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OPERATORS ON HERZ–MORREY SPACES WITH VARIABLE EXPONENTS

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Abstract. This paper studies the nonlinear operators on Herz-Morrey spaces with variable exponents. We obtain our results by extending the extrapolation theory of Rubio de Francia to Herz-Morrey spaces with variable exponents. As applications of our main results, we obtain the boundedness of the spherical maximal functions, the nonlinear commutators of Rochberg and Weiss and the geometrical maximal operators on the Herz-Morrey spaces with variable exponents.

1. Introduction

In this paper, we aim to study the boundedness of nonlinear operators on the Herz-Morrey spaces with variable exponents. The Herz-Morrey spaces with variable exponents are extensions of the Herz spaces, the Herz spaces with variable exponents, the Morrey spaces with variable exponents and the Herz-Morrey spaces.

The Herz-Morrey spaces were firstly appeared in [29]. In [29], the boundedness of the rough singular integral operators on the Herz-Morrey spaces was obtained. Since then, a number of results for the Herz-Morrey spaces were established such as the Hardy inequality and the Hilbert inequality on the Herz-Morrey spaces [51]. An important generalization of the Herz-Morrey spaces is the Herz-Morrey spaces with variable exponents. It is the Herz-Morrey spaces built on the Lebesgue spaces with variable exponents. A substance number of results had been generalized to the Herz-Morrey spaces with variable exponents. The Besov type and the Triebel-Lizorkin type spaces built on the Herz-Morrey spaces with variable exponents were given in [9]. The Hardy-Herz-Morrey spaces with variable exponents were introduced and studied in [48, 49]. The boundedness of the sublinear operators and the mapping properties of the fractional integral operators on the Herz-Morrey spaces with variable exponents were obtained in [25, 46] and [26, 32, 47], respectively. The Sobolev embedding for the Herz-Morrey spaces with variable exponents was established in [31]. The duality for the Herz-Morrey spaces was studied in [31, 33].

In view of the above results, the boundedness of the fractional integral operator, the singular integral operators and some sublinear operators on the Herz-Morrey spaces

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with variable exponents were established. Thus, in this paper, we obtain the mapping properties of some nonlinear operators on the Herz-Morrey spaces with variable exponents. We use the method of extrapolation which is originally introduced by Rubio de Francia in [37, 38, 39]. In order to extend the extrapolation to the Herz-Morrey spaces with variable exponents, we introduce and study the Herz-block space with variable exponents. The classical block spaces were introduced in [2] to study the duality of the classical Morrey spaces. We generalize the classical block space to the Herz-block space with variable exponents to study the norm conjugate formula, the Hölder inequality for the Herz-Morrey spaces with variable exponents and the Herz-block spaces with variable exponents.

By extending the extrapolation to the Herz-Morrey spaces with variable exponents, we obtain a general result for the boundedness of nonlinear operators on the Herz-Morrey spaces with variable exponents. As applications of this general result, we establish the boundedness of the spherical maximal functions, the nonlinear commutators of Rochberg and Weiss and the geometrical maximal operators on the Herz-Morrey spaces with variable exponents.

This paper is organized as follows. The definitions of the Lebesgue spaces with variable exponents, the Herz spaces with variable exponents and the Herz-Morrey spaces with variable exponents are recalled in Section 2. The Herz-block spaces with variable exponents are introduced in Section 3. Some duality results for the Herz-Morrey spaces with variable exponents and the Herz-block spaces with variable exponents are given in this section. The general result on the boundedness of nonlinear operators on the Herz-Morrey spaces with variable exponents is established in Section 4. The applications on the spherical maximal functions, the nonlinear commutators of Rochberg and Weiss and the geometrical maximal operators are also presented in this section.

2. Definition and preliminaries

Let $\mathscr{M}$ and L^1_{loc} denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R}^n$, respectively.

For any $x \in \mathbb{R}^n$ and $r > 0$, define $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$.

We briefly recall the definition of Lebesgue spaces with variable exponent and the class of globally log-Hölder continuous function in the following.

DEFINITION 1. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty]$ be a Lebesgue measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ so that

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leqslant 1\} < \infty$$

where $\mathbb{R}^n_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}$ and

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^n \setminus \mathbb{R}^n_\infty} |f(x)|^{p(x)} dx + \operatorname*{ess\,sup}_{\mathbb{R}^n_\infty} |f(x)|.$$

The function $p(x)$ is called as the exponent function of $L^{p(\cdot)}$.

Whenever $p(\cdot):\mathbb{R}^n\to[1,\infty]$, the Lebesgue space with variable exponent is a Banach function space, see [7, Theorem 3.2.13]. For the definition of Banach function space, the reader is referred to [7, Definition 2.7.7].

The associate space of $L^{p(\cdot)}$ is given in [7, Theorem 3.2.13].

THEOREM 1. *If* $1<p(x)<\infty$, *then the associate space of* $L^{p(\cdot)}$ *is* $L^{p'(\cdot)}(\mathbb{R}^n)$ *where* $p'(\cdot)$ *satisfies* $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$.

We call $p'(x)$ the conjugate function of $p(x)$. Whenever $\sup_{x\in\mathbb{R}^n}p(x)<\infty$, the dual space of $L^{p(\cdot)}$ is equal to the associate space of $L^{p(\cdot)}$, see [7, Theorem 3.4.6].

For any $f\in L^1_{\mathrm{loc}}$, the Hardy-Littlewood maximal operator is defined by

$$\mathrm{M}f(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)|dy$$

where the supremum is taken over all $B\in\mathbb{B}$ containing x.

The following gives the conditions on the exponent functions of the Lebesgue space with variable exponent $L^{p(\cdot)}$ so that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$.

DEFINITION 2. A continuous function g on $\mathbb{R}^n$ is log-Hölder continuous at the origin when

$$|g(x)-g(0)|\leqslant\frac{c_{\log}}{\log(e+1/|x|)},\quad\forall x\in\mathbb{R}^n. \tag{1}$$

We write $g\in C_0^{\log}(\mathbb{R}^n)$ if g is log-Hölder continuous at the origin.

A continuous function g on $\mathbb{R}^n$ is locally log-Hölder continuous if there exists $c_{\log}>0$ such that for all $x,y\in\mathbb{R}^n$ satisfying $|x-y|<\frac{1}{2}$

$$|g(x)-g(y)|\leqslant\frac{c_{\log}}{\log(e+1/|x-y|)}. \tag{2}$$

We denote the class of locally log-Hölder continuous function by $C_{loc}^{\log}(\mathbb{R}^n)$.

Furthermore, a continuous function is globally log-Hölder continuous if $g\in C_{loc}^{\log}(\mathbb{R}^n)$ and there exists $g_\infty\in\mathbb{R}$ so that

$$|g(x)-g_\infty|\leqslant\frac{c_{\log}}{\log(e+|x|)},\quad\forall x\in\mathbb{R}^n. \tag{3}$$

The class of globally log-Hölder continuous function is denoted by $C^{\log}(\mathbb{R}^n)$.

For any Lebesgue measurable function $p(x):\mathbb{R}^n\to(-\infty,\infty]$, define $p_-=\inf_{x\in\mathbb{R}^n}p(x)$ and $p_+=\sup_{x\in\mathbb{R}^n}p(x)$. We write $(p')_+=p'_+$ and $(p')_-=p'_-$.

Whenever $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ and $1<p_-$, the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$.

We now recall the definition of Herz spaces with variable exponents from [1, Definition 3.1]. Let $B_k=\{x\in\mathbb{R}^n:|x|\leqslant 2^k\}$ and $R_k=B_k\backslash B_{k-1}$, $k\in\mathbb{Z}$. Define $\chi_k=\chi_{R_k}$, $k\in\mathbb{Z}$ and $\mathscr{B}=\{B_k:k\in\mathbb{Z}\}$.

DEFINITION 3. Let $0<q\leqslant\infty$, $p(\cdot):\mathbb{R}^n\to(0,\infty)$ be a Lebesgue measurable function and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$.

The Herz space with variable exponent $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}=\left\|\{\|2^{k\alpha(\cdot)}f\chi_k\|_{L^{p(\cdot)}}\}_{k=-\infty}^{\infty}\right\|_{l^q}<\infty.$$

The above definition gives the homogeneous version of the Herz spaces with variable exponent. We also have the inhomogeneous version of the Herz spaces with variable exponents. As the results for them are similar, for brevity, we just consider the homogeneous version only.

The following result shows that under some mild conditions, the characteristic functions of balls belong to $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$.

