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AN IMPROVED SPIRA'S INEQUALITY FOR THE RIEMANN ZETA FUNCTION AND ITS DERIVATIVES IN THE CRITICAL STRIP

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Abstract. The region of validity of Spira's strict inequality, given by $|\zeta(1-s)| = g(s)|\zeta(s)|$ where $g(s) := 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s)$, with $g(s) > 1$, involving the size of the Riemann zeta-function, $\zeta(s)$, at places symmetric with respect to the critical line, is enlarged to the subset $H_{t_*} := H \cap \{t > t_*\}$ of the semi-infinite critical half-strip $H := \{(\sigma,t) \in \mathbf{C} : 1/2 < \sigma < 1, t > 0\}$, where $s = \sigma + it$ and $t_* = 2\pi + \varepsilon = 6.380685^+$. It is conjectured that a smooth line, ℓ, exists in H such that the Spira's inequality holds above ℓ, while the opposite inequality holds below ℓ, and equality holds on ℓ. Moreover, if a nontrivial zero, s_0, of $\zeta(s)$ of multiplicity k exists in H_{t_*}, it is shown that $|\zeta^{(k)}(1-s_0)| > |\zeta^{(k)}(s_0)|$.

1. Introduction

In 1965, R. Spira [12] proved the strict inequality given by

$$|\zeta(1-s)| = g(s)\,|\zeta(s)| \;\text{ with } g(s) > 1, \tag{1}$$

where $g(s) := 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s)$ (see [14, Ch. 2, (2.1.8)]), valid for all σ, t with $1/2 < \sigma < 1$, $t \geqslant 10$, being $s = \sigma + it$, observing that it fails for t around 2π, but *except* for the zeros of the Riemann zeta function $\zeta(s)$ (if they exist).

It was also observed that the strict inequality $|\zeta(1-s)| > |\zeta(s)|$ for $1/2 < \sigma < 1$, $t \geqslant 10$ (that is *not* the same as that in (1)), would imply the validity of the Riemann Hypothesis (RH). Unfortunately, what was proved in [12, Theorem 2, p. 250], and in the weaker form, namely with the sign $\geqslant$, in [9, Note 1, p. 147], [10, Note, p. 115], and for $t \geqslant 2\pi + 1$, is only that $\zeta(s) \neq 0$, $1/2 < \sigma < 1$, $t \geqslant 10$ implies $|\zeta(1-s)| > |\zeta(s)|$, [13, p. 314].

In fact, recall the functional equation, obeyed by the zeta function,

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\,\zeta(1-s), \tag{2}$$

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Paper MIA-26-47

[14, Ch. II, (2.1.1), p. 13], also written, changing s into $1-s$,

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi}{2}s\right)\Gamma(s)\zeta(s),$$

[14, Ch. II, (2.1.8), p. 16], [14, Ch. II, p. 13, eq. (2.1.1)], [2, eq. (4), p. 13]. It implies that if $\zeta(s_0)=0$, also $\zeta(1-s_0)=0$ *whichever* the factor $g(s)$ (either $g(s)>1$, $g(s)=1$, or $g(s)<1$) might be.

Indeed, note that if $a>0$ and $b>0$, then $a=gb>b$ implies that $g>1$, but if $b=0$, then $gb=0$ whichever g might be, hence, e.g., $g>1$ does not imply that $a=gb>b$. In our case, $b=|\zeta(s)|$ vanishes if and only if s is a zero of $\zeta(s)$.

Establishing this kind of inequality is related to studying the so-called "horizontal monotonicity" of the modulus of $\zeta(s)$, of the Riemann xi-function, $\xi(s)$, as well as that of the Dirichlet function, $\eta(s)$. This amounts to investigate the possible monotonic behavior of the aforementioned functions on horizontal lines of the complex plane (σ,t), i.e., as functions of σ for fixed values of t. If one could prove that, e.g., $|\zeta(\sigma+it)|$ grows strictly monotonically with σ as σ increases in $1/2<\sigma<1$, for any fixed t, then, clearly, the validity of the RH would follow. Unfortunately, a monotonic behavior has been proved only outside the critical strip, see [9]. For instance, in [11, Theorem 1, p. 38], the authors proved that the modulus of $\xi(s)$ increases along every horizontal half-line lying in any open right half-plane that contains no zeros of $\xi(s)$. It decreases along each horizontal line contained in any zero-free, open left half-plane. In [6, Theorem 1.4, p. 91] it was proved that the moduli of the three functions $\zeta(s)$, $\xi(s)$, and $\eta(s)$ all decrease with respect to σ in the set $\sigma\leqslant 0$, $|t|\geqslant 8$. Unfortunately, the extension to $\sigma\leqslant 1/2$, for any of the three functions could not be attained.

