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NORM INEQUALITIES FOR PARALLEL SUMS OF OPERATORS

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Abstract. It is shown that if $A:B$ is the parallel sum of the positive definite operators A and B, then

$$\|A:B\| \leqslant \left\| \left(\frac{\|A\|:\|B\|}{\|A\|}\right)^2 A + \left(\frac{\|A\|:\|B\|}{\|B\|}\right)^2 B \right\| \leqslant \frac{1}{2}\left(\|A\|:\|B\| + \frac{\|A\|:\|B\|}{\|A\|+\|B\|}\sqrt{\left(\|A\|-\|B\|\right)^2 + 4\left\|\sqrt{A}\sqrt{B}\right\|^2}\right).$$

These inequalities lead to a considerable improvement of the well-known inequality $\|A:B\| \leqslant \|A\|:\|B\|$ due to Anderson and Duffin (J. Math. Anal. Appl. **26** (1969), 576–594). A lower bound for the norm of $A:B$ is also provided.

1. Introduction

The parallel sum of the two positive definite operators A and B was introduced by Anderson and Duffin [2] as

$$A:B = \left(A^{-1}+B^{-1}\right)^{-1},$$

which is so named because of its origin in and application to the electrical network theory that $\left(r^{-1}+s^{-1}\right)^{-1}$, in short $r:s$, is the resistance arising from resistors r and s in parallel. A norm upper bound for $A:B$ is given in [2, Theorem 25] as

$$\|A:B\| \leqslant \|A\|:\|B\|. \tag{1}$$

Ever since then, the parallel sum and shorted operators as its generalization have been studied for bounded linear operators [1, 3, 6, 7, 15, 16].

In the last few years, the concept of the parallel sum has been extended to several different contexts, including nonnegative forms [10], unbounded operators [13], states of a C^*-algebra [17], linear relations [4] and adjointable operators on Hilbert C^*-modules [8]. The parallel sum and its generalizations have proved to be useful operations in a wide variety of fields, such as electrical networks, statistics, control

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theory, geodetic adjustments, image denoising problems, signal recovery, numerical calculations and so on; see [5, 14] and the references therein.

Although much progress has been made in the study of the parallel sum and its various generalizations, nothing has been done on the improvement of the norm upper bound (1), which is the concern of this paper. In the next section, we establish some inequalities that refine the inequality (1). We also provide a lower bound for the norm of the parallel sum of positive definite operators.

Some other related inequalities for the parallel sum of operators can be found in [9, 12].

2. Results

In order to prove our desired norm inequalities for the parallel sum of positive definite operators, we need the following lemmas. The first lemma contains some of the main properties of the parallel sum of positive definite operators and its proof can be found in [2, 7].

LEMMA 1. *Let A,B,C and D be positive definite operators and $r,s\in\mathbb{R}^+$. Then the following statements hold.*

(i) $0\leqslant A:B=B:A$.

(ii) $A:B\leqslant A$ *and* $A:B\leqslant \frac{A+B}{4}$.

(iii) $(rA):(rB)=r(A:B)$.

(iv) $(rI):(sI)=(r:s)I$.

(v) $A\leqslant C$ *and* $B\leqslant D$ *imply* $A:B\leqslant C:D$.

The second lemma is a variational property for the parallel sum of positive definite operators, which can be found in [2, Lemma 18].

LEMMA 2. *If A and B are positive definite operators on a Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$, then for any $x,y,z\in\mathscr{H}$ such that $x+y=z$*

$$\big\langle (A:B)z,z\big\rangle\leqslant\langle Ax,x\rangle+\langle By,y\rangle.$$

For generalizations of the above lemmas, we refer the reader to [4, 8, 10].

The third lemma contains a norm inequality for sums of positive operators, see [11].

LEMMA 3. *If A and B are positive operators, then*

$$\|A+B\|\leqslant\frac{1}{2}\Big(\|A\|+\|B\|+\sqrt{\big(\|A\|-\|B\|\big)^2+4\big\|\sqrt{A}\sqrt{B}\big\|^2}\Big).$$

Our desired improvements of the inequality (1) can be stated as follow. Note that our first norm inequality can be derived from the operator inequality given in [9, Theorem 2.1], but for the sake of completeness we include a direct proof.

