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INEQUALITY OF HARDY–TYPE FOR n–CONVEX FUNCTION VIA INTERPOLATION POLYNOMIAL AND GREEN FUNCTIONS

DORA POKAZ

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Abstract. We obtain new results on the Hardy-type inequality in the general context, in terms of measure spaces with positive σ-finite measures. The connection is made between the difference operator derived from the Hardy-type inequality on the one hand and the expression containing the interpolating polynomial of Abel-Gontscharoff and the four Green functions on the other hand. We discuss the n-convexity of the function and consider the result depending on the parity of the indexes n and m. Further results are obtained by using the Hölder inequality for conjugate exponents p and q. Finally, we derive upper bounds for the remainder, obtained from the main result, using the Čebyšev functional. The Ostrowski-type bound for the generalized Hardy inequality is also given.

1. Introduction

Green functions, named after the famous British mathematician and physicist George Green are very interesting from different aspects. These functions allow us to solve various types of differential equations, including an ordinary differential equation with initial conditions and more difficult ones such as an inhomogeneous partial differential equation with boundary conditions. Green functions are used for solving wide variety of problems, specifically in quantum field theory, aerodynamics, aeroacoustics, electrodynamics, seismology and statistical field theory.

Here we deal with specific Green functions, based on the article [9]. Throughout the paper, with $\tilde{G}_\gamma$, $\gamma = 1,2,3,4$, we denote the following Green functions defined on $[\alpha,\beta]\times[\alpha,\beta]$ with

$$\tilde{G}_1(t,s) = \begin{cases} \alpha - s, & \alpha \leqslant s \leqslant t; \\ \alpha - t, & t \leqslant s \leqslant \beta. \end{cases} \tag{1}$$

$$\tilde{G}_2(t,s) = \begin{cases} t - \beta, & \alpha \leqslant s \leqslant t; \\ s - \beta, & t \leqslant s \leqslant \beta. \end{cases} \tag{2}$$

$$\tilde{G}_3(t,s) = \begin{cases} t - \alpha, & \alpha \leqslant s \leqslant t; \\ s - \alpha, & t \leqslant s \leqslant \beta. \end{cases} \tag{3}$$

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$$\tilde{G}_4(t,s)=\begin{cases}\beta-s, & \alpha\leqslant s\leqslant t;\\ \beta-t, & t\leqslant s\leqslant\beta.\end{cases} \tag{4}$$

Note that all these functions are continuous and convex with respect to both variables. We proceed with useful result from [15].

LEMMA 1. *For $\phi\in C^2([\alpha,\beta])$, the following identities hold*

$$\phi(t)=\phi(\alpha)+(t-\alpha)\phi'(\beta)+\int_\alpha^\beta \tilde{G}_1(t,s)\phi''(s)ds, \tag{5}$$

$$\phi(t)=\phi(\beta)+(t-\beta)\phi'(\alpha)+\int_\alpha^\beta \tilde{G}_2(t,s)\phi''(s)ds, \tag{6}$$

$$\phi(t)=\phi(\beta)+(t-\alpha)\phi'(\alpha)-(\beta-\alpha)\phi'(\beta)+\int_\alpha^\beta \tilde{G}_3(t,s)\phi''(s)ds, \tag{7}$$

$$\phi(t)=\phi(\alpha)-(\beta-t)\phi'(\beta)+(\beta-\alpha)\phi'(\alpha)+\int_\alpha^\beta \tilde{G}_4(t,s)\phi''(s)ds, \tag{8}$$

where the functions $\tilde{G}_\gamma$, $\gamma=1,\dots,4$, are defined by (1)–(4).

The aim of this article is to give a result related to the general Hardy-type inequality. The classical Hardy inequality from [6] is

$$\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx\leqslant\left(\frac{p}{p-1}\right)^p\int_0^\infty f^p(x)\,dx,\quad p>1, \tag{9}$$

where f is non-negative function such that $f\in L^p(\mathbb{R}_+)$ and $\mathbb{R}_+=(0,\infty)$. The constant $\left(\frac{p}{p-1}\right)^p$ is sharp. Inequality (9) was generalized in many ways, see [7], [11] and [12].