In section 2, Spira's inequality (1), established in [12], will be improved, extending it to the "upper subset" of H, $H_{t_*} := \{(\sigma,t)\in\mathbf{C} : 1/2<\sigma<1, t>t_* := 2\pi+\varepsilon\}$, where $\varepsilon = 0.0975\ldots$, refining a method developed by Saidak and Zvengrowski [9]. Indeed, in [9] a similar inequality was established, independently, but in the weaker form, with the sign $\geqslant$, and for $t\geqslant 2\pi+1$ (actually, $t\geqslant 7$ was enough, as stated there). In [12] Spira proved the inequality for $t\geqslant 10$, while in [1] the result was improved to hold for $t\geqslant 6.8$. Nazardonyavi and Yakubovich adopted a simpler approach to establish Spira's inequality for $t\geqslant 12$, [7].

In the same section it is also conjectured that such inequality is valid on a more general subset of H, namely, $H_\ell := \{(\sigma,t)\in\mathbf{C} : 1/2<\sigma<1, t>L(\sigma)\}$, that is the subset of H above a certain smooth line, ℓ, described by $t=L(\sigma)$, while the opposite inequality holds (in H) below ℓ, and the equality holds on ℓ. Some figures, generated with MATHEMATICA, support these conjectures.

In section 3, it will be established a result similar to Spira's inequality for the (first nonvanishing) derivative of the zeta function *at* the nontrivial zeros, off the critical line (if any of them exist), see (16). More precisely, while it is not known the multiplicity of the nontrivial zeros off the critical line, *if* any of them exist, an inequality similar to Spira's inequality can be shown to hold for the derivatives of $\zeta(s)$ *at* such zeros. If $s_0\in H_{t_*}$ is one of these and is simple, then $\zeta(s_0)=\zeta(1-s_0)=0$, while $\zeta'(s_0)\neq 0$, $\zeta'(1-s_0)\neq 0$. Below, it will be shown that $|\zeta'(1-s_0)|>|\zeta'(s_0)|$. More generally, if s_0 is a nontrivial zero (off the critical line), with multiplicity k, located in subset H_{t_*}, then

$$|\zeta^{(k)}(1-s_0)| > |\zeta^{(k)}(s_0)|. \tag{3}$$

In section 4, finally, the results of the paper are summarized.

2. Improving Saidak-Zvengrowski's result

In 2003, Saidak and Zvengrowski proved that

$$\left|\zeta\left(\frac{1}{2}-\Delta+it\right)\right| \geqslant \left|\zeta\left(\frac{1}{2}+\Delta+it\right)\right| \tag{4}$$

for $0 \leqslant \Delta \leqslant 1/2$, where they set $\Delta := \sigma - 1/2$, for convenience, and $t \geqslant 2\pi + 1$, but actually, as they wrote, $t \geqslant 7$ suffices, [9]. However, they observed that this result is false for $t \leqslant 2\pi$.