PROPOSITION 1. *Let $0<q<\infty$, $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<p_-\leqslant p_+<\infty$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. If $\frac{n}{p(0)}+\alpha_->0$, then for any $B\in\mathbb{B}$, $\chi_B\in\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ and*

$$\|\chi_{B_l}\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}\leqslant C(1+2^{l(\alpha_++\frac{n}{p_\infty})}),\quad l\in\mathbb{N} \tag{4}$$

$$\|\chi_{B_l}\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}\leqslant C2^{l(\alpha_-+\frac{n}{p(0)})},\quad l\in\mathbb{Z}\backslash\mathbb{N}. \tag{5}$$

Proof. When $k>l$, we have $\|2^{k\alpha(\cdot)}\chi_{B_l}\chi_k\|_{L^{p(\cdot)}}=0$. As $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$, according to [7, Corollary 4.5.9], we find that

$$\|2^{k\alpha(\cdot)}\chi_{B_l}\chi_k\|_{L^{p(\cdot)}}\leqslant 2^{k\alpha_+}\|\chi_k\|_{L^{p(\cdot)}}\leqslant C2^{k\alpha_+}2^{\frac{kn}{p_\infty}},\quad 0<k\leqslant l,$$
$$\|2^{k\alpha(\cdot)}\chi_{B_l}\chi_k\|_{L^{p(\cdot)}}\leqslant 2^{k\alpha_-}\|\chi_k\|_{L^{p(\cdot)}}\leqslant C2^{k\alpha_-}2^{\frac{kn}{p(0)}},\quad k\leqslant 0.$$

Consequently, when $l\in\mathbb{N}$, we have

$$\begin{aligned}\|\chi_{B_l}\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}&=\left\|\{\|2^{k\alpha(\cdot)}\chi_{B_l}\chi_k\|_{L^{p(\cdot)}}\}_{k=-\infty}^{\infty}\right\|_{l^q}\\&\leqslant C\left(\sum_{k=-\infty}^{0}2^{k\alpha_-q}2^{\frac{knq}{p(0)}}+C\sum_{k=1}^{l}2^{k\alpha_+q}2^{\frac{knq}{p_\infty}}\right)^{1/q}\\&\leqslant C(1+2^{l(\alpha_++\frac{n}{p_\infty})})\end{aligned}$$

for some $C>0$ because $\frac{n}{p(0)}+\alpha_->0$.

When $l\in\mathbb{Z}\backslash\mathbb{N}$, we have

$$\begin{aligned}\|\chi_{B_l}\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}&=\left\|\{\|2^{k\alpha(\cdot)}\chi_{B_l}\chi_k\|_{L^{p(\cdot)}}\}_{k=-\infty}^{\infty}\right\|_{l^q}\\&\leqslant C\left(\sum_{k=-\infty}^{l}2^{k\alpha_-q}2^{\frac{knq}{p(0)}}\right)^{1/q}\leqslant C2^{l(\alpha_-+\frac{n}{p(0)})}\end{aligned}$$

for some $C>0$ because $\frac{n}{p(0)}+\alpha_->0$. Therefore, $\chi_{B_l}\in \dot K^{\alpha(\cdot)}_{p(\cdot),q}$.

For any $B\in\mathbb{B}$, there is a $l\in\mathbb{Z}$ such that $\chi_B\leqslant \chi_{B_l}$. Thus, we conclude that for any $B\in\mathbb{B}$, $\chi_B\in \dot K^{\alpha(\cdot)}_{p(\cdot),q}$. $\square$

Let us denote the dual space of $\dot K^{\alpha(\cdot)}_{p(\cdot),q}$ by $(\dot K^{\alpha(\cdot)}_{p(\cdot),q})^*$. The following result identifies the dual of $\dot K^{\alpha(\cdot)}_{p(\cdot),q}$.

THEOREM 2. *Let $1<q<\infty$, $p(\cdot):\mathbb{R}^n\to(1,\infty)$ be a Lebesgue measurable function and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. If $\alpha\in C_0^{\log}(\mathbb{R}^n)$ satisfies (3), then*

$$(\dot K^{\alpha(\cdot)}_{p(\cdot),q})^*=\dot K^{-\alpha(\cdot)}_{p'(\cdot),q'}.$$

For the proof of the preceding result, the reader is referred to the proof of [17, Theorem 2.6].

When $-\frac{n}{p(0)}<\alpha_-\leqslant\alpha_+<n(1-\frac{1}{p(0)})$, Proposition 1 assures that for any $B\in\mathbb{B}$, $\chi_B\in \dot K^{\alpha(\cdot)}_{p(\cdot),q}$. As $\frac{1}{p'(0)}=1-\frac{1}{p(0)}$ and $(-\alpha)_-=-\alpha_+$, we see that $\alpha_+<n(1-\frac{1}{p(0)})$ gives $-\frac{n}{p'(0)}<(-\alpha)_-$. Thus, for any $B\in\mathbb{B}$, $\chi_B\in \dot K^{-\alpha(\cdot)}_{p'(\cdot),q'}$. Consequently, Theorem 2 asserts that $\dot K^{\alpha(\cdot)}_{p(\cdot),q}$ is a ball Banach function space. For brevity, we refer the reader to [40] for the definition of ball Banach function space.

In addition, we have the following consequences of Theorem 2.

COROLLARY 1. *Let $1<q<\infty$, $p:\mathbb{R}^n\to(1,\infty)$ be a Lebesgue measurable function and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. If $\alpha\in C_0^{\log}(\mathbb{R}^n)$ satisfies (3), then*

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx\leqslant \|f\|_{\dot K^{\alpha(\cdot)}_{p(\cdot),q}}\|g\|_{\dot K^{-\alpha(\cdot)}_{p'(\cdot),q'}}.$$

COROLLARY 2. *Let $1<q<\infty$ and $p:\mathbb{R}^n\to(1,\infty)$ be a Lebesgue measurable function and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. If $\alpha\in C_0^{\log}(\mathbb{R}^n)$ satisfies (3), then*

$$\|f\|_{\dot K^{\alpha(\cdot)}_{p(\cdot),q}}=\sup_{g\in \dot K^{-\alpha(\cdot)}_{p'(\cdot),q'}}\int_{\mathbb{R}^n}|f(x)g(x)|dx. \tag{6}$$

The following is the boundedness of the Hardy-Littlewood maximal operator on Herz spaces with variable exponents.

THEOREM 3. *Let $0<q\leqslant\infty$, $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<p_-\leqslant p_+<\infty$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. If $\alpha\in C_0^{\log}(\mathbb{R}^n)$ satisfies (3) and*

$$-\frac{n}{p_+}<\alpha_-\leqslant\alpha_+<n\left(1-\frac{1}{p_-}\right), \tag{7}$$

then M is bounded on $\dot K^{\alpha(\cdot)}_{p(\cdot),q}$.

For the proof of Theorem 3, the reader may consult [1, Theorem 4.2 and Corollary 4.7].

Next, we recall a result from [17, Lemma 2.9]. Notice that (7) guarantees that $-\frac{n}{p(0)} < \alpha_- \leqslant \alpha_+ < n(1-\frac{1}{p(0)})$. Thus, for any $B \in \mathbb{B}$, $\chi_B \in \dot{K}^{\alpha(\cdot)}_{p(\cdot),q} \cap \dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}$.

LEMMA 1. *Let $1 < q < \infty$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ satisfying (3). If $\alpha(\cdot) \in C^{\log}_0(\mathbb{R}^n)$ satisfies (3) and (7), then there is a constant $C > 0$ such that for any $B \in \mathbb{B}$,*

$$\|\chi_B\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \|\chi_B\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}} \leqslant C|B|.$$

LEMMA 2. *Let $1 < q < \infty$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ satisfying (3). If $\alpha(\cdot) \in C^{\log}_0(\mathbb{R}^n)$ satisfies (3) and (7), then for any*

$$1 < \theta < \min\left(p'_-, \frac{n}{\frac{n}{p'_-} - \alpha_-}\right), \tag{8}$$

there is a constant $C > 0$ such that for any $j \in \mathbb{Z}$ and $k \in \mathbb{N}$

$$\frac{\|\chi_{B_j}\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}}{\|\chi_{B_{j+k}}\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}} \leqslant C 2^{-(1-\frac{1}{\theta})kn}. \tag{9}$$

Proof. As $p'_+ = (p_-)'$, $p'_- = (p_+)'$, $(-\alpha)_- = -\alpha_+$ and $(-\alpha)_+ = -\alpha_-$, (7) yields $-\alpha_- < n - \frac{n}{p'_-}$ and $\alpha_- < \frac{n}{p'_+} \leqslant \frac{n}{p'_-}$. Thus, $\frac{n}{\frac{n}{p'_-} - \alpha_-} > 1$ and θ is well defined.

