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INEQUALITIES FOR THE RIEMANN ZETA FUNCTION ON THE POSITIVE REALS

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Abstract. In this paper we obtain a sequence of inequalities regarding the Riemann zeta function and its derivative. The simplest special cases of this gives $-\zeta'(s) < \frac{1}{(s-1)^2}$ for $s > 0$ and $\zeta(s) > \frac{1}{s-1} + \gamma$ for $s > 1$.

1. Introduction and statement of results

For $n \in \mathbb{N}$, let

$$F_n(s) = \frac{1}{n} \sum_{m>n} \frac{\log \frac{m}{n}}{(\frac{m}{n})^s}. \tag{1.1}$$

This converges for Re $s > 1$ and is directly related to the Riemann's zeta function. Thus $F_1(s) = -\zeta'(s)$ and more generally,

$$F_n(s) = \left(-\zeta'(s) - \sum_{m=1}^{n} \frac{\log m}{m^s} - \left(\zeta(s) - \sum_{m=1}^{n} \frac{1}{m^s}\right) \log n\right) n^{s-1}. \tag{1.2}$$

Hence, $F_n(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$, which we also denote by $F_n(s)$. However, we shall only be interested in this function for real values of s (see [4] and [6] for the relevant properties of $\zeta(s)$).

The object of this paper is to prove:

THEOREM 1. *For each $n \in \mathbb{N}$ and $s > 0$ such that $s \neq 1$, we have*

$$F_n(s) < F_{2n}(s).$$

Observe that the sum in (1.1) is a Riemann sum approximation to the integral $\int_1^\infty \frac{\log t}{t^s}\, dt$ so that, as $n \to \infty$, for Re $s > 1$,

$$F_n(s) \to \int_1^\infty \frac{\log t}{t^s}\, dt = \frac{1}{(s-1)^2}. \tag{1.3}$$

Using $\sum_{n \leqslant x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$ and $\sum_{n \leqslant x} \frac{\log n}{n^s} = \frac{x^{1-s} \log x}{1-s} - \frac{x^{1-s}}{(1-s)^2} - \zeta'(s) + O(x^{-s} \log x)$ for $0 < \Re s < 1$, we see that (1.2) implies (1.3) also holds for this range. As an immediate consequence, we therefore have

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COROLLARY 1. *For each $n \in \mathbb{N}$ and $s > 0$ such that $s \neq 1$, we have*

$$F_n(s) < \frac{1}{(s-1)^2}.$$

As $F_n(s) = -(n^{s-1}\sum_{m>n}\frac{1}{m^s})'$, this in turn implies

COROLLARY 2. *For each $n \in \mathbb{N}$ and $s > 0$, the function*

$$k_n(s) := n^{s-1}\left(\zeta(s) - \sum_{m=1}^{n}\frac{1}{m^s}\right) - \frac{1}{s-1}$$

is strictly increasing. (At $s = 1$ we take limits as $s \to 1$.) This further implies that, with $H_n = \sum_{r=1}^{n}\frac{1}{r}$ and γ denoting Euler's constant,

$$-\frac{1}{2n} < k_n(s) < \log n + \gamma - H_n \quad (0 < s < 1) \text{ and } \log n + \gamma - H_n < k_n(s) < 0 \quad (s > 1). \tag{1.4}$$

We point out a few special cases, starting with Corollaries 1 and 2. When $n = 1$ these give: for $s > 0$

$$-\zeta'(s) < \frac{1}{(s-1)^2} \quad \text{and} \quad \zeta(s) - \frac{1}{s-1} \quad \text{is strictly increasing,}$$

while (1.4) says

$$\frac{1}{2} < \zeta(s) - \frac{1}{s-1} < \gamma \quad (0 < s < 1) \quad \text{and} \quad \gamma < \zeta(s) - \frac{1}{s-1} < 1 \quad (s > 1).$$

Even these inequalities appear to be new. For example, it follows from the above that for $s > 1$

$$-\frac{\zeta'(s)}{\zeta(s)} < \frac{1}{(s-1)^2}\cdot\frac{1}{\frac{1}{s-1}+\gamma} = \frac{1}{s-1} - \frac{\gamma}{1+\gamma(s-1)};$$

i.e.