As Saidak and Zvengrowski, Spira had stated that the RH is true *if and only if* $|\zeta(1-s)| > |\zeta(s)|$ for $t \geqslant 10$, $1/2 < \sigma < 1$ [12, Theorem 2, p. 250]. Indeed, if one could establish the strict inequality $|\zeta(1-s)| > |\zeta(s)|$ for all points of the critical strip, the validity of the Riemann hypothesis (RH) would follow [9, Note 1, p. 147], [10, Note, p. 115]. This is due to the symmetry around the line $\sigma = 1/2$), which follows from the validity of the functional equation (2), obeyed by $\zeta(s)$. However, if the ratio

$$\alpha(\Delta,t) := \frac{\left|\zeta\left(\frac{1}{2}-\Delta+it\right)\right|}{\left|\zeta\left(\frac{1}{2}+\Delta+it\right)\right|} \tag{5}$$

is defined, where, for convenience, $\alpha(\Delta,t) \equiv \alpha(s-1/2,t)$ is used in place of $g(s)$, proving the *sharper* estimate $\alpha(\Delta,t) > 1$ for all points of the critical strip, is *not* equivalent to prove that $|\zeta(1-s)| > |\zeta(s)|$ in the same region. The equivalence fails at the nontrivial zeros (if any exist).

Here the following is proved.

THEOREM 1. *The strict inequality* $\alpha(\Delta,t) > 1$ *holds for all* σ *and* t *with* $1/2 < \sigma < 1$ *and* $t > t_* := 2\pi + \varepsilon$ $(= 6.380685\ldots)$, t_* *being the unique root of the equation*

$$\varphi(t) := \ln\left(\frac{t}{2\pi}\right) + \frac{1}{2(4t^2+1)} - \frac{2}{4t^2-1} = 0. \tag{6}$$

Therefore, Spira's inequality (1) *holds in the open critical half-strip* $\{(\sigma,t) \in \mathbf{C} : 1/2 < \sigma < 1, t > t_*\}$, except *on the nontrivial zeros of the zeta function (if they exist).*

Proof. In [9, eq. (7), p. 154], the authors first established for the function $\alpha(\Delta,t)$ the form

$$\alpha(\Delta,t) = \frac{(2\pi)^{1/2-\Delta}}{\pi}\left|\Gamma\left(\frac{1}{2}+\Delta+it\right)\right|\left|\sin\left(\frac{\pi}{2}\left(\frac{1}{2}-\Delta+it\right)\right)\right|. \tag{7}$$

They stressed that, strictly speaking, the definition of $\alpha(\Delta,t)$ in (5) is meaningless when $\zeta(1/2+\Delta+it) = 0$. The explicit formula (7), however, shows that the points where this happens actually are not singular points (it has removable singularities). The function $\alpha(\Delta,t)$ does not vanish for $t>0$ and is a real analytic function of Δ,t, namely, the modulus of the holomorphic function

$$A(s) := \frac{(2\pi)^{1/2-\Delta}}{\pi}\Gamma\left(\frac{1}{2}+\Delta+it\right)\sin\left(\frac{\pi}{2}\left(\frac{1}{2}-\Delta,+it\right)\right), \tag{8}$$

see [9, Remark, p. 154].

Then, authors obtained the fine estimate

$$\alpha(\Delta,t) \geqslant \beta(\Delta,t) := \left(\frac{|s|}{2\pi}\right)^{\Delta}\left(1-\frac{4\sigma^3-\sigma}{12t^2}\right), \tag{9}$$

[9, Theorem 3 and Theorem 4, p. 148], and proved that

$$\alpha(\Delta,t) \geqslant 1 \tag{10}$$

for $0 \leqslant \Delta \leqslant 1/2$ and $t \geqslant 2\pi+1$ (actually, for $t \geqslant 7$, see [9, Note 2, p. 148]). This estimate was established in few careful steps, going well beyond what could be obtained using the well-known inequality satisfied by the Euler Gamma function in the complex field due to Lerch [3, pp. 14–15], [5], [4, (1.1), p. 2].

Below, it is possible to go further, beyond the findings of [9], showing the validity of the *strict* inequality $\alpha(\sigma-1/2,t)>1$, for $1/2<\sigma<1$ and $t \geqslant t_* = \pi+\varepsilon$, for some ε. In fact, using (9), one can claim theor

$$\alpha(\Delta,t) \geqslant \beta(\Delta,t) = \left(\frac{\sqrt{\sigma^2+t^2}}{2\pi}\right)^{\sigma-1/2}\left(1-\frac{\sigma(\sigma^2-1/4)}{3t^2}\right) > 1, \tag{11}$$