In addition,

$$-\frac{n}{p'_+} < (-\alpha)_- \leqslant (-\alpha)_+ < n\left(1 - \frac{1}{p'_-}\right). \tag{10}$$

Theorem 3 guarantees that the Hardy-Littlewood maximal operator is bounded on $\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}$. Furthermore, $p'_- > \theta$ assures that $(p'/\theta)_- > 1$. Moreover, (10) assures that

$$-\frac{n}{(p'/\theta)_+} = -\frac{n\theta}{p'_+} < (-\alpha\theta)_- \leqslant (-\alpha\theta)_+ = -\alpha_-\theta. \tag{11}$$

In view of (8), we get

$$\theta < \frac{n}{\frac{n}{p'_-} - \alpha_-}.$$

Thus, we obtain

$$\left(\frac{n}{p'_-} - \alpha_-\right)\theta < n.$$

That is,

$$-\alpha_-\theta < n - \frac{n\theta}{p'_-}.$$

Consequently, (11) gives

$$\begin{aligned}-\frac{n}{(p'/\theta)_+} = -\frac{n\theta}{p'_+} < (-\alpha\theta)_- \leqslant (-\alpha\theta)_+ < n\left(1-\frac{\theta}{p'_-}\right) \\ = n\left(1-\frac{1}{(p'/\theta)_-}\right).\end{aligned}$$

Thus, Theorem 3 asserts that the Hardy-Littlewood maximal operator is bounded on $\dot{K}^{-\theta\alpha(\cdot)}_{p'(\cdot)/\theta,q'/\theta}$.

We have a constant $C>0$ such that for any $k\in\mathbb{N}$

$$C_0\chi_{B_{j+k}}(x)2^{-kn} \leqslant \mathrm{M}\chi_{B_j}(x).$$

Applying the norm $\|\cdot\|_{\dot{K}^{-\theta\alpha(\cdot)}_{p'(\cdot)/\theta,q'/\theta}}$ on both sides of the above inequality, we obtain

$$\begin{aligned}C_0 2^{-kn}\|\chi_{B_{j+k}}\|^{\theta}_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}} &= C_0 2^{-kn}\|\chi_{B_{j+k}}\|_{\dot{K}^{-\theta\alpha(\cdot)}_{p'(\cdot)/\theta,q'/\theta}} \leqslant \|\mathrm{M}\chi_{B_j}\|_{\dot{K}^{-\theta\alpha(\cdot)}_{p'(\cdot)/\theta,q'/\theta}} \\ &\leqslant C_1\|\chi_{B_j}\|_{\dot{K}^{-\theta\alpha(\cdot)}_{p'(\cdot)/\theta,q'/\theta}} = C_1\|\chi_{B_j}\|^{\theta}_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}\end{aligned}$$

for some $C_1>0$.

Corollary 1 and Lemma 1 yield

$$2^{-kn}\frac{2^{\theta n(j+k)}}{\|\chi_{B_{j+k}}\|^{\theta}_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}} \leqslant C\frac{2^{\theta nj}}{\|\chi_{B_j}\|^{\theta}_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}}$$

for some $C>0$. The above inequality gives (9). □

We now recall the definition of the Herz-Morrey spaces with variable exponents from [9, 48, 49].

DEFINITION 4. Let $\lambda \geqslant 0$, $0<q<\infty$, $p(\cdot):\mathbb{R}^n\to(0,\infty]$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ be Lebesgue measurable functions. The Herz-Morrey space with variable exponent $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} = \sup_{k\in\mathbb{Z}} 2^{-k\lambda}\left(\sum_{j=-\infty}^{k}\|2^{j\alpha(\cdot)}f\chi_j\|^q_{L^{p(\cdot)}}\right)^{\frac{1}{q}} < \infty.$$

It is easy to see that

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} = \sup_{k\in\mathbb{Z}} 2^{-k\lambda}\|f\chi_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}. \tag{12}$$

When $\lambda = 0$, the Herz-Morrey space with variable exponent becomes the Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ studied in [1, 17, 18]. In addition, when $\alpha = 0$ and $p = q$, the Herz-Morrey space with variable exponent $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ reduces to the Lebesgue space L^q. When $\alpha(\cdot)$ and $p(\cdot)$ are constant functions, $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ is the classical Herz-Morrey space studied in [29]. In addition, when $\lambda = 0$, $\alpha(\cdot)$ and $p(\cdot)$ are constant functions, then $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ is the classical Herz space studied in [15, 30].

The following result assures that the Herz-Morrey space with variable exponent is nontrivial whenever the exponent function $p(\cdot)$ satisfies some mild conditions.

PROPOSITION 2. *Let* $0 < q < \infty$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *with* $1 < p_- \leqslant p_+ < \infty$ *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. *If*

$$\frac{n}{p(0)} + \alpha_- > \lambda, \tag{13}$$

then for any $B \in \mathbb{B}$, $\chi_B \in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.

Proof. Let $l \in \mathbb{Z}$. In view of (4) and (5), we have

$$2^{-k\lambda}\|\chi_{B_l}\chi_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \leqslant 2^{-k\lambda}\|\chi_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \leqslant C2^{-k\lambda}(1+2^{k(\alpha_+ + \frac{n}{p_\infty})}), \quad k \in \mathbb{N}$$
$$2^{-k\lambda}\|\chi_{B_l}\chi_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \leqslant 2^{-k\lambda}\|\chi_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \leqslant C2^{-k\lambda}2^{k(\alpha_- + \frac{n}{p(0)})}, \quad k \in \mathbb{Z}\backslash\mathbb{N}.$$

As $\|\chi_{B_l}\chi_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} = 0$ when $k > l$, (13) yields

$$\begin{aligned}
\|\chi_{B_l}\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} &= \sup_{k \leqslant l} 2^{-k\lambda}\|\chi_{B_l}\chi_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \\
&\leqslant C\sup\left\{\sup_{0\leqslant k\leqslant l}(2^{-k\lambda}+2^{k(-\lambda+\alpha_+ + \frac{n}{p_\infty})}), \sup_{k<0} 2^{-k\lambda}2^{k(\alpha_- + \frac{n}{p(0)})}\right\} \\
&\leqslant C(1+2^{l(\alpha_+ + \frac{n}{p_\infty})})
\end{aligned}$$

for some $C > 0$.

For any $B \in \mathbb{B}$, there is a $l \in \mathbb{Z}$ such that $\chi_B \leqslant \chi_{B_l}$. Thus, for any $B \in \mathbb{B}$, $\chi_B \in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. $\square$

At the end of this section, we study the r-convexification of the Herz-Morrey space with variable exponent. For any $r > 0$, we write $f \in (M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q})^r$ if

$$\|f\|_{(M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q})^r} = \||f|^r\|^{\frac{1}{r}}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} < \infty.$$

We see that

$$\begin{aligned}
\|f\|_{(M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q})^r} &= \||f|^r\|^{\frac{1}{r}}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \\
&= \sup_{k\in\mathbb{Z}} 2^{-k\lambda/r}\left(\sum_{j=-\infty}^{k}\|2^{j\alpha(\cdot)}|f|^r\chi_j\|^q_{L^{p(\cdot)}}\right)^{\frac{1}{qr}} \\
&= \sup_{k\in\mathbb{Z}} 2^{-k\lambda/r}\left(\sum_{j=-\infty}^{k}\|2^{j\alpha(\cdot)/r}f\chi_j\|^{qr}_{L^{p(\cdot)r}}\right)^{\frac{1}{qr}} \\
&= \|f\|_{M\dot{K}^{\alpha(\cdot)/r,\lambda/r}_{p(\cdot)r,qr}}. \qquad (14)
\end{aligned}$$

Thus, the r-convexification of $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ equals to $M\dot{K}^{\alpha(\cdot)/r,\lambda/r}_{p(\cdot)r,qr}$.

3. Herz-block spaces with variable exponent

We introduce the Herz-block space with variable exponent in this section. We establish some duality results for the Herz-block space with variable exponent and the Herz-Morrey space with variable exponent.

