho\|^2 < \|\lambda+\rho\|^2$$

and the strict Dirac inequality holds for any Schmid module of strictly lower level than s_2.

3. *If $\lambda_6 < 0$ than the basic Dirac inequality fails.*

Proof.

1. We have

$$\lambda - s_{a,b} = \left(-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},-\frac{a}{2}-b,\lambda_6+\frac{a}{2}+b,\lambda_6+\frac{a}{2}+b,-\lambda_6-\frac{a}{2}-b\right),$$

and therefore

$$(\lambda - s_{a,b})^+ = \Big(-\frac{a}{2},\frac{a}{2},\frac{a}{2},\frac{a}{2},\frac{a}{2}+b,\lambda_6+\frac{a}{2}+b,\lambda_6+\frac{a}{2}+b,-\lambda_6-\frac{a}{2}-b\Big)$$
$$= \lambda - \Big(\frac{a}{2},-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},-\frac{a}{2}-b,-\frac{a}{2}-b,-\frac{a}{2}-b,\frac{a}{2}+b\Big).$$

Then the strict Dirac inequality

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 > \|\lambda+\rho\|^2$$

is equivalent to

$$2\big\langle \gamma_{a,b}\,|\,\lambda+\rho\big\rangle < ||\gamma_{a,b}||^2,$$

where $\gamma_{a,b} = \big(\frac{a}{2},-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},-\frac{a}{2}-b,-\frac{a}{2}-b,-\frac{a}{2}-b,\frac{a}{2}+b\big)$ and this inequality is equivalent to

$$-2a^2-4b^2-4ab-10a-8b < 3(\lambda_6-4)(a+2b).$$

Since $\lambda_6 > 2$, $3(\lambda_6-4)(a+2b) > -6(a+2b)$. Furthermore, the inequality

$$-2a^2-4b^2-4ab-10a-8b \leqslant -6(a+2b)$$

holds for all $a,b \in \mathbb{N}_0, a+b \neq 0$. So the strict Dirac inequality holds for any Schmid module $s_{a,b}$.

2. If $0 < \lambda_6 < 2$ then
$$\|(\lambda - s_2)^+ + \rho\|^2 < \|\lambda+\rho\|^2.$$
Since the level of s_2 is equal to two, and the level of as_1+bs_2 is equal to $a+2b$, the only Schmid module of strictly lower level than s_2 is s_1.

   For s_1 we have $\lambda_6 > 0$, which implies
$$\|(\lambda - s_1)^+ + \rho\|^2 > \|\lambda+\rho\|^2.$$

3. If $\lambda_6 < 0$ than the basic Dirac inequality obviously fails since the basic Dirac inequality in Case 1 is equivalent to $\lambda_6 \geqslant 0$. □

THEOREM 2.2. (Case 2) *Let λ be the highest weight of the form* $\lambda = (0,0,0,0,\lambda_5,\lambda_6,\lambda_6,-\lambda_6)$

1. *If* $3\lambda_6-\lambda_5 > 14$ *than λ satisfies the strict Dirac inequality*
$$\|(\lambda - s_{a,b})^+ + \rho\|^2 > \|\lambda+\rho\|^2 \quad \forall a,b \in \mathbb{N}_0, a+b\neq 0.$$

2. *If* $8 < 3\lambda_6-\lambda_5 < 14$ *then*
$$\|(\lambda - s_2)^+ + \rho\|^2 < \|\lambda+\rho\|^2$$
and the strict Dirac inequality holds for any Schmid module of strictly lower level than s_2.

3. *If* $3\lambda_6-\lambda_5<8$ *than the basic Dirac inequality fails.*

Proof.

1. We have

$$\lambda-s_{a,b}=\Big(-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},\lambda_5-\frac{a}{2}-b,\lambda_6+\frac{a}{2}+b,\lambda_6+\frac{a}{2}+b,-\lambda_6-\frac{a}{2}-b\Big),$$

and therefore

$$\begin{aligned}&(\lambda-s_{a,b})^+\\ &=\begin{cases}\big(\frac{a}{2},\frac{a}{2},\frac{a}{2},\frac{a}{2},\lambda_5-\frac{a}{2}-b,\lambda_6+\frac{a}{2}+b,\lambda_6+\frac{a}{2}+b,-\lambda_6-\frac{a}{2}-b\big), \ \lambda_5>a+b\\ \big(\lambda_5-\frac{a}{2}-b,\frac{a}{2},\frac{a}{2},\frac{a}{2},\frac{a}{2},\lambda_6+\frac{a}{2}+b,\lambda_6+\frac{a}{2}+b,-\lambda_6-\frac{a}{2}-b\big), \ b\leqslant\lambda_5\leqslant a+b\\ \big(-\frac{a}{2},\frac{a}{2},\frac{a}{2},\frac{a}{2},-\lambda_5+\frac{a}{2}+b,\lambda_6+\frac{a}{2}+b,\lambda_6+\frac{a}{2}+b,-\lambda_6-\frac{a}{2}-b\big), \ \lambda_5<b\end{cases}\\ &=\begin{cases}\lambda-\big(-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},\frac{a}{2}+b,-\frac{a}{2}-b,-\frac{a}{2}-b,\frac{a}{2}+b\big), \ \lambda_5>a+b\\ \lambda-\big(-\lambda_5+\frac{a}{2}+b,-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},\lambda_5-\frac{a}{2},-\frac{a}{2}-b,-\frac{a}{2}-b,\frac{a}{2}+b\big), \ b\leqslant\lambda_5\leqslant a+b\\ \lambda-\big(\frac{a}{2},-\frac{a}{2},-\frac{a}{2},-\frac{a}{2},2\lambda_5-\frac{a}{2}-b,-\frac{a}{2}-b,-\frac{a}{2}-b,\frac{a}{2}+b\big), \ \lambda_5<b.\end{cases}\end{aligned}$$

Then the strict Dirac inequality

$$\|(\lambda-s_{a,b})^++\rho\|^2>\|\lambda+\rho\|^2$$

is equivalent to

$$\begin{cases}2a^2+4b^2+4ab-10a-32b+(3\lambda_6-\lambda_5)(a+2b)>0, \ \lambda_5>a+b\\ 2a^2+4b^2+4ab-2a-24b+(3\lambda_6-\lambda_5)(a+2b)-8\lambda_5>0, \ b\leqslant\lambda_5\leqslant a+b\\ 2a^2+4b^2+4ab-2a-16b+(3\lambda_6-\lambda_5)(a+2b)-16\lambda_5>0, \ \lambda_5<b\end{cases}$$

Since $3\lambda_6-\lambda_5>14$, then $(3\lambda_6-\lambda_5)(a+2b)>14a+28b$. To prove the strict Dirac inequality it is enough to prove

$$\begin{cases}a^2+2b^2+2ab+2a-2b\geqslant 0, \ \lambda_5>a+b\\ a^2+2b^2+2ab+2a-2b\geqslant 0, \ b\leqslant\lambda_5\leqslant a+b\\ a^2+2b^2+2ab+6a-2b\geqslant 0, \ \lambda_5<b\end{cases}.$$

This is true for all $a,b\in\mathbb{N}_0,(a,b)\neq(0,0)$. So the strict Dirac inequality holds for any Schmid module $s_{a,b}$.

2. If $8<3\lambda_6-\lambda_5<14$ then

$$\|(\lambda-s_2)^++\rho\|^2<\|\lambda+\rho\|^2.$$

Since s_1 is the only Schmid module of strictly lower level than s_2, and for s_1 we have $3\lambda_6-\lambda_5>8$, it follows that

$$\|(\lambda-s_1)^++\rho\|^2>\|\lambda+\rho\|^2.$$

3. If $3\lambda_6 - \lambda_5 < 8$ than the basic Dirac inequality obviously fails since in Case 2 the basic Dirac inequality is equivalent to $3\lambda_6 - \lambda_5 \geqslant 8$. □

LEMMA 2.1. *Let λ be a highest weight such that $(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5) \neq (0,0,0,0,0)$ and*

$$\|(\lambda - s_2)^+ + \rho\|^2 > \|\lambda + \rho\|^2.$$

Then

$$\|(\lambda' - s_2)^+ + \rho\|^2 > \|\lambda' + \rho\|^2,$$

where $\lambda' = (\lambda - s_2)^+$. If $\lambda'_i = 0$ for $i = 1,2,3,4,5$, then

$$\|(\lambda' - s_{a,b})^+ + \rho\|^2 > \|\lambda' + \rho\|^2, \quad \forall a,b \in \mathbb{N}_0, \quad a+b \neq 0.$$

Proof. We have

$$\lambda' = \begin{cases} (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5-1,\lambda_6+1,\lambda_6+1,-\lambda_6-1), & \lambda \text{ as in case 2.1} \\ (\lambda_1,\lambda_2,\lambda_3,\lambda_5-1,\lambda_5,\lambda_6+1,\lambda_6+1,-\lambda_6-1), & \lambda \text{ as in case 2.2} \\ (\lambda_1,\lambda_2,\lambda_5-1,\lambda_5,\lambda_5,\lambda_6+1,\lambda_6+1,-\lambda_6-1), & \lambda \text{ as in case 2.3} \\ (\lambda_1,\lambda_5-1,\lambda_5,\lambda_5,\lambda_5,\lambda_6+1,\lambda_6+1,-\lambda_6-1), & \lambda \text{ as in case 2.4} \\ (\lambda_5-1,\lambda_5,\lambda_5,\lambda_5,\lambda_5,\lambda_6+1,\lambda_6+1,-\lambda_6-1), & \lambda \text{ as in case 2.5} \\ (-\lambda_5+1,\lambda_5,\lambda_5,\lambda_5,\lambda_5,\lambda_6+1,\lambda_6+1,-\lambda_6-1), & \lambda \text{ as in case 2.6} \end{cases}$$

If λ' is as in case 2.1 ($\lambda'_5 \neq \lambda'_4$), then λ is either as in case 2.1 or as in case 2.2. We have

$$\lambda'_5 - 3\lambda'_6 = \begin{cases} \lambda_5 - 3\lambda_6 - 4 < -14 - 4 = -18, & \lambda \text{ as in case 2.1} \\ \lambda_5 - 3\lambda_6 - 3 < -13 - 3 = -16, & \lambda \text{ as in case 2.2} \end{cases}$$

Thus, $\lambda'_5 - 3\lambda'_6 < -14$. It follows that the strict Dirac inequality holds for the second basic Schmid module.