Here we refer to settings and generalization from [7]. We begin by defining the settings that we continue to work with. Let $(\Sigma_1,\Omega_1,\mu_1)$ and $(\Sigma_2,\Omega_2,\mu_2)$ be measure spaces with positive σ-finite measures and A_k be the integral operator defined by

$$A_kf(x):=\frac{1}{K(x)}\int_{\Omega_2}k(x,t)f(t)d\mu_2(t), \tag{10}$$

where $k:\Omega_1\times\Omega_2\to\mathbb{R}$ is a measurable and non-negative kernel, $f:\Omega_2\to\mathbb{R}$ is a measurable function and

$$0<K(x):=\int_{\Omega_2}k(x,t)d\mu_2(t),\quad x\in\Omega_1. \tag{11}$$

Throughout the article we mark an open interval in $\mathbb{R}$ with I. The following result was given in [7] and also [8].

THEOREM 1. *Let u be a weight function, $k(x,y)\geqslant 0$. Assume that $\frac{k(x,y)}{K(x)}u(x)$ is locally integrable on Ω_1 for each fixed $y\in\Omega_2$. Define v by*

$$v(y):=\int\limits_{\Omega_1}\frac{k(x,y)}{K(x)}u(x)d\mu_1(x)<\infty. \tag{12}$$

If Φ is a convex function on the interval $I\subseteq\mathbb{R}$, then the inequality

$$\int\limits_{\Omega_1}\Phi(A_kf(x))u(x)d\mu_1(x)\leqslant\int\limits_{\Omega_2}\Phi(f(y))v(y)d\mu_2(y) \tag{13}$$

holds for all measurable functions $f:\Omega_2\to\mathbb{R}$, such that $Imf\subseteq I$, where A_k is defined by (10)–(11).

Now, we start with the generalized Hardy-type inequality (13). In the settings where A_k is as in (10), a weight function u with v given by (12) and for $\gamma\in\{1,2,3,4\}$, we consider $\tilde{G}_\gamma$ to be as in (1)–(4). In addition, for $\phi\in C^2([\alpha,\beta])$, identities (5)–(8) and some simple calculations yield the following statements from [9]

$$\begin{aligned}&\int\limits_{\Omega_2}\phi(f(y))v(y)d\mu_2(y)-\int\limits_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x)\\ &=\int_\alpha^\beta\left[\int\limits_{\Omega_2}\tilde{G}_\gamma(f(y),s)v(y)d\mu_2(y)-\int\limits_{\Omega_1}\tilde{G}_\gamma(A_kf(x),s)u(x)d\mu_1(x)\right]\phi''(s)ds.\end{aligned} \tag{14}$$

The techniques that we use in the paper are based on classical real analysis and the application of the Abel-Gontscharoff interpolation. The Abel-Gontscharoff interpolation problem in the real case was introduced in 1935 by Whittaker [14] and subsequently by Gontscharoff [5] and Davis [4]. The next theorem presents the Abel-Gontscharoff interpolating polynomial for two points with integral remainder (see [1]).

THEOREM 2. *Let $n,m\in\mathbb{N}$, $n\geqslant 2$, $0\leqslant m\leqslant n-1$ and $\phi\in C^n([\alpha,\beta])$. Then*

$$\phi(u)=Q_{n-1}(\alpha,\beta,\phi,u)+R(\phi,u), \tag{15}$$

where Q_{n-1} is the Abel-Gontscharoff interpolating polynomial for two-points of degree $n-1$, i.e.

$$\begin{aligned}Q_{n-1}(\alpha,\beta,\phi,u)=&\sum_{s=0}^{m}\frac{(u-\alpha)^s}{s!}\phi^{(s)}(\alpha)\\ &+\sum_{r=0}^{n-m-2}\left[\sum_{s=0}^{r}\frac{(u-\alpha)^{m+1+s}(\alpha-\beta)^{r-s}}{(m+1+s)!(r-s)!}\right]\phi^{(m+1+r)}(\beta)\end{aligned}$$

and the remainder is given by

$$R(\phi,u)=\int_\alpha^\beta G_{mn}(u,t)\phi^{(n)}(t)dt,$$

where $G_{mn}(u,t)$ is Green's function defined by

$$G_{mn}(u,t)=\frac{1}{(n-1)!}\begin{cases}\sum\limits_{s=0}^{m}\binom{n-1}{s}(u-\alpha)^{s}(\alpha-t)^{n-s-1}, & \alpha\leqslant t\leqslant u;\\ -\sum\limits_{s=m+1}^{n-1}\binom{n-1}{s}(u-\alpha)^{s}(\alpha-t)^{n-s-1}, & u\leqslant t\leqslant\beta.\end{cases} \tag{16}$$