or, equivalently, taking the logarithms of both sides,

$$\frac{1}{2}\left(\sigma-\frac{1}{2}\right)\ln\left(\frac{\sigma^2+t^2}{(2\pi)^2}\right)+\ln\left(1-\frac{\sigma(\sigma+1/2)(\sigma-1/2)}{3t^2}\right) > 0, \tag{12}$$

restricting to $t>1/2$ (so that $\sigma(\sigma+1/2)(\sigma-1/2)/(3t^2), 1/(4t^2)<1$). Now,

$$\begin{aligned}
\ln\left(\frac{\sigma^2+t^2}{(2\pi)^2}\right) &= \ln\left[\left(\frac{t}{2\pi}\right)^2\left(1+\frac{\sigma^2}{t^2}\right)\right] \\
&= \ln\left(\left(\frac{t}{2\pi}\right)^2\right)+\ln\left(1+\frac{\sigma^2}{t^2}\right) \\
&> 2\ln\left(\frac{t}{2\pi}\right)+\frac{1}{1+t^2/\sigma^2} \\
&> 2\ln\left(\frac{t}{2\pi}\right)+\frac{1}{1+4t^2},
\end{aligned}$$

where the inequality $\ln(1+x) > x/(1+x)$, valid for all $x>-1$, $x \neq 0$ has been used [8, § 4.5 (i), 4.5.1], with $x := \sigma^2/t^2$, confining to $t>1$ $(>\sigma)$, and $1/2<\sigma<1$.

On the other hand,

$$\ln\left(1-\frac{\sigma(\sigma+1/2)(\sigma-1/2}{3t^2}\right) > -\frac{\sigma(\sigma+1/2)(\sigma-1/2)}{3t^2-\sigma(\sigma+1/2)(\sigma-1/2)} > -\frac{2}{4t^2-1},$$

having used $\ln(1-x) > -x/(1-x)$, valid for all $x<1$, $x\neq 0$ [8, § 4.5 (i), 4.5.1], with $x := \sigma(\sigma+1/2)(\sigma-1/2)/(3t^2)$ (confining to $t>1/2$).

Therefore, one can claim more, that is, from (12),

$$\frac{1}{2}\left(\sigma-\frac{1}{2}\right)\ln\left(\frac{\sigma^2+t^2}{(2\pi)^2}\right)+\ln\left(1-\frac{\sigma(\sigma+1/2)(\sigma-1/2}{3t^2}\right)$$

$$> \left(\sigma-\frac{1}{2}\right)\left\{\ln\left(\frac{t}{2\pi}\right)+\frac{1}{2(1+4t^2)}-\frac{2}{4t^2-1}\right\} > 0,$$

that is

$$\varphi(t) := \ln\left(\frac{t}{2\pi}\right)+\frac{1}{2(4t^2+1)}-\frac{2}{4t^2-1} > 0. \tag{13}$$

It is easy to show that the function $\varphi(t)$ is strictly (monotonically) increasing from $-\infty$ to $+\infty$ as t increases from 0 to $+\infty$. Therefore, $\varphi(t)>0$, that is (13) holds, and, *a fortiori*, (12) and (11) hold for $t > t_* := 2\pi+\varepsilon$, where t_* is the unique solution of the equation $\varphi(t)=0$ ($t_* = 6.380685\ldots$, $\varepsilon = 0.0975\ldots$). Being $\varphi(2\pi) = -(48\pi^2+5)/(2(256\pi^4-1)) < 0$ and, for instance, $\varphi(2\pi+0.1) = 0.00649\ldots > 0$, it is inferred that $t_* \in (2\pi, 2\pi+0.1)$, and (13) holds for $t > 2\pi+0.1$.

One may conclude that (11) holds, for all $t > t_* := 2\pi+\varepsilon > 2\pi+0.1$, e.g., *except* on the nontrivial zeros of the zeta function (if they exist) in the critical strip for $|t| > 2\pi+0.1$. Clearly, the value 0.1 can be reduced. Compare these values with the limitations on t given in [9, 12, 1, 7]. □

REMARK 1. One may *conjecture* that a certain smooth line, say ℓ, given by $t = L(\sigma)$ for $1/2<\sigma<1$, exists, implicitly defined by $F(\sigma,t) := \alpha(\sigma-1/2,t) = 1$, such that $F(\sigma,t)$ increases (strictly) monotonically for each fixed value of σ, $1/2<\sigma<1$, as a function of t, for $t>0$. Thus, $F(\sigma,t)>1$ in H above the line ℓ, and $F(\sigma,t)<1$ in H below ℓ.