If λ' is as in case 2.2 ($\lambda'_5 = \lambda'_4 > \lambda'_3$), then λ is either as in case 2.1 or as in case 2.3. We have

$$\lambda'_5 - 3\lambda'_6 = \begin{cases} \lambda_5 - 3\lambda_6 - 4 < -14 - 4 = -18, & \lambda \text{ as in case 2.1} \\ \lambda_5 - 3\lambda_6 - 3 < -12 - 3 = -15, & \lambda \text{ as in case 2.3} \end{cases}$$

Thus, $\lambda'_5 - 3\lambda'_6 < -13$. It follows that the strict Dirac inequality holds for the second basic Schmid module.

If λ' is as in case 2.3 ($\lambda'_5 = \lambda'_4 = \lambda'_3 > \lambda'_2$), then λ is either as in case 2.1 or as in case 2.4. We have

$$\lambda'_5 - 3\lambda'_6 = \begin{cases} \lambda_5 - 3\lambda_6 - 4 < -14 - 4 = -18, & \lambda \text{ as in case 2.1} \\ \lambda_5 - 3\lambda_6 - 3 < -11 - 3 = -14, & \lambda \text{ as in case 2.4} \end{cases}$$

Thus, $\lambda'_5 - 3\lambda'_6 < -12$. It follows that the strict Dirac inequality holds for the second basic Schmid module.

If λ' is as in case 2.4 ($\lambda'_5 = \lambda'_4 = \lambda'_3 = \lambda'_2 > |\lambda'_1|$), then λ is either as in case 2.1 or as in case 2.5 or as in case 2.6. We have

$$\lambda'_5 - 3\lambda'_6 = \begin{cases} \lambda_5 - 3\lambda_6 - 4 < -14 - 4 = -18, \ \lambda \text{ as in case 2.1} \\ \lambda_5 - 3\lambda_6 - 3 < -10 - 3 = -13, \ \lambda \text{ as in case 2.5 or as in case 2.6} \end{cases}$$

Thus, $\lambda'_5 - 3\lambda'_6 < -11$. It follows that the strict Dirac inequality holds for the second basic Schmid module.

If λ' is as in case 2.5 or as in case 2.6 ($\lambda'_5 = \lambda'_4 = \lambda'_3 = \lambda'_2 = |\lambda'_1| > 0$), then λ is either as in case 2.1 or as in case 2.5 (for $\lambda_1 = \frac{1}{2}$) or as in case 2.6 (for $\lambda_1 = \frac{1}{2}$). We have

$$\lambda'_5 - 3\lambda'_6 = \begin{cases} \lambda_5 - 3\lambda_6 - 4 < -14 - 4 = -18, \ \lambda \text{ as in case 2.1} \\ \lambda_5 - 3\lambda_6 - 3 < -10 - 3 = -13, \ \lambda \text{ as in case 2.5 or as in case 2.6} \end{cases}$$

Thus, $\lambda'_5 - 3\lambda'_6 < -10$. It follows that the strict Dirac inequality holds for the second basic Schmid module.

If λ' is as in case 2.7 ($\lambda'_5 = \lambda'_4 = \lambda'_3 = \lambda'_2 = \lambda'_1 = 0$), then $\lambda = (0,0,0,0,1,\lambda_6,\lambda_6,-\lambda_6)$ and $1 - 3\lambda_6 < -14$, that is $\lambda_6 > 5$ and $\lambda'_6 = \lambda_6 + 1 > 6 > 2$. The strict Dirac inequality holds for the second basic Schmid module.

It follows from theorem 2.1 that

$$\|(\lambda' - s_{a,b})^+ + \rho\|^2 - \|\lambda' + \rho\|^2 > 0 \quad \forall a,b \in \mathbb{N}_0 \quad a+b \neq 0. \qquad \square$$

LEMMA 2.2. *Let λ be a highest weight such that $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \neq (0,0,0,0)$ and*

$$\|(\lambda - s_1)^+ + \rho\|^2 > \|\lambda + \rho\|^2.$$

Then

$$\|(\lambda' - s_1)^+ + \rho\|^2 > \|\lambda' + \rho\|^2,$$

where $\lambda' = (\lambda - s_1)^+$. If $\lambda'_i = 0$ for $i = 1,2,3,4$, then

$$\|(\lambda' - s_{a,b})^+ + \rho\|^2 > \|\lambda' + \rho\|^2, \quad \forall a,b \in \mathbb{N}_0, \quad a+b \neq 0.$$

Proof. We have

$$\lambda' = \begin{cases} (\lambda_1-\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6+\frac{1}{2},\lambda_6+\frac{1}{2},-\lambda_6-\frac{1}{2}), \ \lambda \text{ as in case 1.1} \\ (-\lambda_2+\frac{1}{2},\lambda_2+\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6+\frac{1}{2},\lambda_6+\frac{1}{2},-\lambda_6-\frac{1}{2}), \ \lambda \text{ as in case 1.2} \\ (-\lambda_2+\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_2+\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6+\frac{1}{2},\lambda_6+\frac{1}{2},-\lambda_6-\frac{1}{2}), \ \lambda \text{ as in case 1.3} \\ (-\frac{1}{2},\frac{1}{2},\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6+\frac{1}{2},\lambda_6+\frac{1}{2},-\lambda_6-\frac{1}{2}), \ \lambda \text{ as in case 1.4} \\ (-\lambda_2+\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_2+\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6+\frac{1}{2},\lambda_6+\frac{1}{2},-\lambda_6-\frac{1}{2}), \ \lambda \text{ as in case 1.5} \\ (-\lambda_2+\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_2+\frac{1}{2},\lambda_6+\frac{1}{2},\lambda_6+\frac{1}{2},-\lambda_6-\frac{1}{2}), \ \lambda \text{ as in case 1.7} \end{cases}$$

If λ' is as in case 1.1 ($\lambda_1' + \lambda_2' \geqslant 1$), then λ is either as in case 1.1 or as in case 1.2. We have

$$\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' = \begin{cases} \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 4 < -20 - 4 = -24, & \lambda \text{ as in case 1.1} \\ \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 2 < -18 - 2 = -20, & \lambda \text{ as in case 1.2} \end{cases}$$

Thus, $\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' < -20$. It follows that the strict basic Dirac inequality holds.

If λ' is as in case 1.2 ($-\lambda_1' = \lambda_2', \lambda_3' - \lambda_2' \geqslant 1$), then λ is either as in case 1.1 or as in case 1.3. We have

$$\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' = \begin{cases} \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 4 < -20 - 4 = -24, & \lambda \text{ as in case 1.1} \\ \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 2 < -16 - 2 = -18, & \lambda \text{ as in case 1.3} \end{cases}$$

Thus, $\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' < -18$. It follows that the strict basic Dirac inequality holds.

If λ' is as in case 1.3 ($\lambda_3' = \lambda_2' = -\lambda_1', \lambda_2' > 0, \lambda_4' - \lambda_2' \geqslant 1$), then λ is either as in case 1.1 or as in case 1.4 or as in case 1.5. We have

$$\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' = \begin{cases} \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 4 < -20 - 4 = -24, & \lambda \text{ as in case 1.1} \\ \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 2 < -14 - 2 = -16, & \lambda \text{ as in case 1.4} \\ \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 2 < -14 - 2 = -16, & \lambda \text{ as in case 1.5} \end{cases}$$

Thus, $\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' < -16$. It follows that the strict basic Dirac inequality holds.

If λ' is as in case 1.4 ($\lambda_1' = \lambda_2' = \lambda_3' = 0, \lambda_4' > 0$), then λ is either as in case 1.1 or as in case 1.5. We have

$$\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' = \begin{cases} \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 4 < -20 - 4 = -24, & \lambda \text{ as in case 1.1} \\ \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 2 < -14 - 2 = -16, & \lambda \text{ as in case 1.5} \end{cases}$$

Thus, $\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' < -14$. It follows that the strict basic Dirac inequality holds.

If λ' is as in case 1.5 ($\lambda_4' = \lambda_3' = \lambda_2' = -\lambda_1', \lambda_2' > 0, \lambda_5' - \lambda_2' \geqslant 1$), then λ is either as in case 1.1 or as in case 1.4 or as in case 1.7. We have

$$\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' = \begin{cases} \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 4 < -20 - 4 = -24, & \lambda \text{ as in case 1.1} \\ \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 2 < -14 - 2 = -16, & \lambda \text{ as in case 1.4} \\ \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 2 < -12 - 2 = -14, & \lambda \text{ as in case 1.7} \end{cases}$$

Thus, $\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' < -14$. It follows that the strict basic Dirac inequality holds.