THEOREM 1. *Let A and B be positive definite operators on $\mathscr{H}$. Then*

$$\begin{aligned}\|A:B\| &\leqslant \|\alpha^2 A+\beta^2 B\| \\ &\leqslant \frac{1}{2}\Big(\|A\|:\|B\|+\alpha\beta\sqrt{(\|A\|-\|B\|)^2+4\|\sqrt{A}\sqrt{B}\|^2}\Big) \\ &\leqslant \|A\|:\|B\|, \end{aligned} \tag{2}$$

where $\alpha=\frac{\|A\|:\|B\|}{\|A\|}$ *and* $\beta=\frac{\|A\|:\|B\|}{\|B\|}$.

Proof. Let z be any unit vector in $\mathscr{H}$. Put $x=\alpha z$ and $y=\beta z$. Then

$$x+y=(\alpha+\beta)z=\left(\frac{\|B\|}{\|A\|+\|B\|}+\frac{\|A\|}{\|A\|+\|B\|}\right)z=z,$$

and hence $x+y=z$. Therefore, by Lemma 2 and the Cauchy–Schwarz inequality we have

$$\begin{aligned}\big\langle (A:B)z,z\big\rangle &\leqslant \langle Ax,x\rangle+\langle By,y\rangle \\ &=\big\langle \alpha^2 Az,z\big\rangle+\big\langle \beta^2 Bz,z\big\rangle \\ &=\big\langle (\alpha^2 A+\beta^2 B)z,z\big\rangle \leqslant \|\alpha^2 A+\beta^2 B\|,\end{aligned}$$

and hence

$$\big\langle (A:B)z,z\big\rangle \leqslant \|\alpha^2 A+\beta^2 B\|. \tag{3}$$

Now, by taking the supremum over unit vectors $z\in\mathscr{H}$ in (3) and Lemma 1(i), we get

$$\|A:B\| \leqslant \|\alpha^2 A+\beta^2 B\|,$$

which proves the first inequality in (2).

To prove the second inequality in (2), note that from the definition of the parallel sum, we have

$$\alpha^2\|A\|+\beta^2\|B\|=(\|A\|:\|B\|)^2\left(\frac{1}{\|A\|}+\frac{1}{\|B\|}\right)=\frac{(\|A\|:\|B\|)^2}{\|A\|:\|B\|},$$

and so

$$\alpha^2\|A\|+\beta^2\|B\|=\|A\|:\|B\|. \tag{4}$$

Also, we have

$$\alpha^2\|A\|-\beta^2\|B\|=\left(\frac{\|A\|:\|B\|}{\|\sqrt{A}\|\|\sqrt{B}\|}\right)^2\|B\|-\left(\frac{\|A\|:\|B\|}{\|\sqrt{A}\|\|\sqrt{B}\|}\right)^2\|A\|,$$

and hence

$$\alpha^2\|A\| - \beta^2\|B\| = \alpha\beta\big(\|B\| - \|A\|\big). \tag{5}$$

Now, by Lemma 3, (4) and (5), we have

$$\begin{aligned}
\big\|\alpha^2 A + \beta^2 B\big\| &\leqslant \frac{1}{2}\Big(\alpha^2\|A\| + \beta^2\|B\| + \sqrt{\big(\alpha^2\|A\| - \beta^2\|B\|\big)^2 + 4\alpha^2\beta^2\big\|\sqrt{A}\sqrt{B}\big\|^2}\Big)\\
&= \frac{1}{2}\Big(\|A\|:\|B\| + \sqrt{\alpha^2\beta^2\big(\|B\| - \|A\|\big)^2 + 4\alpha^2\beta^2\big\|\sqrt{A}\sqrt{B}\big\|^2}\Big)\\
&= \frac{1}{2}\Big(\|A\|:\|B\| + \alpha\beta\sqrt{\big(\|A\| - \|B\|\big)^2 + 4\big\|\sqrt{A}\sqrt{B}\big\|^2}\Big).
\end{aligned}$$