DEFINITION 5. Let $\lambda \geqslant 0$, $0 < q < \infty$, $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions. For any Lebesgue measurable function b , we write $b \in b\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ if $\operatorname{supp} b \subseteq B_j$ for some $j \in \mathbb{Z}$ and

$$\|b\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \leqslant 2^{-j\lambda}. \qquad (15)$$

We call b a Herz-block. The Herz-block space with variable exponent $B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ is defined as

$$B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q} = \left\{\sum_{k=1}^{\infty}\lambda_k b_k : \sum_{k=1}^{\infty}|\lambda_k| < \infty \text{ and } b_k \in b\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}\right\}. \qquad (16)$$

The Herz-block space with variable exponent $B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ is endowed with the norm

$$\|f\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} = \inf\left\{\sum_{k=1}^{\infty}|\lambda_k| \text{ such that } f = \sum_{k=1}^{\infty}\lambda_k b_k \text{ a.e.}\right\}. \qquad (17)$$

When $p(\cdot) = p$ is a constant function and $p = q$, $p \in (1,\infty)$ and $\alpha(\cdot) \equiv 0$, the Herz-block space becomes the classical block space [2]. For the studies of the block space, see [2, 16].

The followings are the duality results for the Herz-Morrey spaces with variable exponents and the Herz-block spaces with variable exponents. We begin with Hölder inequality.

LEMMA 3. *Let $\lambda \geqslant 0$, $1 < q < \infty$, $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions. We have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \|g\|_{B\dot{K}^{-\alpha(\cdot),\lambda}_{p'(\cdot),q'}}.$$

Proof. Let $g = \sum_{k=1}^{\infty} \lambda_k b_k$ where $\{b_k\} \subset b\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ with $\operatorname{supp} b_k \subset B_j$, $j \in \mathbb{Z}$ and $\sum_{k=1}^{\infty} |\lambda_k| \leqslant 2\|g\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}$.

Corollary 1 assures that

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)b_k(x)|dx = \int_{B_j} |f(x)b_i(x)|dx &\leqslant C\|f\chi_j\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \|b_k\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}} \\ &\leqslant C2^{-j\lambda}\|f\chi_j\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}\end{aligned}$$

for some $C > 0$. In view of (12), we have

$$\int_{\mathbb{R}^n} |f(x)b_k(x)|dx \leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}.$$

Consequently,

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)g(x)|dx &\leqslant \sum_{k=1}^{\infty} |\lambda_k| \int_{\mathbb{R}^n} |f(x)b_k(x)|dx \\ &\leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \sum_{k=1}^{\infty} |\lambda_k| \leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \|g\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}\end{aligned}$$

for some $C > 0$. $\square$

The following gives a condition that guarantees the membership of $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.

PROPOSITION 3. *Let $\lambda \geqslant 0$, $1 < q < \infty$, $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions. If a Lebesgue measurable function f satisfies*

$$\sup_{b \in b\dot{K}^{-\alpha(\cdot),\lambda}_{p'(\cdot),q'}} \int_{\mathbb{R}^n} |f(x)b(x)|dx < \infty,$$

then $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.

Proof. For any $g \in \dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}$ with $\|g\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}} \leqslant 1$ and $j \in \mathbb{Z}$, write $b_{g,j} = 2^{-j\lambda} g\chi_{B_j}$. Obviously, $b_{g,j} \in b\dot{K}^{-\alpha(\cdot),\lambda}_{p'(\cdot),q'}$.

Corollary (2) assures that

$$\sup_{\substack{g\in \dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'} \\ \|g\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}\leqslant 1}} \int_{\mathbb{R}^n} |f(x)b_{g,B}(x)|dx$$
$$= 2^{-j\lambda} \sup_{\substack{g\in \dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'} \\ \|g\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}\leqslant 1}} \int_{\mathbb{R}^n} |f(x)g(x)\chi_j(x)|dx = 2^{-j\lambda}\|\chi_j f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}.$$

By taking supremum over $j\in\mathbb{Z}$, we obtain

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} = \sup_{j\in\mathbb{Z}} 2^{-j\lambda}\|\chi_j f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \leqslant \sup_{b\in b\dot{K}^{-\alpha(\cdot),\lambda}_{p'(\cdot),q'}} \int_{\mathbb{R}^n} |f(x)b(x)|dx < \infty.$$

Thus, $f\in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. □

Next, we present the norm conjugate formula for $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. This formula follows from Lemma 3 and Proposition 3.

PROPOSITION 4. *Let $\lambda\geqslant 0$, $1<q<\infty$, $p(\cdot):\mathbb{R}^n\to(1,\infty)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ be Lebesgue measurable functions. There exist constants $C_0, C_1>0$ such that for any $f\in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$,*

$$C_0\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \leqslant \sup_{b\in b\dot{K}^{-\alpha(\cdot),\lambda}_{p'(\cdot),q'}} \int_{\mathbb{R}^n} |f(x)b(x)|dx \leqslant C_1\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}.$$

Obviously, $\chi_B\in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ and $\chi_B\in B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. Therefore, the above results shows that $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ is a ball Banach function spaces. For simplicity, we refer the reader to [3, 8, 27, 40, 44, 45, 50] for the definition and applications of ball Banach function spaces.

We now establish the boundedness of the Hardy-Littlewood maximal function on $B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. We introduce some notions used in the following theorem. Let $0<q<\infty$, $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<p_-\leqslant p_+<\infty$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. To present the boundedness of the Hardy-Littlewood maximal operator on the Herz-block spaces with variable exponents, we define

$$\kappa_{p(\cdot),\alpha(\cdot)} = \min\left(p_-, \frac{n}{\frac{n}{p_-}+\alpha_+}\right).$$

THEOREM 4. *Let $1<q<\infty$, $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<p_-\leqslant p_+<\infty$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. If $0\leqslant\frac{\lambda}{n}<\frac{1}{(\kappa_{p(\cdot),\alpha(\cdot)})'}$, $\alpha\in C_0^{\log}(\mathbb{R}^n)$ satisfies (3) and (7), then M is bounded on $B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.*

Proof. We first show that M is well defined on $B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. Since $\alpha(\cdot)$ and $p(\cdot)$ satisfy (7), we have

$$\frac{n}{p'(0)}+(-\alpha)_{-}=n-\frac{n}{p(0)}-\alpha_{+}>n-\frac{n}{p_{-}}-\alpha_{+}>0.$$

Proposition 2 guarantees that $\chi_B\in M\dot{K}^{-\alpha(\cdot),\lambda}_{p'(\cdot),q}$, $B\in\mathbb{B}$. Consequently, Lemma 3 assures that there is a constant $C>0$ such that for any $f\in B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$

$$\int_B|f(x)|dx\leqslant C\|\chi_B\|_{M\dot{K}^{-\alpha(\cdot),\lambda}_{p'(\cdot),q}}\|f\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}.$$

Thus, f is a locally integrable function and M is well defined on $B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.

For any $b\in b\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ with $\operatorname{supp} b\subset B_j$, $j\in\mathbb{Z}$, define $g_0=\chi_{B_{j+1}}\mathrm{M}b$, $g_k=\chi_{R_{j+k+1}}\mathrm{M}b$, $k\in\mathbb{N}\backslash\{0\}$. Theorem 3 gives

$$\|g_0\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}=\|\chi_{B_{j+1}}\mathrm{M}b\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}\leqslant\|\mathrm{M}b\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}\leqslant C\|b\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}\leqslant C2^{-(j+1)\lambda}$$

for some $C>0$. Since $\operatorname{supp} g_0\subseteq B_{j+1}$, we find that $g_0=C_0\mathfrak{b}$ where $\mathfrak{b}$ is a Herz-block and C_0 is independent of b and j.

For any $x\in R_{j+k+1}$, $k\in\mathbb{N}\backslash\{0\}$, the definition of the Hardy-Littlewood maximal function, Corollary 1 and Definition 15 yield

$$\begin{aligned}g_k(x)=\mathrm{M}b(x)&\leqslant C\frac{1}{2^{n(j+k)}}\int_{B_j}|b(y)|dy\\&\leqslant C2^{-n(j+k)}\|b\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}\|\chi_{B_j}\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}\leqslant 2^{-n(j+k)}2^{-j\lambda}\|\chi_{B_j}\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}.\end{aligned}$$

As $\operatorname{supp} g_k\subseteq B_{j+k+1}$, by applying the norm $\|\cdot\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}$ on both sides of the above inequality, we obtain

$$\begin{aligned}\|g_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}&\leqslant C\|\chi_{R_{j+k+1}}\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}2^{-n(k+j)}2^{-j\lambda}\|\chi_{B_j}\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}\\&\leqslant C\|\chi_{B_{j+k+1}}\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}2^{-n(k+j)}2^{-j\lambda}\|\chi_{B_j}\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}.\end{aligned}$$

Lemma 1 yields

$$\begin{aligned}\|g_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}}&\leqslant C2^{-j\lambda}\frac{\|\chi_{B_j}\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}}{\|\chi_{B_{j+k+1}}\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}}\\&=C2^{-(j+k+1)\lambda}2^{(k+1)\lambda}\frac{\|\chi_{B_j}\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}}{\|\chi_{B_{j+k+1}}\|_{\dot{K}^{-\alpha(\cdot)}_{p'(\cdot),q'}}}.\end{aligned}$$

Let $\theta \in (1,\infty)$ be selected such that

$$\frac{\lambda}{n} < \frac{1}{\theta'} < \frac{1}{(\kappa_{p(\cdot),\alpha(\cdot)})'}.$$

We find that $\theta < \kappa_{p(\cdot),\alpha(\cdot)}$. Thus, Lemma 2 yields a constant $C>0$ such that

$$\|g_k\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}} \leqslant C 2^{-(j+k+1)\lambda} 2^{-(1-\frac{1}{\theta}-\frac{\lambda}{n})kn}.$$

Therefore,

$$g_k = C2^{-(1-\frac{1}{\theta}-\frac{\lambda}{n})kn}\mathfrak{b}_k$$

where $\mathfrak{b}_k$ is a Herz-block with $\operatorname{supp}\mathfrak{b}_j \subseteq B_{j+k+1}$ and C is a constant independent of b and k.