If λ' is as in case 1.6 ($\lambda_4' = \lambda_3' = \lambda_2' = \lambda_1' = 0, \lambda_5' - \lambda_2' \geqslant 1$), then λ is either as in case 1.1 or as in case 1.7. We have

$$\lambda_5' - 3\lambda_6' = \sum_{i=1}^{5} \lambda_i' - 3\lambda_6' = \begin{cases} \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 4 < -20 - 4 = -24, & \lambda \text{ as in case 1.1} \\ \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 2 < -12 - 2 = -14, & \lambda \text{ as in case 1.7} \end{cases}$$

Thus, $\lambda_5' - 3\lambda_6' < -14$. It follows that the strict Dirac inequality for the second basic Schmid module holds and thus, from the proof of theorem 2.2 we have

$$\|(\lambda' - s_{a,b})^+ + \rho\|^2 - \|\lambda' + \rho\|^2 > 0 \quad \forall a, b \in \mathbb{N}_0, \quad a + b \neq 0.$$

If λ' is as in case 1.7 ($\lambda_5' = \lambda_4' = \lambda_3' = \lambda_2' = -\lambda_1', \lambda_2' > 0$), then λ is either as in case 1.1 or as in case 1.4. We have

$$\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' = \begin{cases} \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 4 < -20 - 4 = -24, \ \lambda \text{ as in case 1.1} \\ \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 2 < -14 - 2 = -16, \ \lambda \text{ as in case 1.4} \end{cases}$$

Thus, $\sum_{i=1}^{5} \lambda_i' - 3\lambda_6' < -12$. It follows that the strict basic Dirac inequality holds.

If λ' is as in case 1.8 ($\lambda_5' = \lambda_4' = \lambda_3' = \lambda_2' = \lambda_1' = 0$), then λ is as in case 1.1. We have

$$-3\lambda_6' = \sum_{i=1}^{5} \lambda_i' - 3\lambda_6' = \sum_{i=1}^{5} \lambda_i - 3\lambda_6 - 4 < -20 - 4 = -24$$

Thus, $\lambda_6' > 8 > 2$. The strict Dirac inequality holds for the second basic Schmid module.

It follows from theorem 2.1 that

$$\|(\lambda' - s_{a,b})^+ + \rho\|^2 - \|\lambda' + \rho\|^2 > 0 \quad \forall a, b \in \mathbb{N}_0 \quad a + b \neq 0. \quad \square$$

THEOREM 2.3. (Case 3) *Let λ be the highest weight as in Case 3, i.e., $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \neq (0,0,0,0)$ such that strict basic Dirac inequality holds. Then*

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall a, b \in \mathbb{N}_0, (a,b) \neq (0,0).$$

Proof. Let λ be as in Case 3, and let us assume that the strict basic Dirac inequality holds. First we will prove that in this case we have

$$\|(\lambda - s_{0,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall b \in \mathbb{N}. \tag{2.1}$$

Let us denote $\lambda' = (\lambda - s_2)^+$. We have already proved that if λ is in Case 3 and the strict basic Dirac inequality holds, then the strict Dirac inequality also holds for s_2. So we have

$$\|\lambda' + \rho\|^2 > \|\lambda + \rho\|^2.$$

Let us assume that $b > 1$. Let $w \in W_{\mathfrak{k}}$ be such that $\lambda - s_2 = w(\lambda - s_2)^+$. From Lemma 1.1 we have

$$\begin{aligned} \|(\lambda - s_{0,b})^+ + \rho\|^2 &= \|(\lambda - s_2 - s_{0,b-1})^+ + \rho\|^2 = \|(w(\lambda - s_2)^+ - s_{0,b-1})^+ + \rho\|^2 \\ &\geqslant \|((\lambda - s_2)^+ - s_{0,b-1})^+ + \rho\|^2 = \|(\lambda' - s_{0,b-1})^+ + \rho\|^2. \end{aligned}$$

It follows from the last two inequalities that

$$\|(\lambda - s_{0,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > \|(\lambda' - s_{0,b-1})^+ + \rho\|^2 - \|\lambda' + \rho\|^2 \quad \forall b > 1 \tag{2.2}$$

If $\lambda_i' = 0$ for $i = 1,2,3,4,5$, then it follows from lemma 2.1 that

$$\|(\lambda' - s_{0,b-1})^+ + \rho\|^2 > \|\lambda' + \rho\|^2, \quad \forall b > 1,$$

and it follows from (2.2) that

$$\|(\lambda - s_{0,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0, \quad \forall b > 1.$$

Since $\|\lambda' + \rho\|^2 > \|\lambda + \rho\|^2$, we have

$$\|(\lambda - s_{0,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0, \quad \forall b \in \mathbb{N}.$$

If $(\lambda_1', \lambda_2', \lambda_3', \lambda_4', \lambda_5') \neq (0,0,0,0,0)$ and if $b > 2$ then it follows from lemma 2.1 and from (2.2) that

$$\|(\lambda' - s_{0,b-1})^+ + \rho\|^2 - \|\lambda' + \rho\|^2 > \|(\lambda'' - s_{0,b-2})^+ + \rho\|^2 - \|\lambda'' + \rho\|^2,$$

where $\lambda'' = (\lambda' - s_2)^+$. By induction, it follows

$$\|(\lambda - s_{0,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall b \in \mathbb{N}.$$

Now we will prove that if λ is as in Case 3, and the strict basic Dirac inequality holds, then

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall a, b \in \mathbb{N}_0, (a,b) \neq (0,0).$$

Let us denote $\tilde{\lambda} = (\lambda - s_1)^+$. We have

$$\|\tilde{\lambda} + \rho\|^2 > \|\lambda + \rho\|^2.$$

Let us assume that $a > 1$ or $a = 1$, $b > 0$. Let $\tilde{w} \in W_{\mathfrak{k}}$ be such that $\lambda - s_1 = \tilde{w}(\lambda - s_1)^+$. It follows from Lemma 1.1 that

$$\begin{aligned}\|(\lambda - s_{a,b})^+ + \rho\|^2 &= \|(\lambda - s_1 - s_{a-1,b})^+ + \rho\|^2 = \|(\tilde{w}(\lambda - s_1)^+ - s_{a-1,b})^+ + \rho\|^2 \\ &\geqslant \|((\lambda - s_1)^+ - s_{a-1,b})^+ + \rho\|^2 = \|(\tilde{\lambda} - s_{a-1,b})^+ + \rho\|^2.\end{aligned}$$

It follows from the last two inequalities that

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > \|(\tilde{\lambda} - s_{a-1,b})^+ + \rho\|^2 - \|\tilde{\lambda} + \rho\|^2. \tag{2.3}$$

If $\tilde{\lambda}_i = 0$ for $i = 1,2,3,4$, then it follows from lemma 2.2 that

$$\|(\tilde{\lambda} - s_{a-1,b})^+ + \rho\|^2 > \|\tilde{\lambda} + \rho\|^2,$$

and it follows from (2.3) that

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall a, b \in \mathbb{N}_0, \ a + b \neq 0.$$

If $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4) \neq (0,0,0,0)$ and $a > 1$, then it follows from lemma 2.2 and from (2.3) that

$$\|(\tilde{\lambda} - s_{a-1,b})^+ + \rho\|^2 - \|\tilde{\lambda} + \rho\|^2 > \|(\bar{\lambda} - s_{a-2,b})^+ + \rho\|^2 - \|\bar{\lambda} + \rho\|^2,$$

where $\bar{\lambda} = (\tilde{\lambda} - s_1)^+$. By induction and by (2.1), it follows that

$$\|(\lambda - s_{a,b})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall a, b \in \mathbb{N}_0, (a,b) \neq (0,0). \quad \square$$

2.2. Dirac inequality for $\mathfrak{e}_7$

The basic Schmid $\mathfrak{k}$-modules in $S(\mathfrak{p}^-)$ have lowest weights $-s_i$, $i=1,2,3$, where

$$\begin{aligned}
s_1 &= \beta_1 = (0,0,0,0,0,0,-1,1),\\
s_2 &= \beta_1+\beta_2 = (0,0,0,0,1,1,-1,1),\\
s_3 &= \beta_1+\beta_2+\beta_3 = (0,0,0,0,0,2,-1,1).
\end{aligned}$$

The highest weight $(\mathfrak{g},K)$-modules have highest weight of the form

$$\begin{aligned}
&\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,-\lambda_7), \quad |\lambda_1| \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \lambda_4 \leqslant \lambda_5,\\
&\lambda_i - \lambda_j \in \mathbb{Z},\ 2\lambda_i \in \mathbb{Z},\ 1 \leqslant i \leqslant j \leqslant 5\\
&\frac{1}{2}\left(\lambda_8-\lambda_7-\lambda_6+\sum_{i=1}^{5}(-1)^{n(i)}\lambda_i\right) \in \mathbb{N}_0, \quad \sum_{n=1}^{5} n(i) \text{ even},
\end{aligned}$$

which can be written more shortly as

$$\begin{aligned}
&\lambda = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,-\lambda_7), \quad |\lambda_1| \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \lambda_4 \leqslant \lambda_5,\\
&\lambda_i - \lambda_j \in \mathbb{Z},\ 2\lambda_i \in \mathbb{Z},\ 1 \leqslant i \leqslant j \leqslant 5\\
&\frac{1}{2}(\lambda_8-\lambda_7-\lambda_6-\lambda_5-\lambda_4-\lambda_3-\lambda_2+\lambda_1) \in \mathbb{N}_0.
\end{aligned}$$

In this case

$$\rho = \left(0,1,2,3,4,5,-\frac{17}{2},\frac{17}{2}\right).$$

The basic necessary condition for unitarity is the Dirac inequality

$$||(\lambda-s_1)^+ + \rho||^2 \geqslant ||\lambda+\rho||^2.$$

As before, we write $(\lambda-s_1)^+ = \lambda-\gamma_1$. Then the Dirac inequality is equivalent to

$$2\langle\gamma_1\,,\,\lambda+\rho\rangle \leqslant \|\gamma_1\|^2.$$

We have

$$\begin{aligned}
\lambda - s_1 &= (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7+1,-\lambda_7-1)\\
\lambda+\rho &= \left(\lambda_1,\lambda_2+1,\lambda_3+2,\lambda_4+3,\lambda_5+4,\lambda_6+5,\lambda_7-\frac{17}{2},-\lambda_7+\frac{17}{2}\right)
\end{aligned}$$

There are two basic cases.