REMARK 1. Further, for $\alpha\leqslant t,u\leqslant\beta$ the following inequalities hold

$$(-1)^{n-m-1}\frac{\partial^{s}G_{mn}(u,t)}{\partial u^{s}}\geqslant 0,\quad 0\leqslant s\leqslant m,$$
$$(-1)^{n-s}\frac{\partial^{s}G_{mn}(u,t)}{\partial u^{s}}\geqslant 0,\quad m+1\leqslant s\leqslant n-1.$$

Further in the article, we state our results for the class of n-convex functions, a more general class of functions that contains convex functions as a special case. We recall the basic definition and some properties of n-convex functions.

DEFINITION 1. The n-th order divided difference, $n\in\mathbb{N}_0$, of a function $\phi:[\alpha,\beta]\to\mathbb{R}$ at mutually distinct points $x_0,x_1,\dots,x_n\in[\alpha,\beta]$ is defined recursively by

$$[x_i;\phi]=\phi(x_i),\quad i=0,\dots,n$$
$$[x_0,\dots,x_n;\phi]=\frac{[x_1,\dots,x_n;\phi]-[x_0,\dots,x_{n-1};\phi]}{x_n-x_0}.$$

The value $[x_0,\dots,x_n;\phi]$ is independent of the order of the points $x_0,\dots,x_n$. A function $f:[\alpha,\beta]\to\mathbb{R}$ is n-convex if all its n-th order divided differences are non-negative, i. e. $[x_0,\dots,x_n;f]\geqslant 0$ for all choices $x_i\in[\alpha,\beta]$. Thus, 0-convex functions are non-negative and 1-convex functions are non-decreasing, while 2-convex functions are convex in the classical sense. An n-times differentiable is n-convex if and only if its n-derivative is non-negative (see [13]).

After the Introduction, in Section 2 we state and prove the main result involving the Hardy difference operator in the general settings and the Abel-Gontscharoff interpolating polynomial. We discuss the cases where indexes n and m are numbers with opposite parity. We conclude the section with the application of the Hölder inequality. Finally, Section 3 is devoted to results concerning the determination of upper bounds for remainders.

2. Main result

In the main result we establish the connection between the Hardy difference operator derived from(13) and the expression that involves Abel-Gontscharoff interpolating polynomial (15) and Green functions (1)–(4).

THEOREM 3. *Let $(\Sigma_1,\Omega_1,\mu_1)$ and $(\Sigma_2,\Omega_2,\mu_2)$ be measure spaces with positive σ-finite measures. Let $u:\Omega_1\to\mathbb{R}$, be a weight function and v defined by (12). Let A_kf, K be defined by (10) and (11) respectively and for $\gamma\in\{1,2,3,4\}$, $\tilde{G}_\gamma$ is as in (1)–(4). Finally, let $n,m\in\mathbb{N}$, $n\geqslant 4$, $0\leqslant m\leqslant n-3$, G_{mn} be defined by (16) and $\phi: I\to\mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous and $\alpha,\beta\in I$, $\alpha<\beta$. Then*

$$\begin{aligned}
&\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y)-\int_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x)\\
&=\int_\alpha^\beta D\tilde{G}_\gamma(f,z)\times\sum_{s=0}^{m}\frac{(z-\alpha)^s}{s!}\phi^{(s+2)}(\alpha)dz+\int_\alpha^\beta D\tilde{G}_\gamma(f,z)\\
&\quad\times\sum_{r=0}^{n-m-4}\left[\sum_{s=0}^{r}\frac{(z-\alpha)^{m+s+1}(-1)^{r-s}(\beta-\alpha)^{r-s}}{(m+s+1)!(r-s)!}\right]\phi^{(m+r+3)}(\beta)dz\\
&\quad+\int_\alpha^\beta\phi^{(n)}(t)\left(\int_\alpha^\beta D\tilde{G}_\gamma(f,z)\times G_{m,n-2}(z,t)dz\right)dt,
\end{aligned}\tag{17}$$

where

$$D\tilde{G}_\gamma(f,z)=\int_{\Omega_2}\tilde{G}_\gamma(f(y),z)v(y)d\mu_2(y)-\int_{\Omega_1}\tilde{G}_\gamma(A_kf(x),z)u(x)d\mu_1(x)\tag{18}$$