As $\sum_{k=1}^{\infty} 2^{-(1-\frac{1}{\theta}-\frac{\lambda}{n})kn} < \infty$, we find that

$$\mathrm{M}b = \sum_{k=0}^{\infty} g_k \in B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}. \tag{18}$$

Let $f \in B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. We have a family of Herz-block $\{b_i\}_{i=0}^{\infty}$ and $\{\lambda_i\}_{i=0}^{\infty}$ such that $f = \sum_{i=0}^{\infty}\lambda_i b_i$ and $\sum_{i=0}^{\infty}|\lambda_i| \leqslant 2\|f\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}$. For any $k\in\mathbb{N}$, (18) gives

$$\mathrm{M}b_i = \sum_{k=0}^{\infty}\gamma_{i,k}b_{i,k}$$

for some Herz-blocks $\{b_{i,k}\}_{k=0}^{\infty}$ with $\sum_{k=0}^{\infty}|\gamma_{i,k}| < C$ where C is independent of i and k.

Let $F = \sum_{i=0}^{\infty}|\lambda_i|\mathrm{M}b_i$. We see that $F = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}|\lambda_i|\gamma_{i,k}b_{i,k}$ and

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}|\lambda_i||\gamma_{i,k}| \leqslant C\sum_{i=0}^{\infty}|\lambda_i| \leqslant C\|f\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \tag{19}$$

for some $C>0$. As $b_{i,k}$, $0\leqslant i,k<\infty$ are Herz-blocks, we find that $F\in B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ with $\|F\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \leqslant C\|f\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}$.

Write $D(x) = \frac{\mathrm{M}f(x)}{F(x)}$ when $F(x)\neq 0$ and $D(x)=0$ when $F(x)=0$. As

$$\mathrm{M}f \leqslant \sum_{k=0}^{\infty}|\lambda_i|\mathrm{M}b_i(x) = F(x),$$

we have $|D(x)|\leqslant 1$, $x\in\mathbb{R}^n$. Consequently, $Db_{i,k}$, $0\leqslant i,k<\infty$ are Herz-blocks and $\operatorname{supp}Db_{i,k}\subseteq\operatorname{supp}b_{i,k}$. Since

$$\mathrm{M}f(x) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\lambda_i\gamma_{i,k}b_{i,k}(x)D(x),$$

we find that $\mathrm{M}f \in B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ and (19) gives

$$\|\mathrm{M}f\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \leqslant \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} |\lambda_i||\gamma_{i,k}| \leqslant C\|f\|_{B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}$$

for some $C>0$. Thus, the Hardy-Littlewood maximal operator is bounded on $B\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. $\square$

4. Main results

In this section, we establish a general result for the boundedness of nonlinear operators on Herz-Morrey spaces with variable exponents. We obtain this result by extending the extrapolation to Herz-Morrey spaces with variable exponents. In order to use the extrapolation, we need the Muckenhoput class of weight functions.

DEFINITION 6. For $1<p<\infty$, a locally integrable function $\omega:\mathbb{R}^n \to [0,\infty)$ is said to be an A_p weight if

$$[\omega]_{A_p} = \sup_{B\in\mathbb{B}} \left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega:\mathbb{R}^n \to [0,\infty)$ is said to be an A_1 weight if there is a constant $C>0$ such that for any $B\in\mathbb{B}$

$$\frac{1}{|B|}\int_B \omega(y)dy \leqslant C\omega(x), \quad a.e.\, x\in B.$$

The infimum of all such C is denoted by $[\omega]_{A_1}$. We define $A_\infty = \cup_{p\geqslant 1}A_p$.

For any $p\in(0,\infty)$, $\omega\in A_\infty$, the weighted Lebesgue space $L^p(\omega)$ consists of all $f\in\mathscr{M}$ satisfying $\|f\|_{L^p(\omega)} = (\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx)^{1/p}<\infty$.

For any $\delta\in(0,1]$, define $\mathrm{M}_\delta(h) = (\mathrm{M}(|h|^{\frac{1}{\delta}}))^\delta$. Let $q\in(0,\infty)$, $\delta\in(0,1]$ and $p_0\in(0,q)$. Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $p_0<p_-\leqslant p_+<\infty$ and $(p_+/p_0)'>\frac{1}{\delta}$. The operator $\mathscr{R}_\delta$ is defined by

$$\mathscr{R}_\delta h = \sum_{k=0}^{\infty}\frac{\mathrm{M}^k_\delta(h)}{2^k\|\mathrm{M}^k_\delta\|_{B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}}}, \quad h\in L^1_{loc},$$

where M^k_δ is the k^{th} iterations of the operator M_δ, $\|\mathrm{M}^k_\delta\|_{B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}}$ is the operator norm of M^k_δ on $B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}$ and $\mathrm{M}^0_\delta(h)=|h|$. The followings are the boundedness of M_δ and $\mathscr{R}_\delta$ on the local block spaces with variable exponents.

The following give several useful properties for the operator $\mathscr{R}_\delta$.

PROPOSITION 5. *Let* $q\in(1,\infty)$, $\delta\in(0,1]$ *and* $p_0\in(0,q)$. *Let* $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ *with* $p_0<p_-\leqslant p_+<\infty$ *and* $\alpha\in C_0^{\log}(\mathbb{R}^n)$ *satisfy* (3) *and*

$$n\left(\frac{1-\delta}{p_0}-\frac{1}{p_+}\right)<\alpha_-\leqslant\alpha_+<n\left(\frac{1}{p_0}-\frac{1}{p_-}\right). \tag{20}$$

If $\min((p_+/p_0)',(q/p_0)')>\frac{1}{\delta}$ *and*

$$0\leqslant\frac{\lambda}{n}<\frac{\delta}{p_0(\kappa_{\delta(p(\cdot)/p_0)',-p_0\alpha(\cdot)/\delta})'}, \tag{21}$$

then M_δ *is bounded on* $B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}$.

The operator $\mathscr{R}_\delta$ *is well defined on* $B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}$ *and there is a constant* $C>0$ *such that for any* $h\in B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}$,

$$|h(x)|\leqslant\mathscr{R}_\delta h(x) \tag{22}$$

$$\|\mathscr{R}_\delta h\|_{B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}}\leqslant 2\|h\|_{B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}} \tag{23}$$

$$[(\mathscr{R}_\delta h)^{\frac{1}{\delta}}]_{A_1}\leqslant C\|\mathrm{M}_\delta\|^{\frac{1}{\delta}}_{B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}}. \tag{24}$$

Proof. As M_δ is sublinear, to obtain the boundedness of M on the Herz-block space with variable exponent $B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}$, it suffices to show that there is a constant $C>0$ such that for any $b\in b\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}$

$$\|\mathrm{M}_\delta(b)\|_{B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}}\leqslant C. \tag{25}$$

Let $b\in b\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}$ with $\operatorname{supp} b\subset B_j$, $j\in\mathbb{Z}$.