Case 1.1: $\frac{1}{2}(\lambda_8-\lambda_7-\lambda_6-\lambda_5-\lambda_4-\lambda_3-\lambda_2+\lambda_1) \geqslant 1$. In this case $\gamma_1 = s_1$. The basic inequality is equivalent to

$$\lambda_7 \geqslant 8.$$

Case 1.2: $\frac{1}{2}(\lambda_8-\lambda_7-\lambda_6-\lambda_5-\lambda_4-\lambda_3-\lambda_2+\lambda_1)=0$. We have

$$s_{\alpha_1}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7+1,-\lambda_7-1)$$
$$=\left(\lambda_1+\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}\right).$$

In this case we have eight subcases.

Case 1.2.1: $\frac{1}{2}(\lambda_8-\lambda_7-\lambda_6-\lambda_5-\lambda_4-\lambda_3-\lambda_2+\lambda_1)=0,\ \lambda_1<\lambda_2$.
In this case

$$(\lambda-s_1)^+=\left(\lambda_1+\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}\right)$$

and $\gamma_1=\frac{1}{2}(-1,1,1,1,1,1,-1,1)$. The basic inequality is equivalent to

$$\lambda_7\geqslant\frac{15}{2}.$$

Case 1.2.2: $\frac{1}{2}(\lambda_8-\lambda_7-\lambda_6-\lambda_5-\lambda_4-\lambda_3-\lambda_2+\lambda_1)=0,\ \lambda_1=\lambda_2<\lambda_3$.
In this case

$$(\lambda-s_1)^+=\left(\lambda_2-\frac{1}{2},\lambda_2+\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}\right)$$

and $\gamma_1=\frac{1}{2}(1,-1,1,1,1,1,-1,1)$. The basic inequality is equivalent to

$$\lambda_7\geqslant 7.$$

Case 1.2.3: $\frac{1}{2}(\lambda_8-\lambda_7-\lambda_6-\lambda_5-\lambda_4-\lambda_3-\lambda_2+\lambda_1)=0,\ 0<\lambda_1=\lambda_2=\lambda_3<\lambda_4$.
In this case

$$(\lambda-s_1)^+=\left(\lambda_3-\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_3+\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}\right)$$

and $\gamma_1=\frac{1}{2}(1,1,-1,1,1,1,-1,1)$. The basic inequality is equivalent to

$$\lambda_7\geqslant\frac{13}{2}.$$

Case 1.2.4: $\frac{1}{2}(\lambda_8-\lambda_7-\lambda_6-\lambda_5-\lambda_4-\lambda_3-\lambda_2+\lambda_1)=0,\ 0=\lambda_1=\lambda_2=\lambda_3<\lambda_4$.
In this case

$$(\lambda-s_1)^+=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}\right)$$

and $\gamma_1 = \frac{1}{2}(-1,-1,-1,1,1,1,-1,1)$. The basic inequality is equivalent to

$$\lambda_7 \geqslant 6.$$

Case 1.2.5: $\frac{1}{2}(\lambda_8 - \lambda_7 - \lambda_6 - \lambda_5 - \lambda_4 - \lambda_3 - \lambda_2 + \lambda_1) = 0,\ 0 < \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 < \lambda_5$.

In this case

$$(\lambda - s_1)^+ = \left(\lambda_4 - \frac{1}{2}, \lambda_4 - \frac{1}{2}, \lambda_4 - \frac{1}{2}, \lambda_4 + \frac{1}{2}, \lambda_5 - \frac{1}{2}, \lambda_6 - \frac{1}{2}, \lambda_7 + \frac{1}{2}, -\lambda_7 - \frac{1}{2}\right)$$

and $\gamma_1 = \frac{1}{2}(1,1,1,-1,1,1,-1,1)$. The basic inequality is equivalent to

$$\lambda_7 \geqslant 6.$$

Case 1.2.6: $\frac{1}{2}(\lambda_8 - \lambda_7 - \lambda_6 - \lambda_5 - \lambda_4 - \lambda_3 - \lambda_2 + \lambda_1) = 0,\ 0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 < \lambda_5$. We have

$$\begin{aligned} s_{\alpha_1} s_{\varepsilon_2 - \varepsilon_1} s_{\varepsilon_3 + \varepsilon_4} &\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \lambda_5 - \frac{1}{2}, \lambda_6 - \frac{1}{2}, \lambda_7 + \frac{1}{2}, -\lambda_7 - \frac{1}{2}\right) \\ &= (0,0,0,0,\lambda_5 - 1, \lambda_6 - 1, \lambda_7, -\lambda_7). \end{aligned}$$

In this case

$$(\lambda - s_1)^+ = (0,0,0,0,\lambda_5 - 1, \lambda_6 - 1, \lambda_7, -\lambda_7)$$

and $\gamma_1 = (0,0,0,0,1,1,0,0)$. The basic inequality is equivalent to

$$\lambda_7 \geqslant 4.$$

Case 1.2.7: $\frac{1}{2}(\lambda_8 - \lambda_7 - \lambda_6 - \lambda_5 - \lambda_4 - \lambda_3 - \lambda_2 + \lambda_1) = 0,\ 0 < \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5$.

In this case

$$(\lambda - s_1)^+ = \left(\lambda_5 - \frac{1}{2}, \lambda_5 - \frac{1}{2}, \lambda_5 - \frac{1}{2}, \lambda_5 - \frac{1}{2}, \lambda_5 + \frac{1}{2}, \lambda_6 - \frac{1}{2}, \lambda_7 + \frac{1}{2}, -\lambda_7 - \frac{1}{2}\right)$$

and $\gamma_1 = \frac{1}{2}(1,1,1,1,-1,1,-1,1)$. The basic inequality is equivalent to

$$\lambda_7 \geqslant \frac{11}{2}.$$

Case 1.2.8: $\frac{1}{2}(\lambda_8 - \lambda_7 - \lambda_6 - \lambda_5 - \lambda_4 - \lambda_3 - \lambda_2 + \lambda_1) = 0,\ 0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5$. We have

$$\begin{aligned} s_{\varepsilon_5 - \varepsilon_1} s_{\alpha_1} s_{\varepsilon_2 + \varepsilon_3} s_{\varepsilon_4 + \varepsilon_5} &\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \lambda_6 - \frac{1}{2}, \lambda_7 + \frac{1}{2}, -\lambda_7 - \frac{1}{2}\right) \\ &= (0,0,0,0,1,\lambda_6 - 1, \lambda_7, -\lambda_7). \end{aligned}$$

In this case

$$(\lambda - s_1)^+ = (0,0,0,0,1,\lambda_6 - 1,\lambda_7,-\lambda_7)$$

and $\gamma_1 = (0,0,0,0,-1,1,0,0)$. The basic inequality is equivalent to

$$\lambda_7 \geqslant 0.$$

Now we are going to see in which cases the Dirac inequality holds for s_2. We have

$$\lambda - s_2 = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5 - 1,\lambda_6 - 1,\lambda_7 + 1,-\lambda_7 - 1).$$

We write $(\lambda - s_2)^+ = \lambda - \gamma_2$. Then the Dirac inequality for s_2

$$\|(\lambda - s_2)^+ + \rho\|^2 \geqslant \|\lambda + \rho\|^2$$

is equivalent to

$$2\langle \gamma_2, \lambda + \rho\rangle \leqslant \|\gamma_2\|^2$$

There are seven cases.

Case 2.1: $\lambda_5 > \lambda_4$. In this case $\gamma_2 = s_2$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + \lambda_6 - 2\lambda_7 + 24 \leqslant 0.$$

Case 2.2: $\lambda_5 = \lambda_4 > \lambda_3$. In this case

$$(\lambda - s_2)^+ = (\lambda_1,\lambda_2,\lambda_3,\lambda_5 - 1,\lambda_5,\lambda_6 - 1,\lambda_7 + 1,-\lambda_7 - 1)$$

and $\gamma_2 = (0,0,0,1,0,1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + \lambda_6 - 2\lambda_7 + 23 \leqslant 0.$$

Case 2.3: $\lambda_5 = \lambda_4 = \lambda_3 > \lambda_2$. In this case

$$(\lambda - s_2)^+ = (\lambda_1,\lambda_2,\lambda_5 - 1,\lambda_5,\lambda_5,\lambda_6 - 1,\lambda_7 + 1,-\lambda_7 - 1)$$

and $\gamma_2 = (0,0,1,0,0,1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + \lambda_6 - 2\lambda_7 + 22 \leqslant 0.$$

Case 2.4: $\lambda_5 = \lambda_4 = \lambda_3 = \lambda_2 > |\lambda_1|$. In this case

$$(\lambda - s_2)^+ = (\lambda_1,\lambda_5 - 1,\lambda_5,\lambda_5,\lambda_5,\lambda_6 - 1,\lambda_7 + 1,-\lambda_7 - 1)$$

and $\gamma_2 = (0,1,0,0,0,1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + \lambda_6 - 2\lambda_7 + 21 \leqslant 0.$$

Case 2.5: $\lambda_5 = \lambda_4 = \lambda_3 = \lambda_2 = \lambda_1 > 0$. We have two subcases:

Case 2.5.1: $\lambda_5=\lambda_4=\lambda_3=\lambda_2=\lambda_1>0$, $\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5-\lambda_6-\lambda_7+\lambda_8)\geqslant 1$.