Proof. We prove the statement for the case of Green function $\tilde{G}_1$. Applying the (5) from Lemma 1 to both expressions on the left-hand side in (17), we obtain

$$\begin{aligned}
\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y)&=\int_{\Omega_2}\left[\phi(\alpha)+(f(y)-\alpha)\phi'(\beta)\right]v(y)d\mu_2(y)\\
&\quad+\int_{\Omega_2}\left[\int_\alpha^\beta\tilde{G}_1(f(y),z)\phi''(z)dz\right]v(y)d\mu_2(y),
\end{aligned}\tag{19}$$

$$\begin{aligned}
\int_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x)&=\int_{\Omega_1}\left[\phi(\alpha)+(A_kf(x)-\alpha)\phi'(\beta)\right]u(x)d\mu_1(x)\\
&\quad+\int_{\Omega_1}\left[\int_\alpha^\beta\tilde{G}_1(A_kf(x),z)\phi''(z)dz\right]u(x)d\mu_1(x).
\end{aligned}\tag{20}$$

Now, if we subtract (20) from (19), we get

$$\begin{aligned}
&\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y)-\int_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x)\qquad(21)\\
&=\int_{\Omega_2}\left[\int_\alpha^\beta\tilde{G}_1(f(y),z)\phi''(z)dz\right]v(y)d\mu_2(y)\\
&\quad-\int_{\Omega_1}\left[\int_\alpha^\beta\tilde{G}_1(A_kf(x),z)\phi''(z)dz\right]u(x)d\mu_1(x),
\end{aligned}$$

since

$$\int_{\Omega_2} \left[\phi(\alpha)+(f(y)-\alpha)\phi'(\beta)\right] v(y)d\mu_2(y) \\ -\int_{\Omega_1} \left[\phi(\alpha)+(A_k f(x)-\alpha)\phi'(\beta)\right] u(x)d\mu_1(x) \\ =0. \tag{22}$$

The reason for equation (22) lies in the following equations

$$\int_{\Omega_2} v(y)d\mu_2(y) = \int_{\Omega_2}\int_{\Omega_1} \frac{k(x,y)}{K(x)} u(x)d\mu_1(x)d\mu_2(y) \\ = \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y)d\mu_2(y)d\mu_1(x) = \int_{\Omega_1} u(x)d\mu_1(x)$$

and

$$\int_{\Omega_1} A_k f(x)u(x)d\mu_1(x) = \int_{\Omega_1} \left[\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)d\mu_2(y)\right] u(x)d\mu_1(x) \\ = \int_{\Omega_2} f(y)v(y)d\mu_2(y).$$

Additionaly, by using Fubini theorem on (21), we get the result for the Green function $\tilde{G}_1$. Similarly, from (6), (7) and (8), we obtain the combined expression (14) for all four Green functions.

Further, for function ϕ'' in (14) we take the Abel-Gontscharoff interpolating polynomial. We substitute ϕ with ϕ'' and n with $n-2$ in the expression

$$\phi(z) = \sum_{s=0}^{m} \frac{(z-\alpha)^s}{s!}\phi^{(s)}(\alpha) \\ + \sum_{r=0}^{n-m-2} \left[\sum_{s=0}^{r} \frac{(z-\alpha)^{m+1+s}(-1)^{r-s}(\beta-\alpha)^{r-s}}{(m+1+s)!(r-s)!}\right] \phi^{(m+1+r)}(\beta) \\ + \int_{\alpha}^{\beta} G_{mn}(z,t)\phi^{(n)}(t)dt$$

and get

$$\phi''(z)=\sum_{s=0}^{m}\frac{(z-\alpha)^s}{(s)!}\phi^{(s+2)}(\alpha) + \sum_{r=0}^{n-m-4}\left[\sum_{s=0}^{r}\frac{(z-\alpha)^{m+s+1}(-1)^{r-s}(\beta-\alpha)^{r-s}}{(m+s+1)!(r-s)!}\right]\phi^{(m+r+3)}(\beta) + \int_{\alpha}^{\beta} G_{m,n-2}(z,t)\phi^{(n)}(t)dt. \tag{23}$$

Finally, our statement (17) follows from (14) and (23). □

THEOREM 4. *Suppose that u, v A_k ,$\tilde{G}_\gamma$, $D\tilde{G}_\gamma$ for $\gamma\in\{1,2,3,4\}$ and G_{mn}, $n\geqslant 4,\ 0\leqslant m\leqslant n-3$, be as in* Theorem 3. *If $\phi: I\to\mathbb{R}$ is n-convex, and*