Therefore, Spira's inequality would hold above ℓ, the opposite inequality would hold below ℓ, and equality would hold on ℓ. One may also conjecture that $F(\sigma,t)$ decreases monotonically as σ increases from $1/2$ to 1, and $\alpha(0) = g(1/2) \approx 6.29$ and $\alpha(1/2) = g(1) \approx 2\pi$, see Fig. 2.

Figures 1, 2, and 3, support all these conjectures. In Fig.s 1, 2, and 3, several level curves $F(\sigma,t) = \mathit{const.}$, generated with MATHEMATICA, are shown. Observe also the location of the ordinate 2π in Fig. 2.

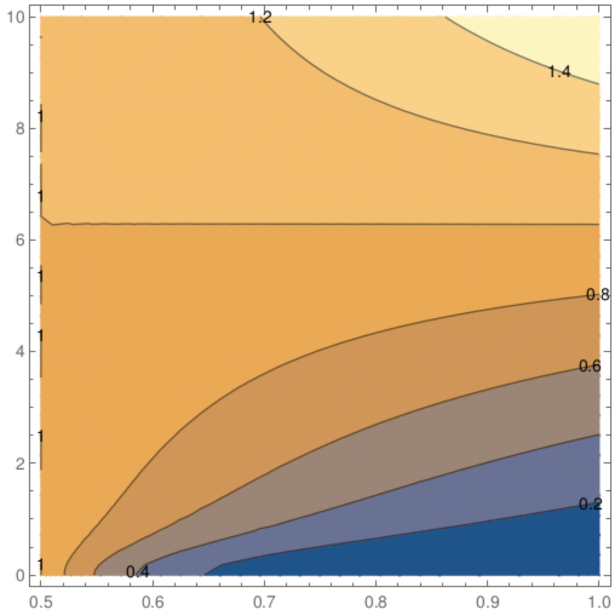

Figure 1: *Level curves* $\alpha(\Delta,t) = c$ *for various* c*, showing regions where* $\alpha(\Delta,t) > 1$*,* $\alpha(\Delta,t) = 1$ *(line* ℓ *in the text), or* $\alpha(\Delta,t) < 1$.

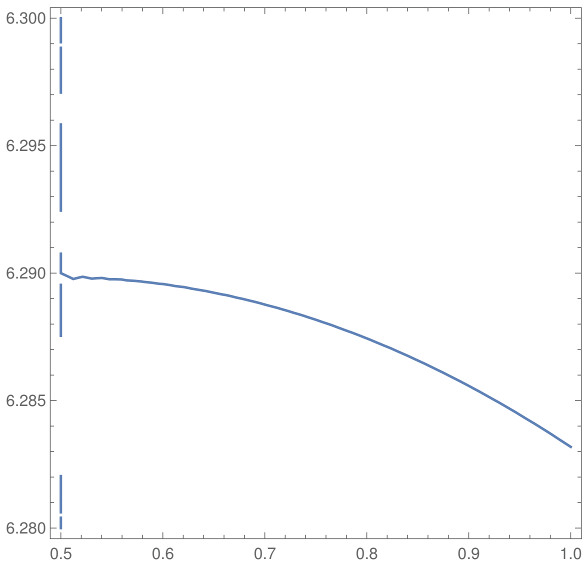

Figure 2: *Enlarged view of the level curve* $\alpha = 1$ *(line* ℓ *in the text).*