In this case

$$(\lambda-s_2)^+=(\lambda_5-1,\lambda_5,\lambda_5,\lambda_5,\lambda_5,\lambda_6-1,\lambda_7+1,-\lambda_7-1)$$

and $\gamma_2=(1,0,0,0,0,1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5+\lambda_6-2\lambda_7+20\leqslant 0.$$

Case 2.5.2: $\lambda_5=\lambda_4=\lambda_3=\lambda_2=\lambda_1>0$, $\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5-\lambda_6-\lambda_7+\lambda_8)=0$. We have

$$\begin{aligned}&s_{\alpha_1}(\lambda_5-1,\lambda_5,\lambda_5,\lambda_5,\lambda_5,\lambda_6-1,\lambda_7+1,-\lambda_7-1)\\&=\left(\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{3}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}\right)\end{aligned}$$

In this case

$$(\lambda-s_2)^+=\left(\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{3}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}\right)$$

and $\gamma_2=\frac{1}{2}(1,1,1,1,1,3,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5+\lambda_6-2\lambda_7+19\leqslant 0.$$

Case 2.6: $\lambda_5=\lambda_4=\lambda_3=\lambda_2=-\lambda_1>0$. In this case

$$(\lambda-s_2)^+=(-\lambda_5+1,\lambda_5,\lambda_5,\lambda_5,\lambda_5,\lambda_6-1,\lambda_7+1,-\lambda_7-1)$$

and $\gamma_2=(-1,0,0,0,0,1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5+\lambda_6-2\lambda_7+20\leqslant 0.$$

Case 2.7: $\lambda_5=\lambda_4=\lambda_3=\lambda_2=\lambda_1=0$. We have two subcases:

Case 2.7.1: $\lambda_5=\lambda_4=\lambda_3=\lambda_2=\lambda_1=0$, $\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5-\lambda_6-\lambda_7+\lambda_8)\geqslant 1$.

In this case

$$(\lambda-s_2)^+=(0,0,0,0,1,\lambda_6-1,\lambda_7+1,-\lambda_7-1)$$

and $\gamma_2=(0,0,0,0,-1,1,-1,1)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5+\lambda_6-2\lambda_7+16\leqslant 0.$$

Case 2.7.2: $\lambda_5 = \lambda_4 = \lambda_3 = \lambda_2 = \lambda_1 = 0$, $\frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 + \lambda_8) = 0$. We have

$$\begin{aligned} &s_{\varepsilon_5-\varepsilon_4} s_{\varepsilon_4+\varepsilon_5}(0,0,0,0,-1,\lambda_6-1,\lambda_7+1,-\lambda_7-1) \\ &\quad = (0,0,0,0,1,\lambda_6-1,\lambda_7+1,-\lambda_7-1) \\ &s_{\alpha_1} s_{\varepsilon_2-\varepsilon_1} s_{\varepsilon_3+\varepsilon_4} s_{\alpha_1}(0,0,0,0,1,\lambda_6-1,\lambda_7+1,-\lambda_7-1) \\ &\quad = (0,0,0,0,0,\lambda_6-2,\lambda_7,-\lambda_7). \end{aligned}$$

In this case

$$(\lambda - s_2)^+ = (0,0,0,0,0,\lambda_6-2,\lambda_7,-\lambda_7)$$

and $\gamma_2 = (0,0,0,0,0,2,0,0)$. The Dirac inequality for s_2 is equivalent to

$$\lambda_5 + \lambda_6 - 2\lambda_7 + 8 \leqslant 0,$$

i.e. $\lambda_7 \geqslant 2$.

Now we are going to see in which cases the Dirac inequality holds for s_3. We have

$$\lambda - s_3 = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6-2,\lambda_7+1,-\lambda_7-1),$$

and therefore $(\lambda - s_3)^+ = \lambda - s_3$. Then the Dirac inequality for s_3

$$\|(\lambda - s_3)^+ + \rho\|^2 \geqslant \|\lambda + \rho\|^2$$

is equivalent to

$$2\langle s_3, \lambda + \rho\rangle \leqslant \|s_3\|^2,$$

i.e.,

$$\lambda_6 - \lambda_7 + 12 \leqslant 0.$$

It is easy to see that in cases $1.1, 1.2.1, 1.2.2, 1.2.3, 1.2.4, 1.2.5$ or $1.2.7$ if the Dirac inequality holds for s_1 then it also holds for s_2. Let us assume that the Dirac inequality holds for s_1. We have

$$\lambda_5 + \lambda_6 \leqslant \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - 2\lambda_7 \leqslant -2\lambda_7,$$

i.e.

$$\lambda_5 + \lambda_6 - 2\lambda_7 \leqslant -4\lambda_7 \leqslant (-4)\cdot\frac{11}{2} = -22,$$

and therefore the Dirac inequality obviously holds for s_2 if λ is in one of the cases 2.3, 2.4, 2.5, 2.6 or 2.7. If λ is in case 2.1 or in case 2.2 and also in one of the cases $1.1, 1.2.1, 1.2.2, 1.2.3, 1.2.4$ or $1.2.5$ (if λ is in case 2.1 or 2.2, then λ can not be in case $1.2.7$) and the Dirac inequality holds for s_1 then $\lambda_7 \geqslant 6$ and therefore

$$\lambda_5 + \lambda_6 - 2\lambda_7 \leqslant -4\lambda_7 \leqslant (-4)\cdot 6 = -24,$$

so the Dirac inequality holds for s_2.

Furthermore, in cases $1.1, 1.2.1, 1.2.2, 1.2.3, 1.2.4, 1.2.5$ or $1.2.7$ if the Dirac inequality holds for s_1 then it also holds for s_3, since $\lambda_6 \leqslant \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - 2\lambda_7 \leqslant -2\lambda_7$ and therefore

$$\lambda_6 - \lambda_7 + 12 \leqslant -3\lambda_7 + 12 \leqslant (-3)\cdot\frac{11}{2} + 12 < 0.$$

Therefore, we have three basic cases:

Case 1: $\lambda_i = 0,\ i \in \{1,2,3,4,5\}, \lambda_6 = -2\lambda_7$ (case 1.2.8)
In this case the basic Dirac inequality can be written as

$$\lambda_7 \geqslant 0.$$

The Dirac inequality for the second basic Schmid module is equivalent to

$$\lambda_7 \geqslant 2.$$

The Dirac inequality for the third basic Schmid module is equivalent to

$$\lambda_7 \geqslant 4.$$

It is clear that if the Dirac inequality holds for the third basic Schmid module, then it automatically holds for the first and the second basic Schmid module.

Case 2: $\lambda_i = 0,\ i \in \{1,2,3,4\},\ \lambda_5 > 0,\ -\lambda_5 - \lambda_6 - 2\lambda_7 = 0$ (case 1.2.6)
In this case the basic Dirac inequality can be written as

$$\lambda_7 \geqslant 4.$$

The Dirac inequality for the second basic Schmid module is equivalent to

$$\lambda_7 \geqslant 6.$$

The Dirac inequality for the third basic Schmid module is equivalent to

$$\lambda_6 - \lambda_7 + 12 \leqslant 0.$$

If the Dirac inequality holds for the second basic Schmid module, then it automatically holds for the first and the third basic Schmid module, since

$$\lambda_6 - \lambda_7 + 12 = -\lambda_5 - 3\lambda_7 + 12 \leqslant -3\lambda_7 + 12 \leqslant -18 + 12 < 0.$$

Case 3: λ is of type $1.1, 1.2.1, 1.2.2, 1.2.3, 1.2.4, 1.2.5$ or $1.2.7$. The Dirac inequality for the second and the third Schmid module is automatically satisfied if the basic Dirac inequality holds.

Let

$$\begin{aligned} s_{a,b,c} &= as_1 + bs_2 + cs_3 \\ &= (0,0,0,0,b,b+2c,-a-b-c,a+b+c),\ a,b,c \in \mathbb{N}_0,\ a+b+c > 0 \end{aligned}$$

be a general Schmid module.

THEOREM 2.4. (Case 1) *Let* λ *be the highest weight of the form* $\lambda = (0,0,0,0,0,-2\lambda_7,\lambda_7,-\lambda_7)$.

1. *If* $\lambda_7 > 4$ *then* λ *satisfies the strict Dirac inequality for any Schmid module* $s_{a,b,c}$, *i.e.*
$$\|(\lambda - s_{a,b,c})^+ + \rho\|^2 > \|\lambda+\rho\|^2,\ a,b,c \in \mathbb{N}_0,\ (a,b,c) \neq (0,0,0)$$
2. *If* $2 < \lambda_7 < 4$ *then*
$$\|(\lambda - s_3)^+ + \rho\|^2 < \|\lambda+\rho\|^2$$
and the strict Dirac inequality holds for any Schmid module of strictly lower level than s_3.
3. *If* $0 < \lambda_7 < 2$ *then*
$$\|(\lambda - s_2)^+ + \rho\|^2 < \|\lambda+\rho\|^2$$
and the strict Dirac inequality holds for any Schmid module of strictly lower level than s_2.
4. *If* $\lambda_7 < 0$ *than the basic Dirac inequality fails.*

Proof.