We find that $|b|^{\frac{1}{\delta}}\in b\dot{K}^{-p_0\alpha(\cdot)/\delta,p_0\lambda/\delta}_{\delta(p(\cdot)/p_0)',\delta(q/p_0)'}$. Let θ be selected so that

$$\frac{p_0\lambda}{n\delta}<\frac{1}{\theta'}<\frac{1}{(\kappa_{\delta(p(\cdot)/p_0)',-p_0\alpha(\cdot)/\delta})'}.$$

We have $\theta<\kappa_{\delta(p(\cdot)/p_0)',-p_0\alpha(\cdot)/\delta}$ and $\frac{p_0\lambda}{n\delta}<1-\frac{1}{\theta}$. The proof of Theorem 4 yields $\{b_j\}_{j=0}^{\infty}\in b\dot{K}^{-p_0\alpha(\cdot)/\delta,p_0\lambda/\delta}_{\delta(p(\cdot)/p_0)',\delta(q/p_0)'}$ such that

$$(\mathrm{M}_\delta(b))^{\frac{1}{\delta}}=\mathrm{M}(|b|^{\frac{1}{\delta}})\leqslant C\sum_{j=0}^{\infty}2^{(1-\frac{1}{\theta}-\frac{\lambda}{n\delta})j}b_j.$$

Since $\delta\in(0,1]$, we get

$$|\mathrm{M}_\delta(b)|\leqslant C\left(\sum_{j=0}^{\infty}2^{(1-\frac{1}{\theta}-\frac{\lambda}{n\delta})j}b_j\right)^{\delta}\leqslant C\sum_{j=0}^{\infty}2^{(1-\frac{1}{\theta}-\frac{\lambda}{n\delta})\delta j}|b_j|^{\delta}$$

for some $C>0$ independent of b. As $b_j \in b\dot{K}^{-p_0\alpha(\cdot)/\delta, p_0\lambda/\delta}_{\delta(p(\cdot)/p_0)', \delta(q/p_0)'}$, we have $|b_j|^\delta \in b\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}$. Consequently, $\mathrm{M}_\delta(b) \in B\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}$ and (25) holds. That is, $\mathscr{R}_\delta$ is well defined on $B\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}$.

Moreover, the definition of $\mathscr{R}_\delta$ yields (22) and (23). As M_δ is sublinear, for any $h \in B\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}$, we find that

$$\begin{aligned}(\mathrm{M}(\mathscr{R}_\delta h)^{\frac{1}{\delta}})^\delta = \mathrm{M}_\delta(\mathscr{R}_\delta h) &\leqslant \sum_{k=0}^{\infty} \frac{\mathrm{M}_\delta^{k+1}(h)}{2^k \|\mathrm{M}_\delta^k\|_{B\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}}} \\ &\leqslant 2\|\mathrm{M}_\delta\|_{B\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}} \mathscr{R}_\delta h.\end{aligned}$$

Thus, $(\mathscr{R}_\delta h)^{\frac{1}{\delta}} \in A_1$ and (24) is valid. $\square$

We are now ready to establish the boundedness of nonlinear operators on the Herz-Morrey spaces with variable exponents.

THEOREM 5. *Let $q \in (1,\infty)$, $\delta \in (0,1]$ and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \leqslant p_+ < \infty$. Let $\alpha \in C_0^{\log}(\mathbb{R}^n)$ satisfy (3). Suppose that there exists a $p_0 \in (0, \min(p_-, q))$ satisfying $\min((p_+/p_0)', (q/p_0)') > \frac{1}{\delta}$, (20) and (21). If for any $\omega \in A_1$, the operator $T : L^{p_0}(\omega^\delta) \to L^{p_0}(\omega^\delta)$ is bounded, then T can be extended to be a bounded operator on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.*

Proof. For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$ and $h \in B\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}$, (14), Lemma 3 and (23) guarantee that

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^{p_0} \mathscr{R}_\delta h(x)dx &\leqslant \||f|^{p_0}\|_{M\dot{K}^{p_0\alpha(\cdot),\lambda}_{p(\cdot)/p_0, q/p_0}} \|\mathscr{R}_\delta h\|_{B\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}} \\ &\leqslant C\|f\|^{p_0}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \|h\|_{B\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}} < \infty.\end{aligned}$$

Therefore,

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q} \hookrightarrow \bigcap_{h \in B\dot{K}^{-p_0\alpha(\cdot), p_0\lambda}_{(p(\cdot)/p_0)', (q/p_0)'}} L^{p_0}(\mathscr{R}_\delta h). \tag{26}$$

Thus, for any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$, Tf is well defined. Consequently, T is a well defined operator on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.

Propositions (3) and (4) assure that

$$\begin{aligned}\|Tf\|^{p_0}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} &= \||Tf|^{p_0}\|_{M\dot{K}^{p_0\alpha(\cdot),\lambda}_{p(\cdot)/p_0, q/p_0}} \\ &\leqslant C \sup\left\{\int_{\mathbb{R}^n} |Tf(x)|^{p_0}|h(x)|dx : \|h\|_{b\dot{K}^{-p_0\alpha(\cdot)/\delta, p_0\lambda/\delta}_{\delta(p(\cdot)/p_0)', \delta(q/p_0)'}} \leqslant 1\right\}\end{aligned} \tag{27}$$

for some $C>0$.

In view of (22), we find that

$$\|Tf\|^{p_0}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \leqslant C\sup\left\{\int_{\mathbb{R}^n}|Tf(x)|^{p_0}\mathscr{R}_\delta h(x)dx : \|h\|_{b\dot{K}^{-p_0\alpha(\cdot)/\delta,p_0\lambda/\delta}_{\delta(p(\cdot)/p_0)',\delta(q/p_0)'}}\leqslant 1\right\}.$$

As (24) asserts that $(\mathscr{R}_\delta h)^{\frac{1}{\delta}}\in A_1$, we find that $T: L^{p_0}(\mathscr{R}_\delta h)\to L^{p_0}(\mathscr{R}_\delta h)$ is bounded. Therefore,

$$\|Tf\|^{p_0}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \leqslant C\sup\left\{\int_{\mathbb{R}^n}|f(x)|^{p_0}\mathscr{R}_\delta h(x)dx : \|h\|_{b\dot{K}^{-p_0\alpha(\cdot)/\delta,p_0\lambda/\delta}_{\delta(p(\cdot)/p_0)',\delta(q/p_0)'}}\leqslant 1\right\}.$$

Lemma 3, (23) and (27) give

$$\begin{aligned}\|Tf\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} &\leqslant C\left(\int_{\mathbb{R}^n}|f(x)|^{p_0}\mathscr{R}_\delta h(x)dx\right)^{1/p_0}\\ &\leqslant C\||f|^{p_0}\|^{1/p_0}_{M\dot{K}^{p_0\alpha(\cdot),\lambda}_{p(\cdot)/p_0,q/p_0}}\|\mathscr{R}_\delta h\|^{1/p_0}_{B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}}\\ &\leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}\|h\|^{1/p_0}_{B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}}\leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}.\end{aligned}$$

We have the last inequality because for any $h\in b\dot{K}^{-p_0\alpha(\cdot)/\delta,p_0\lambda/\delta}_{\delta(p(\cdot)/p_0)',\delta(q/p_0)'}$, $\|h\|_{B\dot{K}^{-p_0\alpha(\cdot),p_0\lambda}_{(p(\cdot)/p_0)',(q/p_0)'}}$ $\leqslant 1$. Thus, we establish the boundedness of T on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. $\square$

The above theorem does not require that T is a linear operator or sublinear operator. The reader is referred to [17, 18, 19, 20, 21, 22] for the studies of nonlinear operators on Herz spaces with variable exponents, weighted Morrey spaces with variable exponents, Morrey-Banach spaces and grand Morrey spaces.

In the next section, we apply our main result to the spherical maximal functions, the nonlinear commutators of Rochberg and Weiss and the geometric maximal operators. The spherical maximal function is a sublinear operator while the nonlinear commutator of Rochberg and Weiss and the geometric maximal operator are nonlinear operators.

4.1. Spherical maximal functions

Let $n\geqslant 3$. The spherical maximal function is defined as

$$\mathrm{S}f(x)=\sup_{r>0}\left|\int_{S^{n-1}}f(x-ry)d\sigma(y)\right|$$

where S^{n-1} is the unit sphere on $\mathbb{R}^n$ and $d\sigma$ is the normalized surface measure on S^{n-1}, see [43]. The boundedness of the spherical maximal function provide applications on the studies of partial differential equations, see [28]. The reader is referred to

[11] for the boundedness of the spherical maximal function on Lebesgue spaces with variable exponents. Furthermore, we have the boundedness of the spherical maximal function on Morrey spaces with variable exponents and Herz spaces with variable exponents, see [21, Corollary 4.3] and [18], respectively.

The weighted norm inequalities for the spherical maximal functions are established in [4, Theorem 3.1].