(i) $n-m$ is odd number, then the inequality

$$\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y)-\int_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x) \geqslant \int_{\alpha}^{\beta} D\tilde{G}_\gamma(f,z)\times\sum_{s=0}^{m}\frac{(z-\alpha)^s}{s!}\phi^{(s+2)}(\alpha)dz+\int_{\alpha}^{\beta} D\tilde{G}_\gamma(f,z) \times\sum_{r=0}^{n-m-4}\left[\sum_{s=0}^{r}\frac{(z-\alpha)^{m+s+1}(-1)^{r-s}(\beta-\alpha)^{r-s}}{(m+s+1)!(r-s)!}\right]\phi^{(m+r+3)}(\beta)dz \tag{24}$$

holds.

(ii) $n-m$ is even number, then the inequality

$$\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y)-\int_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x) \leqslant \int_{\alpha}^{\beta} D\tilde{G}_\gamma(f,z)\times\sum_{s=0}^{m}\frac{(z-\alpha)^s}{s!}\phi^{(s+2)}(\alpha)dz+\int_{\alpha}^{\beta} D\tilde{G}_\gamma(f,z) \times\sum_{r=0}^{n-m-4}\left[\sum_{s=0}^{r}\frac{(z-\alpha)^{m+s+1}(-1)^{r-s}(\beta-\alpha)^{r-s}}{(m+s+1)!(r-s)!}\right]\phi^{(m+r+3)}(\beta)dz \tag{25}$$

holds.

Proof. (i) Since $\tilde{G}_\gamma(\cdot,\cdot)$ is continuous and convex with respect to both variables for each $\gamma\in\{1,2,3,4\}$, according to the (13), inequality

$$D\tilde{G}_\gamma(f,z)\geqslant 0$$

holds for every $t \in [\alpha,\beta]$. On the other hand $\tilde{G}_\gamma$ given by (1) - (4) are non-negative and so is ϕ^n since ϕ is n-convex. By definition (16) of the function G_{mn}, according to the Remark for $s=0$ we can conclude that $(-1)^{n-m-1}G_{mn} \geqslant 0$. If $n-m-1$ is even, then G_{mn} is non-negative. On the other hand, if $n-m-1$ is odd, then G_{mn} is not positive. In (17) we have expression containing $G_{m,n-2}$. So, if $n-m-3$ is even, then $G_{m,n-2} \geqslant 0$. Instead of the condition $n-m-3$ is even, we can observe the equivalent condition where $n-m$ is odd and the inequality (24) follows from the Theorem 3.

(ii) Similarly, if $n-m-3$ is odd, then $G_{m,n-2} \leqslant 0$ so the inequality (25) holds. □

In other words, condition (i) of the Theorem 4 is satisfied when n and m are numbers of opposite parity. One of them is even and the other one is odd. Analogously, condition (ii) means that numbers n and m are both even or both odd.

In further study, we consider Hölder inequality for conjugate exponents p and q. As usual we suppose that $1 \leqslant p,q \leqslant \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. The symbol $\|\cdot\|_p$ denotes the standard $L^p([\alpha,\beta])$ norm of a function, i. e.

$$\| g \|_p = \left(\int_\alpha^\beta |g(s)|^p ds \right)^{\frac{1}{p}}$$

for $1 \leqslant p < \infty$, while $\|g\|_\infty$ is the essential supremum of g.

To simplify notation, for $\gamma \in \{1,2,3,4\}$ we introduce the abbreviation $B_\gamma : [\alpha,\beta] \to \mathbb{R}$ in the form:

$$B_\gamma(t) = \int_\alpha^\beta D\tilde{G}_\gamma(f,z) \times G_{m,n-2}(z,t)\, dz, \tag{26}$$

where we assume that all the terms appearing in B_γ satisfy the assumptions of Theorem 3.