1. We have
$$\lambda - s_{a,b,c} = (0,0,0,0,-b,-2\lambda_7 - b - 2c, \lambda_7 + a + b + c, -\lambda_7 - a - b - c)$$
$$s_{\varepsilon_5-\varepsilon_1} s_{\alpha_1} s_{\varepsilon_4-\varepsilon_1} s_{\varepsilon_4+\varepsilon_5} s_{\varepsilon_2+\varepsilon_3} s_{\alpha_1} s_{\varepsilon_5-\varepsilon_4} s_{\varepsilon_4+\varepsilon_5} (\lambda - s_{a,b,c}) = (0,0,0,0,a,-2\lambda_7 - a - 2b - 2c, \lambda_7 + c, -\lambda_7 - c)$$
and therefore
$$(\lambda - s_{a,b,c})^+ = (0,0,0,0,a,-2\lambda_7 - a - 2b - 2c, \lambda_7 + c, -\lambda_7 - c) = \lambda - (0,0,0,0,-a,a+2b+2c,-c,c).$$
Then the strict Dirac inequality
$$\|(\lambda - s_{a,b,c})^+ + \rho\|^2 > \|\lambda+\rho\|^2$$
is equivalent to
$$2\left\langle \gamma_{a,b,c}, \lambda+\rho \right\rangle < ||\gamma_{a,b,c}||^2,$$
where $\gamma_{a,b,c} = (0,0,0,0,-a,a+2b+2c,-c,c)$ and this inequality is equivalent to
$$2(-2\lambda_7(a+2b+3c) + a + 10b + 27c) < a^2 + (a+2b+2c)^2 + 2c^2.$$
Since $\lambda_7 > 4$, $-2\lambda_7(a+2b+3c) < -8(a+2b+3c)$. We see that the inequality
$$2(-8(a+2b+3c) + a + 10b + 27c) \leqslant a^2 + (a+2b+2c)^2 + 2c^2$$
holds for all $a,b,c \in \mathbb{N}_0, a+b+c \neq 0$. So the strict Dirac inequality holds for any Schmid module $s_{a,b,c}$.

2. If $2<\lambda_7<4$ then
$$\|(\lambda-s_3)^+ +\rho\|^2<\|\lambda+\rho\|^2.$$
Since the level of s_i is equal to i where $i\in\{1,2,3\}$, and the level of $as_1+bs_2+cs_3$ is equal to $a+2b+3c$, the only Schmid modules of strictly lower level than s_3 are s_1,s_2 and $2s_1$. For $s_i, i\in\{1,2\}$, we have $\lambda_7>2>0$, i.e.
$$\|(\lambda-s_i)^+ +\rho\|^2>\|\lambda+\rho\|^2.$$
We have $(\lambda-2s_1)^+=\lambda-(0,0,0,0,-2,2,0,0)$. Therefore, the strict Dirac inequality for $2s_1$ is equivalent to $\lambda_7>-\frac{1}{2}$, which is true since $2<\lambda_7<4$.

3. If $0<\lambda_7<2$ then
$$\|(\lambda-s_2)^+ +\rho\|^2<\|\lambda+\rho\|^2.$$
Since the level of s_2 is equal to 2 and the level of $as_1+bs_2+cs_3$ is equal to $a+2b+3c$, the only Schmid module of strictly lower level than s_2 is s_1. For s_1 we have $\lambda_7>0$, which implies
$$\|(\lambda-s_1)^+ +\rho\|^2>\|\lambda+\rho\|^2.$$

4. If $\lambda_7<0$ than the basic Dirac inequality obviously fails since in Case 1 the basic Dirac inequality is equivalent to $\lambda_7\geqslant 0$. □

THEOREM 2.5. (Case 2) *Let λ be the highest weight of the form $\lambda=(0,0,0,0,\lambda_5,\lambda_6,\lambda_7,-\lambda_7)$ such that $\lambda_5>0$ and $-\lambda_5-\lambda_6-2\lambda_7=0$.*

1. *If $\lambda_7>6$ than λ satisfies the strict Dirac inequality for any Schmid module $s_{a,b,c}$, i.e.*
$$\|(\lambda-s_{a,b,c})^+ +\rho\|^2>\|\lambda+\rho\|^2,\ a,b,c\in\mathbb{N}_0,\ (a,b,c)\neq(0,0,0)$$

2. *If $4<\lambda_7<6$ then*
$$\|(\lambda-s_2)^+ +\rho\|^2<\|\lambda+\rho\|^2$$
and the strict Dirac inequality holds strictly for any Schmid module of strictly lower level than s_2.

3. *If $\lambda_7<4$ than the basic Dirac inequality fails.*

Proof.

1. We have
$$\lambda-s_{a,b,c}=(0,0,0,0,\lambda_5-b,\lambda_6-b-2c,\lambda_7+a+b+c,-\lambda_7-a-b-c)$$
$$s_{\alpha_1}s_{\varepsilon_3+\varepsilon_4}s_{\varepsilon_2-\varepsilon_1}s_{\alpha_1}(\lambda-s_{a,b,c})$$
$$=(0,0,0,0,\lambda_5-a-b,\lambda_6-a-b-2c,\lambda_7+b+c,-\lambda_7-b-c)$$

and therefore

$$(\lambda - s_{a,b,c})^+ = \begin{cases} (0,0,0,0,\lambda_5 - a - b, \lambda_6 - a - b - 2c, \lambda_7 + b + c, -\lambda_7 - b - c),\ \lambda_5 > a+b \\ (0,0,0,0,-\lambda_5+a+b, \lambda_6-a-b-2c, \lambda_7+b+c, -\lambda_7-b-c),\ b \leqslant \lambda_5 \leqslant a+b \\ s_{\alpha_1} s_{\varepsilon_3+\varepsilon_4} s_{\varepsilon_2-\varepsilon_1} s_{\alpha_1}(0,0,0,0,-\lambda_5+a+b, \lambda_6-a-b-2c,\ \lambda_7+b+c, -\lambda_7-b-c) \\ = (0,0,0,0,a,\lambda_5+\lambda_6-a-2b-2c, \lambda_5+\lambda_7+c, -\lambda_5-\lambda_7-c),\ \lambda_5 < b \end{cases}$$

$$= \begin{cases} \lambda - (0,0,0,0,a+b,a+b+2c,-b-c,b+c),\ \lambda_5 > a+b \\ \lambda - (0,0,0,0,2\lambda_5 - a - b, a+b+2c, -b-c, b+c),\ b \leqslant \lambda_5 \leqslant a+b \\ \lambda - (0,0,0,0,\lambda_5 - a, -\lambda_5 + a + 2b + 2c, -\lambda_5 - c, \lambda_5 + c),\ \lambda_5 < b. \end{cases}$$

Then the strict Dirac inequality

$$\|(\lambda - s_{a,b,c})^+ + \rho\|^2 > \|\lambda + \rho\|^2$$

is equivalent to

$$\begin{cases} -2\lambda_5 c - 2\lambda_7(a+2b+3c) + 9a + 26b + 27c \\ \qquad < (a+b)^2 + 2(a+b)c + 2c^2 + (b+c)^2, \\ \lambda_5 > a+b \\ -2\lambda_5 c - 2\lambda_7(a+2b+3c) + 8\lambda_5 + a + 18b + 27c \\ \qquad < (a+b)^2 + 2(a+b)c + 2c^2 + (b+c)^2, \\ b \leqslant \lambda_5 \leqslant a+b \\ -2\lambda_5 c - 2\lambda_7(a+2b+3c) + 16\lambda_5 + a + 10b + 27c \\ \qquad < a^2 + 2a(b+c) + c^2 + 2(b+c)^2, \\ \lambda_5 < b \end{cases}$$

Let us assume that $\lambda_7 > 6$. Since $\lambda_5 \geqslant 0$, to prove the strict Dirac inequality it is enough to prove

$$\begin{cases} -3a + 2b - 9c \leqslant (a+b)^2 + 2(a+b)c + 2c^2 + (b+c)^2,\ \lambda_5 > a+b \\ -3a + 2b - 9c \leqslant (a+b)^2 + 2(a+b)c + 2c^2 + (b+c)^2,\ b \leqslant \lambda_5 \leqslant a+b \\ -11a + 2b - 9c \leqslant a^2 + 2a(b+c) + c^2 + 2(b+c)^2,\ \lambda_5 < b \end{cases}.$$

This is true for all $a,b,c \in \mathbb{N}_0, (a,b,c) \neq (0,0,0)$. So the strict Dirac inequality holds for any Schmid module $s_{a,b,c}$.

2. If $4 < \lambda_7 < 6$ then
$$\|(\lambda - s_2)^+ + \rho\|^2 < \|\lambda + \rho\|^2.$$
Since s_1 is the only Schmid module of strictly lower level than s_2 and for s_1 we have $\lambda_7 > 4$, it follows that
$$\|(\lambda - s_1)^+ + \rho\|^2 > \|\lambda + \rho\|^2.$$

3. If $\lambda_7 < 4$ than the basic Dirac inequality obviously fails since in Case 2 the basic Dirac inequality is equivalent to $\lambda_7 \geqslant 4$. $\square$

LEMMA 2.3. *Let λ be a highest weight such that*

$$\|(\lambda - s_3)^+ + \rho\|^2 > \|\lambda + \rho\|^2.$$

Then

$$\|(\lambda' - s_3)^+ + \rho\|^2 > \|\lambda' + \rho\|^2,$$

where $\lambda' = (\lambda - s_3)^+$.

Proof. We have

$$\lambda' = (\lambda - s_3)^+ = \lambda - s_3 = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 - 2, \lambda_7 + 1, -\lambda_7 - 1).$$

The strict Dirac inequality

$$\|(\lambda' - s_3)^+ + \rho\|^2 > \|\lambda' + \rho\|^2$$

is equivalent to

$$\lambda'_6 - \lambda'_7 + 12 < 0$$

and this is equivalent to

$$\lambda_6 - \lambda_7 + 9 < 0,$$

which is true since

$$\lambda_6 - \lambda_7 + 12 < 0. \quad \square$$

LEMMA 2.4. *Let λ be a highest weight such that* $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (0,0,0,0,0)$ *and*

$$\|(\lambda - s_2)^+ + \rho\|^2 > \|\lambda + \rho\|^2.$$

Then

$$\|(\lambda - s_{0,b,0})^+ + \rho\|^2 > \|\lambda + \rho\|^2 \quad \forall b \in \mathbb{N}.$$

Proof. We have $\lambda - s_{0,b,0} = (0,0,0,0,-b,\lambda_6 - b, \lambda_7 + b, -\lambda_7 - b)$. Now we have two cases.