To prove the third inequality in (2), we have

$$\begin{aligned}
&\frac{1}{2}\Big(\|A\|:\|B\| + \alpha\beta\sqrt{\big(\|A\| - \|B\|\big)^2 + 4\big\|\sqrt{A}\sqrt{B}\big\|^2}\Big)\\
&\qquad \leqslant \frac{1}{2}\Big(\|A\|:\|B\| + \alpha\beta\sqrt{\big(\|A\| - \|B\|\big)^2 + 4\|A\|\|B\|}\Big)\\
&\qquad = \frac{1}{2}\Big(\|A\|:\|B\| + \alpha\beta\sqrt{\big(\|A\| + \|B\|\big)^2}\Big)\\
&\qquad = \frac{1}{2}\Big(\|A\|:\|B\| + \frac{\big(\|A\|:\|B\|\big)^2}{\|A\|\|B\|}\big(\|A\| + \|B\|\big)\Big)\\
&\qquad = \frac{1}{2}\Big(\|A\|:\|B\| + \frac{\big(\|A\|:\|B\|\big)^2}{\|A\|:\|B\|}\Big) = \|A\|:\|B\|.
\end{aligned}$$

Therefore the proof of the theorem is completed. □

REMARK 1. The following example shows that the inequalities in Theorem 1 are nontrivial improvements. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Then simple computations show that $\|A\|:\|B\| = 1.2$, $\alpha = 0.6$, $\beta = 0.4$, $A:B = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{bmatrix}$, $\sqrt{A}\sqrt{B} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2 \end{bmatrix}$ and $\alpha^2 A + \beta^2 B = \begin{bmatrix} 0.84 & 0 \\ 0 & 1.04 \end{bmatrix}$. Therefore,

$$\begin{aligned}
\big\|A:B\big\| = 1 &< \big\|\alpha^2 A + \beta^2 B\big\| = 1.04\\
&< \frac{1}{2}\Big(\|A\|:\|B\| + \alpha\beta\sqrt{\big(\|A\| - \|B\|\big)^2 + 4\big\|\sqrt{A}\sqrt{B}\big\|^2}\Big) \cong 1.0947\\
&< \|A\|:\|B\| = 1.2.
\end{aligned}$$

In the following theorem, we employ Lemma 2 to provide a lower bound for the norm of the parallel sum of positive definite operators.

THEOREM 2. *Let A and B be positive definite operators. Then*

$$\sqrt{\frac{\left\|A^{-1}\right\|^{-2}:\left\|B^{-1}\right\|^{-2}}{2}} \leqslant \|A:B\|. \tag{6}$$

Proof. First note that since $\left\|C^{-1}\right\|^{-2} I \leqslant C^2$ for any positive definite operator C, by Lemma 1 (iv)–(v), we get

$$\left(\left\|A^{-1}\right\|^{-2}:\left\|B^{-1}\right\|^{-2}\right) I \leqslant A^2 : B^2. \tag{7}$$

Now, let z be any unit vector in $\mathscr{H}$. Put $x = A^{-1}(A:B)z$ and $y = B^{-1}(A:B)z$. We have

$$x + y = A^{-1}(A:B)z + B^{-1}(A:B)z = \left(A^{-1}+B^{-1}\right)(A:B)z = z.$$

Thus, $x+y=z$. So by (7), Lemma 2, Lemma 1(i) and the Cauchy–Schwarz inequality we have,

$$\begin{aligned}
\left\|A^{-1}\right\|^{-2}:\left\|B^{-1}\right\|^{-2} &= \left\langle \left(\left\|A^{-1}\right\|^{-2}:\left\|B^{-1}\right\|^{-2}\right) z, z\right\rangle \\
&\leqslant \left\langle \left(A^2:B^2\right) z, z\right\rangle \\
&\leqslant \langle A^2 x, x\rangle + \langle B^2 y, y\rangle \\
&= \left\langle A^2 A^{-1}(A:B)z, A^{-1}(A:B)z\right\rangle + \left\langle B^2 B^{-1}(A:B)z, B^{-1}(A:B)z\right\rangle \\
&= \left\langle A^{-1} A(A:B)z, (A:B)z\right\rangle + \left\langle B^{-1} B(A:B)z, (A:B)z\right\rangle \\
&= \left\langle (A:B)z, (A:B)z\right\rangle + \left\langle (A:B)z, (A:B)z\right\rangle \leqslant 2\|A:B\|^2,
\end{aligned}$$

and hence

$$\left\|A^{-1}\right\|^{-2}:\left\|B^{-1}\right\|^{-2} \leqslant 2\|A:B\|^2. \tag{8}$$

The desired inequality (6) now follows from (8). □

REMARK 2. The inequality in Theorem 2 is sharp. Indeed, for $A=B=I$ we have $\sqrt{\frac{\left\|A^{-1}\right\|^{-2}:\left\|B^{-1}\right\|^{-2}}{2}} = \|A:B\| = \frac{1}{2}$.

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