COROLLARY 1. *Let $n,m \in \mathbb{N}$, $n \geqslant 4$, $0 \leqslant m \leqslant n-3$ and $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous and $\phi^{(n)} \in L^p[\alpha,\beta]$ for α, $\beta \in I$, $\alpha < \beta$. Further, let A_k be as in (10), $\gamma \in \{1,2,3,4\}$, $\tilde{G}_\gamma$ as in (1)–(4), $D\tilde{G}_\gamma$ as in (18), G_{mn} as in (16), B_γ* as in (26) and u a weight function with v given by (12). If (p,q) is a pair of conjugate exponents, then

$$|R(\phi)| \leqslant \|\phi^{(n)}\|_p \times \|B_\gamma(t)\|_q, \tag{27}$$

holds, where

$$\begin{aligned} R(\phi) = &\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) - \int_\alpha^\beta D\tilde{G}_\gamma(f,z) \\ &\times \left[\sum_{s=0}^{m} \frac{(z-\alpha)^s}{(s)!} \phi^{(s+2)}(\alpha) + \sum_{r=0}^{n-m-4} \left[\sum_{s=0}^{r} \frac{(z-\alpha)^{m+s+1}(-1)^{r-s}(\beta-\alpha)^{r-s}}{(m+s+1)!(r-s)!} \right] \right. \\ &\left. \times \phi^{(m+r+3)}(\beta) \right] dt. \end{aligned} \tag{28}$$

Proof. Applying the Hölder inequality on (17) considering the notation (28) and abbreviation (26) we get

$$\begin{aligned}|R(\phi)| &= \left|\int_\alpha^\beta \phi^{(n)}(t)\left[\int_\alpha^\beta D\tilde{G}_\gamma(f,z)\times G_{m,n-2}(z,t)\,dz\right]dt\right| \\ &\leqslant \|\phi^{(n)}\|_p \times \left(\int_\alpha^\beta \left|\int_\alpha^\beta D\tilde{G}_\gamma(f,z)\times G_{m,n-2}(z,t)dz\right|^q dt\right)^{\frac{1}{q}} \\ &= \|\phi^{(n)}\|_p \times \left(\int_\alpha^\beta |B_\gamma(t)|^q\,dt\right)^{\frac{1}{q}} = \|\phi^{(n)}\|_p \times \|B_\gamma(t)\|_q\end{aligned}$$

and obtain the required inequality. □

REMARK 2. As special cases for boundary values p and q, from the inequality (27) we get the following inequalities

$$|R(\phi)| \leqslant \max|\phi^{(n)}(t)| \times \left|\int_\alpha^\beta B_\gamma(t)\,dt\right|$$

and

$$|R(\phi)| \leqslant \max\left|\int_\alpha^\beta B_\gamma(t)\,dt\right| \times \left|\int_\alpha^\beta \phi^{(n)}(t)dt\right|.$$

3. Upper bounds for remainders

Consider the Čebyšev functional,

$$T(h,g) = \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)g(t)dt - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt\right)\cdot\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt\right)$$

for Lebesgue integrable functions $h,g:[\alpha,\beta]\to\mathbb{R}$. It plays a crucial role in the construction of the upper bounds. The next two theorems from [3] provide Grüss and Ostrowski type inequalities involving the above functional.

THEOREM 5. *Let $h,g:[\alpha,\beta]\to\mathbb{R}$ be two absolutely continuous functions with $(\cdot-\alpha)(\beta-\cdot)(h')^2, (\cdot-\alpha)(\beta-\cdot)(g')^2 \in L([\alpha,\beta])$. Then*

$$|T(h,g)| \leqslant \frac{1}{\sqrt{2}}|T(h,h)|^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (s-\alpha)(\beta-s)[g'(s)]^2 ds\right)^{\frac{1}{2}}. \tag{29}$$

The constant $\frac{1}{\sqrt{2}}$ is the best possible in (29).

THEOREM 6. *Assume that $g:[\alpha,\beta]\to\mathbb{R}$ is monotonic non-decreasing and $h:[\alpha,\beta]\to\mathbb{R}$ is absolutely continuous with $h'\in L^\infty([\alpha,\beta])$. Then*

$$|T(h,g)| \leqslant \frac{1}{2(\beta-\alpha)}\left\|h'\right\|_\infty \int_\alpha^\beta (s-\alpha)(\beta-s)dg(s). \tag{30}$$

The constant $\frac{1}{2}$ is the best possible in (30) .

We apply the Theorem 5 to get upper bound for the remainder obtained according to the main result (17).