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} > \frac{\gamma}{1+\gamma(s-1)} \quad (s > 1). \tag{1.5}$$

The same argument gives $\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} < \frac{\gamma}{1+\gamma(s-1)}$ for $0 < s < 1$. Inequality (1.5) improves a recent result by Alzer and Kwong [1] (itself an improvement on an earlier inequality by Delange [5]) where the RHS in (1.5) is $\frac{\gamma}{s}$.

Theorem 1 says something more directly about the function $\eta(s) = (1-2^{1-s})\zeta(s)$. Observe that

$$F_{2n}(s) - F_n(s) = \frac{1}{2n}\sum_{m>2n}\frac{\log\frac{m}{2n}}{(\frac{m}{2n})^s} - \frac{1}{n}\sum_{m>n}\frac{\log\frac{m}{n}}{(\frac{m}{n})^s} = \frac{1}{2n}\sum_{m>2n}(-1)^{m-1}\frac{\log\frac{m}{2n}}{(\frac{m}{2n})^s},$$

by splitting the first sum into even m and odd m. Further note that this is

$$-\frac{1}{2n}\left(\sum_{m>2n}\frac{(-1)^{m-1}}{(m/2n)^s}\right)'.$$

Thus Theorem 1 implies that for every $n \in \mathbb{N}$

$$\ell_n(s) := (2n)^{s-1}\left(\eta(s) - \sum_{m=1}^{2n} \frac{(-1)^{m-1}}{m^s}\right) \quad \text{is strictly decreasing on } (0,\infty).$$

This can be compared to the result that $\eta(s)$ is strictly concave on $(0,\infty)$, see [2]. Note that $\ell_n(0) = \frac{1}{4n}$ while $\ell_n(\infty) = 0$, so $0 < \ell_n(s) < \frac{1}{4n}$.

There have been a number of results on inequalities for $\zeta(s)$ (eg [1], [2]) often in conjunction with results on $\Gamma(s)$. See for example, [3], where inequalities and conjectures are given for the Hurwitz zeta function on $(1,\infty)$. These complement our results.

2. Proof of Theorem 1

Before we start the proof, we have the following Mellin transform representation of F_n:

$$F_n(s) = \frac{1}{(s-1)^2} - sn^{s-1}\int_n^\infty \frac{E_n(x)}{x^{s+1}}\,dx, \tag{2.1}$$

where

$$E_n(x) = \int_n^x \log\frac{t}{n}\,d\{t\} = \{x\}\log\frac{x}{n} - \int_n^x \frac{\{t\}}{t}\,dt.$$

Equation (2.1) holds initially for Re $s > 1$, but since the integral on the right converges absolutely for Re $s > 0$, it holds for all such s except $s = 1$. To see (2.1), note that, with $L_n(x) = \sum_{n<m\leqslant x} \log\frac{m}{n}$,

$$F_n(s) = sn^{s-1}\int_n^\infty \frac{L_n(x)}{x^{s+1}}\,dx.$$

But

$$L_n(x) = \int_n^x \log\frac{t}{n}\,d[t] = x\log\frac{x}{n} - x + n - E_n(x),$$

on writing $[t] = t - \{t\}$. Thus

$$F_n(s) = sn^s\int_n^\infty \frac{\frac{x}{n}\log\frac{x}{n} - \frac{x}{n} + 1}{x^{s+1}}\,dx - sn^{s-1}\int_n^\infty \frac{E_n(x)}{x^{s+1}}\,dx.$$

The first integral is $s\int_1^\infty \frac{y\log y - y + 1}{y^{s+1}}dx = \frac{1}{(s-1)^2}$, proving (2.1).