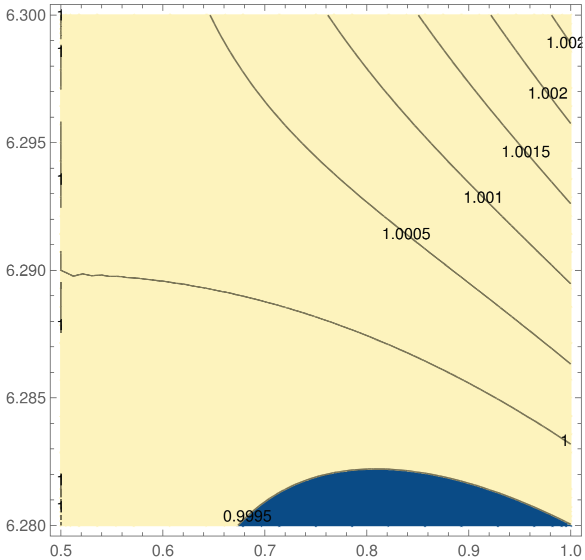

Figure 3: *Enlarged view of the regions where* $\alpha(\Delta,t) > 1$, $\alpha(\Delta,t) = 1$, *and* $\alpha(\Delta,t) < 1$, *better showing the line where* $\alpha(\Delta,t) = 1$ *(line* ℓ *in the text).*

3. Spira's inequality for the derivatives of the zeta function

A similar inequality for derivatives of the zeta function was also considered by Saidak [10], where he observed that *if*

$$|\zeta'(1/2-\Delta+it)| \geqslant |\zeta'(1/2+\Delta+it)|$$

would hold for $0 \leqslant \Delta \leqslant 1/2$, $t \geqslant 2\pi+1$, then the validity of the RH would follow [10, Theorem 3.1, p. 116]. The same was noticed to be true, as it was recalled above, for the zeta function, in place of their first derivatives, *but* with the strict inequality sign.

Writing

$$\zeta(1-s) = A(s)\,\zeta(s), \tag{14}$$

for $s \in H$, so that $\alpha(s-1/2,t) \equiv A(s)$ (see (8), if s_0 is a simple nontrivial zero, off the critical line (if it exists), obtains, differentiating,

$$-\zeta'(1-s_0) = A'(s_0)\,\zeta(s_0) + A(s_0)\,\zeta'(s_0) = A(s_0)\,\zeta'(s_0).$$

It follows that the first derivative of the zeta function satisfies the same relation due to Spira,

$$\zeta'(1-s_0)| = |A(s_0)|\,|\zeta'(s_0)| \equiv \alpha(s-1/2)\,|\zeta'(s_0)| > |\zeta'(s_0)|, \tag{15}$$

at the zero $s_0 \in H_{t_*}$. It s_0 has multiplicity k, then, differentiating both sides of (14), being $\zeta^{(j)}(s_0) = 0$ for $j = 0,1,\ldots,k-1$, while $\zeta^{(k)}(s_0) \neq 0$, yields

$$(-1)^k \zeta^{(k)}(1-s_0) = A(s_0)\,\zeta^{(k)}(s_0),$$

so that one can infer

$$|\zeta^{(k)}(1-s_0)| = |A(s_0)|\,|\zeta^{(k)}(s_0)| \equiv \alpha(s-1/2)\,|\zeta^{(k)}(s_0)| > |\zeta^{(k)}(s_0)|, \qquad (16)$$

where $t_* = 2\pi + \varepsilon = 6.380685^+$, if $s_0 \in S_{t_*}$. One can stress that these inequalities can be established immediately by merely exploiting the results obtained above for the zeta function. In addition, the conjectures of Remark 1 also hold in this case for the same token.

4. Summary

In this paper, refining an argument due to Saidak and Zvengrowski [9], the validity of Spira's inequality (1), first established in [12], was extended to the subset $H_{t_*} := H \cap \{t > t_*\}$ of the semi-infinite critical half-strip $H := \{(\sigma,t) \in \mathbf{C} : 1/2 < \sigma < 1, t > 0\}$, where $t_* = 2\pi + \varepsilon = 6.380685^+$. This inequality does not necessarily hold at the nontrivial zeros (if they exist). It is conjectured that this inequality holds in H, above some smooth line, ℓ, and that the opposite inequality holds below it, and equality on it. A Spira-type inequality for the kth derivative of the zeta function *at* any nontrivial zero off the critical line, it is also established, if k is its multiplicity (assuming that such a zero exists).

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