Case 1: $-\lambda_6 - 2\lambda_7 - 2b \geqslant 0$.
In this case

$$(\lambda - s_{0,b,0})^+ = (0,0,0,0,b,\lambda_6 - b, \lambda_7 + b, -\lambda_7 - b) = \lambda - (0,0,0,0,-b,b,-b,b).$$

The strict Dirac inequality

$$\|(\lambda - s_{0,b,0})^+ + \rho\|^2 > \|\lambda + \rho\|^2$$

is equivalent to

$$2\langle \gamma, \lambda + \rho \rangle < \|\gamma\|^2,$$

where $\gamma = (0,0,0,0,-b,b,-b,b)$ and the last inequality is equivalent to

$$\lambda_6 - 2\lambda_7 + 18 < 2b.$$

Since in this case $-\lambda_6 - 2\lambda_7 - 2b \geqslant 0$, then λ is not in case 2.7.2.. Therefore, λ is in case 2.7.1. Since the strict Dirac inequality holds for the second basic Schmid module, we have $\lambda_6 - 2\lambda_7 + 16 < 0$ and therefore $\lambda_6 - 2\lambda_7 + 18 < 2 \leqslant 2b$.

Case 2: $-\lambda_6 - 2\lambda_7 - 2b < 0$.
Then

$$s_{\alpha_1} s_{\varepsilon_3+\varepsilon_4} s_{\varepsilon_2-\varepsilon_1} s_{\alpha_1}(0,0,0,0,b,\lambda_6 - b,\lambda_7 + b,-\lambda_7 - b) = \left(0,0,0,0,-\frac{\lambda_6+2\lambda_7}{2},\frac{\lambda_6}{2}-\lambda_7-2b,-\frac{\lambda_6}{2},\frac{\lambda_6}{2}\right),$$

so

$$(\lambda - s_{0,b,0})^+ = \left(0,0,0,0,-\frac{\lambda_6+2\lambda_7}{2},\frac{\lambda_6}{2}-\lambda_7-2b,-\frac{\lambda_6}{2},\frac{\lambda_6}{2}\right) = \lambda - \gamma',$$

where $\gamma' = \left(0,0,0,0,\frac{\lambda_6}{2}+\lambda_7,\frac{\lambda_6}{2}+\lambda_7+2b,\lambda_7+\frac{\lambda_6}{2},-\lambda_7-\frac{\lambda_6}{2}\right)$. The strict Dirac inequality

$$\|(\lambda - s_{0,b,0})^+ + \rho\|^2 > \|\lambda + \rho\|^2$$

is equivalent to

$$-2(\lambda_6 + 2\lambda_7) < b\left(\lambda_7 - \frac{\lambda_6}{2} + b - 5\right). \tag{2.4}$$

Since in this case we have $-\lambda_6 - 2\lambda_7 < 2b$, it is enough to prove

$$4b \leqslant b\left(\lambda_7 - \frac{\lambda_6}{2} + b - 5\right).$$

The last inequality is equivalent to

$$\lambda_6 - 2\lambda_7 + 18 \leqslant 2b.$$

If λ is in case 2.7.1, then we have

$$\lambda_6 - 2\lambda_7 + 16 < 0,$$

since we assumed that the strict Dirac inequality holds for the second basic Schmid module. Therefore

$$\lambda_6 - 2\lambda_7 + 18 < 2 \leqslant 2b.$$

If λ is in case 2.7.2, then we have $\lambda_6 + 2\lambda_7 = 0$, so inequality (2.4) is equivalent to

$$\lambda_6 - 2\lambda_7 < 2b - 10.$$

Since we assumed that the strict Dirac inequality holds for the second basic Schmid module, we have

$$\lambda_6 - 2\lambda_7 < -8$$

and therefore

$$\lambda_6 - 2\lambda_7 < 2 - 10 \leqslant 2b - 10. \quad \square$$

LEMMA 2.5. *Let λ be a highest weight such that $(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)\neq(0,0,0,0,0)$ and*

$$\|(\lambda-s_2)^+ +\rho\|^2>\|\lambda+\rho\|^2.$$

Then

$$\|(\lambda'-s_2)^+ +\rho\|^2>\|\lambda'+\rho\|^2,$$

where $\lambda'=(\lambda-s_2)^+$. If $\lambda_i'=0$ for $i=1,2,3,4,5$, then

$$\|(\lambda'-s_{0,b,0})^+ +\rho\|^2>\|\lambda'+\rho\|^2,\quad \forall b\in\mathbb{N}.$$

Proof. We have

$$\lambda'=\begin{cases}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5-1,\lambda_6-1,\lambda_7+1,-\lambda_7-1), & \lambda \text{ as in case 2.1}\\ (\lambda_1,\lambda_2,\lambda_3,\ \lambda_5-1,\lambda_5,\lambda_6-1,\lambda_7+1,-\lambda_7-1), & \lambda \text{ as in case 2.2}\\ (\lambda_1,\lambda_2,\lambda_5-1,\lambda_5,\lambda_5,\lambda_6-1,\lambda_7+1,-\lambda_7-1), & \lambda \text{ as in case 2.3}\\ (\lambda_1,\lambda_5-1,\lambda_5,\lambda_5,\lambda_5,\lambda_6-1,\lambda_7+1,-\lambda_7-1), & \lambda \text{ as in case 2.4}\\ (\lambda_5-1,\lambda_5,\lambda_5,\lambda_5,\lambda_5,\lambda_6-1,\lambda_7+1,-\lambda_7-1), & \lambda \text{ as in case 2.5.1.}\\ (\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{3}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}), & \lambda \text{ as in case 2.5.2.}\\ (-\lambda_5+1,\lambda_5,\lambda_5,\lambda_5,\lambda_5,\lambda_6-1,\lambda_7+1,-\lambda_7-1), & \lambda \text{ as in case 2.6}\end{cases}$$

Therefore,

$$\lambda_5'+\lambda_6'-2\lambda_7'=\begin{cases}\lambda_5+\lambda_6-2\lambda_7-4, & \lambda \text{ as in case 2.1}\\ \lambda_5+\lambda_6-2\lambda_7-3, & \lambda \text{ as in case 2.2, 2.3, 2.4, 2.5.1, 2.5.2, 2.6}\end{cases}$$

Since the strict Dirac inequality holds for the second basic Schmid module, we have

$$\lambda_5'+\lambda_6'-2\lambda_7'<\begin{cases}-28, & \lambda \text{ as in case 2.1}\\ -26, & \lambda \text{ as in case 2.2}\\ -25, & \lambda \text{ as in case 2.3}\\ -24, & \lambda \text{ as in case 2.4}\\ -23, & \lambda \text{ as in case 2.5.1 or 2.6}\\ -22, & \lambda \text{ as in case 2.5.2..}\end{cases}$$

It is clear that

$$\|(\lambda'-s_2)^+ +\rho\|^2>\|\lambda'+\rho\|^2$$

if λ is in one of the cases $2.1, 2.2, 2.3$ or 2.4. If λ is as in case $2.5.1$ or 2.6, then λ' is not as in case 2.1 and therefore

$$\|(\lambda'-s_2)^+ +\rho\|^2>\|\lambda'+\rho\|^2.$$

If λ is as in case $2.5.2$, then λ' is not as in case 2.1 or 2.2 and therefore

$$\|(\lambda'-s_2)^+ +\rho\|^2>\|\lambda'+\rho\|^2.$$

So the strict Dirac inequality holds for the second basic Schmid module for the weight λ'.

If $\lambda_5' = \lambda_4' = \lambda_3' = \lambda_2' = \lambda_1' = 0$, then it follows from lemma 2.4 that

$$\|(\lambda' - s_{0,b,0})^+ + \rho\|^2 > \|\lambda' + \rho\|^2 \quad \forall b \in \mathbb{N}. \quad \square$$

LEMMA 2.6. *Let λ be a highest weight such that λ is as in case 3 and*

$$\|(\lambda - s_1)^+ + \rho\|^2 > \|\lambda + \rho\|^2.$$

Then

$$\|(\lambda' - s_1)^+ + \rho\|^2 > \|\lambda' + \rho\|^2,$$

where $\lambda' = (\lambda - s_1)^+$. If λ' is as in Case 1 or Case 2, then

$$\|(\lambda' - s_{a,b,c})^+ + \rho\|^2 > \|\lambda' + \rho\|^2, \quad \forall a,b,c \in \mathbb{N}_0, \quad a+b+c \neq 0.$$

Proof. We have

$$\lambda' = \begin{cases} (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7+1,-\lambda_7-1), \ \lambda \text{ as in case } 1.1 \\ (\lambda_1+\frac{1}{2},\lambda_2-\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}), \ \lambda \text{ as in case } 1.2.1 \\ (\lambda_2-\frac{1}{2},\lambda_2+\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}), \ \lambda \text{ as in case } 1.2.2 \\ (\lambda_3-\frac{1}{2},\lambda_3-\frac{1}{2},\lambda_3+\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}), \ \lambda \text{ as in case } 1.2.3 \\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}), \ \lambda \text{ as in case } 1.2.4 \\ (\lambda_4-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_4-\frac{1}{2},\lambda_4+\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}), \ \lambda \text{ as in case } 1.2.5 \\ (\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5-\frac{1}{2},\lambda_5+\frac{1}{2},\lambda_6-\frac{1}{2},\lambda_7+\frac{1}{2},-\lambda_7-\frac{1}{2}), \ \lambda \text{ as in case } 1.2.7 \end{cases}$$