THEOREM 7. *Let $n \in \mathbb{N}$, $n \geqslant 4$, $\gamma \in \{1,2,3,4\}$, $D\tilde{G}_\gamma$, B_γ, R are as in (18), (26), (28), respectively and $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ is absolutely continuous with $(\cdot-\alpha)(\beta-\cdot)(\phi^{(n+1)})^2 \in L([\alpha,\beta])$. If $(\cdot-\alpha)(\beta-\cdot)(B'_\gamma)^2 \in L([\alpha,\beta])$, then the remainder*

$$\rho(\phi;\alpha,\beta) = R(\phi) - \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta B_\gamma(t)dt \tag{31}$$

is bounded by

$$|\rho(\phi;\alpha,\beta)| \leqslant \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}\left|T(B_\gamma,B_\gamma)\right|^{\frac{1}{2}}\left(\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right)^{\frac{1}{2}}. \tag{32}$$

Proof. From (17) and (31) we conclude

$$\rho(\phi;\alpha,\beta) = \int_\alpha^\beta B_\gamma(t)\phi^{(n)}(t)dt - \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta B_\gamma(t)dt. \tag{33}$$

Assumptions of Theorem 5 are satisfied for $h = B_\gamma$ and $g = \phi^{(n)}$, so together with (33) we calculate

$$\begin{aligned}
&\frac{1}{\beta-\alpha}|\rho(\phi;\alpha,\beta)| \\
&= \left|\frac{1}{\beta-\alpha}\int_\alpha^\beta B_\gamma(t)\phi^{(n)}(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta B_\gamma(t)dt\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(t)dt\right| \\
&\leqslant \frac{1}{\sqrt{2}}\left|T(B_\gamma,B_\gamma)\right|^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{34}$$

Therefore from (34) we get (32). □

Application of the Theorem 6 yields the following result, again concerning the upper bound for the remainder ρ defined by (31).

THEOREM 8. *Let $n \in \mathbb{N}$, $n \geqslant 4$, B_γ be as in (26) and $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ is monotonic non-decreasing. If B_γ is absolutely continuous with $B'_\gamma \in L^\infty([\alpha,\beta])$, then the remainder $\rho(\phi;\alpha,\beta)$ given by (31) is bounded by*

$$\begin{aligned}
&|\rho(\phi;\alpha,\beta)| \\
&\leqslant \|B'_\gamma\|_\infty\left[\frac{(\beta-\alpha)\left(\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right)}{2} - \left\{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right\}\right].
\end{aligned} \tag{35}$$

Proof. Assumptions of Theorem 6 are satisfied for $h = B_\gamma$ and $g = \phi^{(n)}$, so, taking into account (33), we have

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta B_\gamma(t)\phi^{(n)}(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta B_\gamma(t)dt\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(t)dt\right|$$
$$\leqslant \frac{1}{2(\beta-\alpha)}\left\|B'_\gamma\right\|_\infty \int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt. \tag{36}$$

Simple calculation yields

$$\begin{aligned}&\int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt\\ &=\int_\alpha^\beta [2t-(\alpha+\beta)]\,\phi^{(n)}(t)dt\\ &=(\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right]-2\left[\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right].\end{aligned}$$

Finally, inserting the last expression in (36) and taking into account (33) we get (35). □

The last theorem gives Ostrowski-type bound for the generalized Hardy's inequality. About Ostrowski-type inequalities can be found i.e. in [2] and [10].

THEOREM 9. *Let $n \in \mathbb{N}$, $n \geqslant 4$, $D\tilde{G}_\gamma$ and B_γ be as in (18) and (26), respectively. Let $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\|\phi^{(n)}\|_p < \infty$. Then*

$$\left|\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1}\phi(A_k f(x))u(x)d\mu_1(x) - \int_\alpha^\beta D\tilde{G}_\gamma(f,z)\right.$$
$$\times\left[\sum_{s=0}^{m}\frac{(z-\alpha)^s}{(s)!}\phi^{(s+2)}(\alpha)+\sum_{r=0}^{n-m-4}\left[\sum_{s=0}^{r}\frac{(z-\alpha)^{m+s+1}(-1)^{r-s}(\beta-\alpha)^{r-s}}{(m+s+1)!(r-s)!}\right]\right.$$
$$\left.\left.\times\phi^{(m+r+3)}(\beta)\right]dt\right| \leqslant \left\|\phi^{(n)}\right\|_p \left\|B_\gamma\right\|_q.$$

The constant on the right hand side is sharp when $1 < p \leqslant \infty$ and the best possible when $p = 1$.

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Dora Pokaz
University of Zagreb
Faculty of Civil Engineering
Fra Andrije Kačića Miošića 26, 10000 Zagreb, Croatia
e-mail: dora.pokaz@grad.unizg.hr