THEOREM 6. *Let $n \geqslant 3$ and $\frac{n}{n-1} < p < \infty$. For any $\omega \in A_1$, there is a constant $C > 0$ such that*

$$\|\mathrm{S} f\|_{L^p(\omega^{\frac{n-2}{n-1}})} \leqslant C\|f\|_{L^p(\omega^{\frac{n-2}{n-1}})} \quad \forall f \in L^p(\omega^{\frac{n-2}{n-1}}). \tag{28}$$

The reader is referred to [18] for the boundedness of the spherical maximal functions on the Herz spaces with variable exponents.

By applying Theorem 5 with $\delta = \frac{n-2}{n-1}$ on Theorem 28, we obtain the boundedness of the spherical maximal functions on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.

THEOREM 7. *Let $q \in (\frac{n}{n-1}, \infty)$ and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $\frac{n}{n-1} < p_- \leqslant p_+ < \infty$. Let $\alpha \in C_0^{\log}(\mathbb{R}^n)$ satisfy (3) and*

$$n\left(\frac{1}{n} - \frac{1}{p_+}\right) < \alpha_- \leqslant \alpha_+ < n\left(\frac{n-1}{n} - \frac{1}{p_-}\right). \tag{29}$$

If $\min((\frac{p_+(n-1)}{n})', (\frac{q(n-1)}{n})') > \frac{n-1}{n-2}$ and

$$0 \leqslant \frac{\lambda}{n} < \frac{n-2}{(n-1)(\kappa_{(n-2)((n-1)p(\cdot)/n)'/(n-1), -(n-1)\alpha(\cdot)/(n-2)})'}, \tag{30}$$

then S can be extended to be a bounded operator on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.

Proof. Since the function $p' = f(p) = \frac{p}{p-1}$ is continuous on $(1, \infty)$ and for any fixed $p(\cdot), \alpha(\cdot)$ and δ, the function $q(x) = \frac{\delta}{x(\kappa_{\delta(p(\cdot)/x)', -x\alpha(\cdot)/\delta})'}$ is continuous on $[1, \infty)$, we can select a $p_0 \in (\frac{n}{n-1}, \min(p_-, q))$ such that $\min((p_+/p_0)', (q/p_0)') > \frac{n-1}{n-2}$,

$$n\left(\frac{1}{(n-1)p_0} - \frac{1}{p_+}\right) < \alpha_- \leqslant \alpha_+ < n\left(\frac{1}{p_0} - \frac{1}{p_-}\right)$$

and

$$0 \leqslant \frac{\lambda}{n} < \frac{n-2}{p_0(n-1)(\kappa_{(n-2)(p(\cdot)/p_0)'/(n-1), -p_0(n-1)\alpha(\cdot)/(n-2)})'}.$$

With this selected p_0, we can apply Theorem 5 and (28) to obtain the boundedness of S on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. $\square$

We give an example for the above theorem. Let $\alpha(\cdot) \in C_0^{\log}(\mathbb{R}^n)$ satisfy (3) and (29) and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. If $\alpha_- \geqslant 0$, $\frac{n}{n-1} < p_- \leqslant p_+ < n$, $1 < q < n$ and

$$0 \leqslant \lambda < \frac{n(n-p_+)}{(n-1)p_+}, \tag{31}$$

then S is bounded on $M\dot{K}_{p(\cdot),q}^{\alpha(\cdot),\lambda}$.

Since $(\frac{n-1}{n-2})' = n-1$, $p_+, q \in (1,n)$ gives $\min((\frac{p_+ n}{n-1})', (\frac{qn}{n-1})') > \frac{n-1}{n-2}$. As $\alpha_- > 0$, we find that

$$\frac{1}{(\kappa_{(n-2)(p(\cdot)(n-1)/n)'/(n-1),-(n-1)\alpha(\cdot)/(n-2)})'} = \frac{1}{(((n-2)(p(\cdot)(n-1)/n)'/(n-1))_-)'}.$$

Since

$$\begin{aligned} \frac{n-2}{n-1}\left(\frac{p(\cdot)(n-1)}{n}\right)' &= \frac{n-2}{n-1}\left(\frac{p_+(n-1)}{n}\right)' \\ &= \frac{n-2}{n-1}\frac{(n-1)p_+}{p_+n-p_+-n} \\ &= \frac{(n-2)p_+}{p_+n-p_+-n}, \end{aligned}$$

we have

$$\left(\frac{n-2}{n-1}\left(\frac{p(\cdot)(n-1)}{n}\right)'\right)' = \frac{(n-2)p_+}{n-p_+}.$$

Consequently,

$$\frac{n-2}{(n-1)(\kappa_{(n-2)((n-1)p(\cdot)/n)'/(n-1),-(n-1)\alpha(\cdot)/(n-2)})'} = \frac{n-p_+}{(n-1)p_+}.$$

Thus, (30) is equivalent to (31). Notice that the condition $p_+ < n$ guarantees that (31) is well defined.

In particular, if $\alpha \geqslant 0$, $1 < p, q < n$ and $0 \leqslant \lambda < \frac{n(n-p)}{(n-1)p}$, then S is bounded on the Herz-Morrey space $M_{p,q}^{\alpha,\lambda}$.

For the boundedness of the spherical maximal functions on Herz spaces with variable exponents and grand Morrey spaces, see [18, 23].

4.2. Nonlinear commutators of Rochberg and Weiss

Let C_0^∞ and $\mathscr{D}'$ be the space of smooth functions with compact supports and the class of distributions in $\mathbb{R}^n$, respectively. A linear operator $T : C_0^\infty \to \mathscr{D}'$ is a Calderón-Zygmund operator, if T is bounded on L^2 and there exists a kernel $C, \delta > 0$ and

$K(x,y):\mathbb{R}^{2n}\backslash\{(x,x):x\in\mathbb{R}^n\}\to\mathbb{R}$ such that for any $f\in C_0^\infty$ and $x\notin \mathrm{supp} f$,

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy,$$

where K satisfies

$$\begin{aligned}|K(x,y)|&\leqslant C|x-y|^{-n}, \quad x\neq y,\\ |K(x,y)-K(z,y)|&\leqslant C|x-z|^{\delta}|x-y|^{-n-\delta}, \quad |x-z|\leqslant |x-y|/2,\\ |K(x,y)-K(x,z)|&\leqslant C|y-z|^{\delta}|x-y|^{-n-\delta}, \quad |y-z|\leqslant |x-y|/2.\end{aligned}$$

For any Calderón-Zygmund operator T, we define

$$Nf=T(f\log|f|)-Tf\log|Tf|. \tag{32}$$

This nonlinear commutator was introduced by Rochberg and Weiss in [36]. This nonlinear operator yields applications on the estimates of the Jacobian and the weak minima of variational integrals, see [12] and [24], respectively.

The weighted norm inequalities for nonlinear commutators of Rochberg and Weiss are established in [35, Theorem 1.3].

THEOREM 8. *Let $p\in(1,\infty)$ and $\omega\in A_1$. There is a constant $C>0$ such that*

$$\int_{\mathbb{R}^n}|Nf(x)|^p\omega(x)dx\leqslant C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx.$$

For the proof of the above theorem, see [35, Theorem 1.3]. For the boundedness of N on Morrey-Banach spaces which includes the classical Morrey space, the Orlicz-Morrey spaces and the Morrey spaces with variable exponent, see [20].

Notice that the above theorem is a special case of the results obtained in [35, Theorem 1.3]. The results given in [35, Theorem 1.3] are valid for $\omega\in A_\infty$ and $p\in(0,\infty)$ where the right hand side is replaced by $\int_{\mathbb{R}^n}(\mathrm{M}^2f(x))^p\omega(x)dx$. As M is bounded on $L^p(\omega)$ provided that $\omega\in A_p$ and $A_1\subset A_p$, Theorem 8 follow from [35, Theorem 1.3].

By applying Theorem 5 to the nonlinear commutator N with $\delta=1$, Theorem 8 yields the boundedness of N on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.

THEOREM 9. *Let $q\in(1,\infty)$ and $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<p_-\leqslant p_+<\infty$. Let $\alpha\in C_0^{\log}(\mathbb{R}^n)$ satisfy (3) and (7).*

1. *If $0\leqslant\alpha_-$ and $0\leqslant\lambda<\frac{n}{p_+}$, then N is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.*

2. *If $\alpha_-<0$ and $0\leqslant\lambda<\alpha_-+\frac{n}{p_+}$, then N is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.*

Proof. When $\alpha_-\geqslant 0$, $\kappa_{p'(\cdot),-\alpha(\cdot)}=p'_-$. As $\frac{1}{p'_-}+\frac{1}{p_+}=1$, $0\leqslant\lambda<\frac{n}{p_+}$ is equivalent to

$$0\leqslant\lambda<\frac{n}{(\kappa_{p'(\cdot),-\alpha(\cdot)})'}. \tag{33}$$

When $\alpha_- < 0$, then $\kappa_{p'(\cdot),-\alpha(\cdot)} = \frac{np'_-}{n-\alpha_- p'_-}$ and hence,

$$\frac{1}{(\kappa_{p'(\cdot),-\alpha(\cdot)})'} = 1 - \frac{1}{p'_-} + \frac{\alpha_-}{n} = \frac{1}{p_+} + \frac{\alpha_-}{n}.$$

Thus, $0 \leqslant \lambda < \alpha_- + \frac{n}{p_+}$ is also equivalent to (33).