Since

$$\|(\lambda - s_1)^+ + \rho\|^2 > \|\lambda + \rho\|^2,$$

it follows that

$$\lambda_7' > \begin{cases} 9, \ \lambda \text{ as in case } 1.1 \\ 8, \ \lambda \text{ as in case } 1.2.1 \\ 7+\frac{1}{2}, \ \lambda \text{ as in case } 1.2.2 \\ 7, \ \lambda \text{ as in case } 1.2.3 \\ 6+\frac{1}{2}, \ \lambda \text{ as in case } 1.2.4 \\ 6+\frac{1}{2}, \ \lambda \text{ as in case } 1.2.5 \\ 6, \ \lambda \text{ as in case } 1.2.7 \end{cases} \tag{2.5}$$

It is clear that

$$\|(\lambda' - s_1)^+ + \rho\|^2 > \|\lambda' + \rho\|^2$$

if λ is as in case 1.1 or 1.2.1. If λ is as in case 1.2.2, then λ' is not as in case 1.1. Also if λ is as in case 1.2.3, then λ' is neither as in case 1.1 nor as in case 1.2.1. If λ is as in case 1.2.4 or 1.2.5, then λ' is not in any of the cases 1.1, 1.2.1, 1.2.2. If λ

is in case $1.2.7$, then λ' is in none of the cases $1.1, 1.2.1, 1.2.2, 1.2.3$. It follows from (2.5) that

$$\|(\lambda' - s_1)^+ + \rho\|^2 > \|\lambda' + \rho\|^2.$$

Furthermore, it follows from (2.5) that if λ is as in Case 3, then $\lambda'_7 > 6$. Therefore, it follows from the proof of theorem 2.4 and the proof of theorem 2.5 that if λ' is as in Case 1 (case 1.2.8.) or as in Case 2 (case 1.2.6.), then

$$\|(\lambda' - s_{a,b,c})^+ + \rho\|^2 > \|\lambda' + \rho\|^2 \quad \forall a,b,c \in \mathbb{N}_0,\ a+b+c \neq 0. \quad \square$$

THEOREM 2.6. (Case 3) *Let λ be the highest weight as in Case 3 such that the strict basic Dirac inequality holds. Then*

$$\|(\lambda - s_{a,b,c})^+ + \rho\|^2 > \|\lambda + \rho\|^2,\ a,b,c \in \mathbb{N}_0,\ (a,b,c) \neq (0,0,0)$$

Proof. Let λ be as in Case 3, and let us assume that the strict basic Dirac inequality holds. First we will prove that in this case we have

$$\|(\lambda - s_{0,b,0})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall b \in \mathbb{N}. \tag{2.6}$$

Let us denote $\lambda' = (\lambda - s_2)^+$. We have already proved that if λ is in Case 3 and the strict basic Dirac inequality holds, then the strict Dirac inequality also holds for s_2. So we have

$$\|\lambda' + \rho\|^2 > \|\lambda + \rho\|^2. \tag{2.7}$$

If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ then (2.6) obviously follows from Lemma 2.4. Let us assume that $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \neq (0,0,0,0,0)$. From Lemma 2.5 it follows that

$$\|(\lambda' - s_2)^+ + \rho\|^2 > \|\lambda' + \rho\|^2. \tag{2.8}$$

Let us assume that $b > 1$ ((2.6) obviously holds for $b = 1$ since $s_{0,b,0} = s_2$). Let $w \in W_{\mathfrak{k}}$ be such that $\lambda - s_2 = w(\lambda - s_2)^+$. From Lemma 1.1 we have

$$\begin{aligned}
\|(\lambda - s_{0,b,0})^+ + \rho\|^2 &= \|(\lambda - s_2 - s_{0,b-1,0})^+ + \rho\|^2 \\
&= \|(w(\lambda - s_2)^+ - s_{0,b-1,0})^+ + \rho\|^2 \\
&\geqslant \|((\lambda - s_2)^+ - s_{0,b-1,0})^+ + \rho\|^2 \\
&= \|(\lambda' - s_{0,b-1,0})^+ + \rho\|^2.
\end{aligned}$$

It follows from the last inequality and (2.7) that

$$\|(\lambda - s_{0,b,0})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > \|(\lambda' - s_{0,b-1,0})^+ + \rho\|^2 - \|\lambda' + \rho\|^2. \tag{2.9}$$

If $\lambda'_i = 0$ for $i = 1,2,3,4,5$, then it follows from Lemma 2.4 and (2.8)

$$\|(\lambda' - s_{0,b-1,0})^+ + \rho\|^2 > \|\lambda' + \rho\|^2.$$

and by (2.9)

$$\|(\lambda - s_{0,b,0})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0.$$

If $(\lambda_1', \lambda_2', \lambda_3', \lambda_4', \lambda_5') \neq (0,0,0,0,0)$ and $b = 2$, then (2.6) follows from (2.9) and (2.8).

If $(\lambda_1', \lambda_2', \lambda_3', \lambda_4', \lambda_5') \neq (0,0,0,0,0)$ and $b > 2$, then we have

$$\|(\lambda' - s_{0,b-1,0})^+ + \rho\|^2 - \|\lambda' + \rho\|^2 > \|(\lambda'' - s_{0,b-2,0})^+ + \rho\|^2 - \|\lambda'' + \rho\|^2,$$

where $\lambda'' = (\lambda' - s_2)^+$. By induction, it follows that

$$\|(\lambda - s_{0,b,0})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall b \in \mathbb{N}.$$

Now we will prove that

$$\|(\lambda - s_{0,b,c})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall b, c \in \mathbb{N}_0,\ b + c \neq 0. \tag{2.10}$$

Let us denote $\lambda''' = (\lambda - s_3)^+$. Since λ is in Case 3, it is easy to check that λ''' is also in Case 3. We have already proved that if λ is in Case 3 and the strict basic Dirac inequality holds, then the strict Dirac inequality also holds for s_3. So we have

$$\|\lambda''' + \rho\|^2 > \|\lambda + \rho\|^2.$$

Let $w' \in W_k$ be such that $\lambda - s_3 = w'(\lambda - s_3)^+$. From Lemma (1.1) we have

$$\begin{aligned}\|(\lambda - s_{0,b,c})^+ + \rho\|^2 &= \|(\lambda - s_3 - s_{0,b,c-1})^+ + \rho\|^2 \\ &= \|(w'(\lambda - s_3)^+ - s_{0,b,c-1})^+ + \rho\|^2 \\ &\geqslant \|((\lambda - s_3)^+ - s_{0,b,c-1})^+ + \rho\|^2 \\ &= \|(\lambda''' - s_{0,b,c-1})^+ + \rho\|^2,\end{aligned}$$

if $c > 1$. From the last two inequalities it follows that

$$\|(\lambda - s_{0,b,c})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > \|(\lambda''' - s_{0,b,c-1})^+ + \rho\|^2 - \|\lambda''' + \rho\|^2 \\ \forall b, c \in \mathbb{N}_0, b + c \neq 0 \tag{2.11}$$

Now (2.10) follows from Lemma 2.3, (2.2) and (2.6) by induction on c.

Now we will prove that if λ is in Case 3, and the strict basic Dirac inequality holds, then

$$\|(\lambda - s_{a,b,c})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall a, b, c \in \mathbb{N}_0, (a,b,c) \neq (0,0,0).$$

Lat us assume that $a > 0$ (if $a = 0$, the last inequality is exactly (2.10)). Let us denote $\tilde{\lambda} = (\lambda - s_1)^+$. We have

$$\|\tilde{\lambda} + \rho\|^2 > \|\lambda + \rho\|^2.$$

Let $\tilde{w} \in W_k$ be such that $\lambda - s_1 = \tilde{w}(\lambda - s_1)^+$. From Corollary 2.9 we have

$$\begin{aligned}\|(\lambda - s_{a,b,c})^+ + \rho\|^2 &= \|(\lambda - s_1 - s_{a-1,b,c})^+ + \rho\|^2 \\ &= \|(\tilde{w}(\lambda - s_1)^+ - s_{a-1,b,c})^+ + \rho\|^2 \\ &\geqslant \|((\lambda - s_1)^+ - s_{a-1,b,c})^+ + \rho\|^2 \\ &= \|(\tilde{\lambda} - s_{a-1,b,c})^+ + \rho\|^2.\end{aligned}$$

From the last two inequalities it follows that

$$\|(\lambda - s_{a,b,c})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > \|(\tilde{\lambda} - s_{a-1,b,c})^+ + \rho\|^2 - \|\tilde{\lambda} + \rho\|^2. \tag{2.12}$$

If $\tilde{\lambda}$ is in Case 1 or Case 2, then it follows from lemma 2.6 that

$$\|(\tilde{\lambda} - s_{a-1,b,c})^+ + \rho\|^2 > \|\tilde{\lambda} + \rho\|^2,$$

and from (2.12) it follows that

$$\|(\lambda - s_{a,b,c})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall a,b,c \in \mathbb{N}_0, a+b+c \neq 0.$$

If $\tilde{\lambda}$ is in Case 3 and $a > 1$ then it follows from Lemma 2.6 and (2.12) that

$$\|(\tilde{\lambda} - s_{a-1,b,c})^+ + \rho\|^2 - \|\tilde{\lambda} + \rho\|^2 > \|(\bar{\lambda} - s_{a-2,b,c})^+ + \rho\|^2 - \|\bar{\lambda} + \rho\|^2,$$

where $\bar{\lambda} = (\tilde{\lambda} - s_1)^+$. By induction on a and by (2.10), it follows

$$\|(\lambda - s_{a,b,c})^+ + \rho\|^2 - \|\lambda + \rho\|^2 > 0 \quad \forall a,b,c \in \mathbb{N}_0, (a,b,c) \neq (0,0,0). \quad \square$$

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