As the functions $p' = f(p) = \frac{p}{p-1}$ and $h(x) = \frac{1}{x(\kappa_{(p(\cdot)/x)',-x\alpha(\cdot)})'}$ are continuous on $[1,\infty)$, we can select a $p_0 \in (1, \min(p_-, q))$ such that $\min((p_+/p_0)', (q/p_0)') > 1$ and

$$0 \leqslant \frac{\lambda}{n} < \frac{1}{p_0(\kappa_{(p(\cdot)/p_0)',-p_0\alpha(\cdot)})'}.$$

Notice that when $\delta = 1$, (20) becomes (7). Therefore, we can apply Theorem 5 with $\delta = 1$ to obtain the boundedness of N on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. $\square$

In particular, we have the boundedness of the nonlinear commutator of Rochberg and Weiss on Herz spaces with variable exponent and Herz-Morrey spaces.

COROLLARY 3. *Let $q \in (1,\infty)$ and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Let $\alpha \in C_0^{\log}(\mathbb{R}^n)$ satisfy (3) and (7). If $1 < p_- \leqslant p_+ < \infty$, then N is bounded on $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$.*

COROLLARY 4. *Let $\alpha \in \mathbb{R}$ and $p, q \in (1,\infty)$.*

1. *If $0 \leqslant \alpha < n\left(1 - \frac{1}{p}\right)$ and $0 \leqslant \lambda < \frac{n}{p}$, then N is bounded on $MK^{\alpha,\lambda}_{p,q}$.*

2. *If $-\frac{n}{p} < \alpha < 0$ and $0 \leqslant \lambda < \alpha + \frac{n}{p}$, then N is bounded on $MK^{\alpha,\lambda}_{p,q}$.*

Proof. When $\alpha \geqslant 0$, $\alpha < n\left(1 - \frac{1}{p}\right)$ shows that (7) is satisfied. In addition, we have $\kappa_{p',-\alpha} = p'$ and (33) is fulfilled. Thus, N is bounded on $MK^{\alpha,\lambda}_{p,q}$. When $\alpha < 0$, $-\frac{n}{p} < \alpha < 0$ and $0 \leqslant \lambda < \alpha + \frac{n}{p}$ guarantee that (7) and (33) are satisfied. Therefore, Theorem 9 yields the boundedness of N on $MK^{\alpha,\lambda}_{p,q}$. $\square$

4.3. Geometric maximal functions

For any Lebesgue measurable function f on $\mathbb{R}^n$, the geometric maximal operator $M_0 f$ is defined as

$$M_0 f(x) = \sup_I \exp\left(\frac{1}{|I|}\int_I \log|f(y)|dy\right)$$

where the supremum is taken over all cubes I containing x with their sides are parallel to the coordinates axes.

Moreover, for any locally integrable function f, M_0^* is defined by

$$M_0^* f(x) = \lim_{r\to 0}(M(|f|^r))^{1/r}(x).$$

The minimal operator is given by

$$\mathfrak{m}f(x)=\inf_I \frac{1}{|I|}\int_I |f(y)|dy$$

where the infimum is taken over all cubes containing x. The minimal operator is related with the reverse Hölder class, see [5].

The reader is referred to [17] for the boundedness of the geometrical maximal operators on the Herz spaces with variable exponents.

We have the following weighted norm inequalities for M_0, M_0^* and $\mathfrak{m}$ from [42], [6, Theorems 1.7 and 3.1], respectively.

THEOREM 10. *Let $\omega:\mathbb{R}^n\to(0,\infty)$ be a Lebesgue measurable function. We have $\omega\in A_\infty$ if and only if for any $0<p<\infty$,*

$$\int_{\mathbb{R}^n}(M_0f(x))^p\omega(x)dx\leqslant C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx,\quad \forall f\in L^p(\omega).$$

THEOREM 11. *Let $\omega:\mathbb{R}^n\to(0,\infty)$ be a Lebesgue measurable function. We have $\omega\in A_\infty$ if and only if for any $0<p<\infty$,*

$$\int_{\mathbb{R}^n}(M_0^*f(x))^p\omega(x)dx\leqslant C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx,\quad \forall f\in L^p(\omega).$$

THEOREM 12. *Let $p>0$ and $\omega\in A_\infty$. There is a constant $C>0$ such that for any $f\in\mathscr{M}$ with $\frac{1}{f}\in L^p(\omega)$, we have*

$$\int_{\mathbb{R}^n}\left(\frac{1}{\mathfrak{m}f(x)}\right)^p\omega(x)dx\leqslant C\int_{\mathbb{R}^n}\frac{1}{|f(x)|^p}\omega(x)dx$$

Theorems 5, 10, 11 and 12 give the following results on the boundedness of the geometrical maximal operators and the minimal operator on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.

THEOREM 13. *Let $q\in(1,\infty)$ and $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $0<p_-\leqslant p_+<\infty$. Let $\alpha\in C_0^{\log}(\mathbb{R}^n)$ satisfy (3) and (7).*

1. *If $0\leqslant\alpha_-$ and $0\leqslant\lambda<\frac{n}{p_+}$, then M_0 is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.*
2. *If $\alpha_-<0$ and $0\leqslant\lambda<\alpha_-+\frac{n}{p_+}$, then M_0 is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.*

Proof. We select a $p_0\in(0,\min(p_-,q))$ such that $\min((p_+/p_0)',(q/p_0)')>1$. When $\alpha_-\geqslant 0$, for any $p_0\in(0,p_-)$, $\kappa_{(p(\cdot)/p_0)',-p_0\alpha(\cdot)}=(p(\cdot)/p_0)'_-=(p_+/p_0)'$,

$$\frac{1}{p_0(\kappa_{(p(\cdot)/p_0)',-p_0\alpha(\cdot)})'}=\frac{1}{p_+}.$$

Similarly, when $\alpha_- < 0$, we have

$$\frac{1}{p_0\big(\kappa_{(p(\cdot)/p_0)',-p_0\alpha(\cdot)}\big)'} = \frac{1}{p_+} + \frac{\alpha_-}{n}.$$

The above equalities guarantee that

$$0 \leqslant \frac{\lambda}{n} < \frac{1}{p_0\big(\kappa_{(p(\cdot)/p_0)',-p_0\alpha(\cdot)}\big)'}.$$

Since (20) becomes (7) when $\delta = 1$. Theorems 5 and 10 yield the boundedness of M_0 on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$. $\square$

THEOREM 14. *Let $q \in (1,\infty)$ and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \leqslant p_+ < \infty$. Let $\alpha \in C_0^{\log}(\mathbb{R}^n)$ satisfy (3) and (7).*

1. *If $0 \leqslant \alpha_-$ and $0 \leqslant \lambda < \frac{n}{p_+}$, then M_0^* is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.*

2. *If $\alpha_- < 0$ and $0 \leqslant \lambda < \alpha_- + \frac{n}{p_+}$, then M_0^* is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$.*

THEOREM 15. *Let $q \in (1,\infty)$ and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \leqslant p_+ < \infty$. Let $\alpha \in C_0^{\log}(\mathbb{R}^n)$ satisfy (3) and (7).*

1. *If $0 \leqslant \alpha_-$ and $0 \leqslant \lambda < \frac{n}{p_+}$, then there is a constant $C > 0$ such that for any $\frac{1}{f} \in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$,*

$$\left\|\frac{1}{\mathfrak{m}f}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \leqslant C\left\|\frac{1}{f}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}.$$

2. *If $\alpha_- < 0$ and $0 \leqslant \lambda < \alpha_- + \frac{n}{p_+}$, then there is a constant $C > 0$ such that for any $\frac{1}{f} \in M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}$,*

$$\left\|\frac{1}{\mathfrak{m}f}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}} \leqslant C\left\|\frac{1}{f}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p(\cdot),q}}.$$

As the proofs of Theorems 14 and 15 are similar to the proof of Theorem 13, we omit them and leave them to the reader.

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