We note that $E_n(x)$ is right continuous with jump discontinuities at integer points and that on intervals $(k, k+1)$,

$$E_n'(x) = \log\frac{x}{n}.$$

Proof of Theorem 1. From (2.1), we obtain

$$F_{2n}(s) = \frac{1}{(s-1)^2} - s(2n)^{s-1}\int_{2n}^\infty \frac{E_{2n}(x)}{x^{s+1}}\,dx = \frac{1}{(s-1)^2} - sn^{s-1}\int_n^\infty \frac{\frac{1}{2}E_{2n}(2x)}{x^{s+1}}\,dx.$$

Thus we find

$$F_{2n}(s) - F_n(s) = sn^{s-1} \int_n^\infty \frac{\Delta_n(x)}{x^{s+1}}\, dx, \tag{2.2}$$

where $\Delta_n(x) = E_n(x) - \frac{1}{2}E_{2n}(2x)$. Hence $\Delta(\cdot)$ is right-continuous with jump discontinuities at points $k,\ k+\frac{1}{2}$ $(k \in \mathbb{N},\ k \geqslant n)$ and for x not of this form

$$\Delta_n'(x) = E_n'(x) - E_{2n}'(2x) = \log\frac{x}{n} - \log\frac{2x}{2n} = 0;$$

i.e. $\Delta_n(\cdot)$ is *constant* on intervals $[k, k+\frac{1}{2})$ and $[k+\frac{1}{2}, k+1)$ for each integer $k \geqslant n$. These constants are given by

$$\Delta_n(k) = -\frac{1}{2}\sum_{r=n+1}^{k} \log\Big(\frac{2r}{2r-1}\Big)$$

$$\Delta_n\Big(k+\frac{1}{2}\Big) = \frac{1}{2}\sum_{r=n}^{k} \log\Big(\frac{2r+1}{2r}\Big).$$

To show these, use $E_n(x) = x\log\frac{x}{n} - x + n - \sum_{n<m\leqslant x} \log\frac{m}{n}$. Then

$$\Delta_n(x) = \frac{1}{2}\sum_{2n<m\leqslant 2x} \log\frac{m}{2n} - \sum_{n<m\leqslant x} \log\frac{m}{n}.$$

Now take $x = k$ and $k+\frac{1}{2}$ in turn. Then for the first sum on the right above, split the sum into m even and m odd. For $x = k$ we get

$$\begin{aligned}
\Delta_n(k) &= \frac{1}{2}\sum_{r=n+1}^{k}\Big(\log\frac{2r}{2n} + \log\frac{2r-1}{2n}\Big) - \sum_{r=n+1}^{k}\log\frac{r}{n} \\
&= \frac{1}{2}\sum_{r=n+1}^{k}\Big(\log\frac{2r-1}{2n} - \log\frac{2r}{2n}\Big) \\
&= \frac{1}{2}\sum_{r=n+1}^{k}\log\Big(\frac{2r-1}{2r}\Big)
\end{aligned}$$

For $x = k+\frac{1}{2}$, noting that $2x = 2k+1$, we get

$$\begin{aligned}
\Delta_n\left(k+\frac{1}{2}\right) &= \frac{1}{2}\sum_{r=n+1}^{k}\log\frac{2r}{2n} + \frac{1}{2}\sum_{r=n}^{k}\log\frac{2r+1}{2n} - \sum_{r=n+1}^{k}\log\frac{r}{n} \\
&= \frac{1}{2}\log\frac{2n+1}{2n} + \frac{1}{2}\sum_{r=n+1}^{k}\Big(\log\frac{2r+1}{2n} - \log\frac{2r}{2n}\Big) \\
&= \frac{1}{2}\log\frac{2n+1}{2n} + \frac{1}{2}\sum_{r=n+1}^{k}\log\frac{2r+1}{2r} \\
&= \frac{1}{2}\sum_{r=n}^{k}\log\Big(\frac{2r+1}{2r}\Big).
\end{aligned}$$

In particular, we note that

(a) $\Delta_n(n) = 0$ and $\Delta_n(k) < 0$ for $k > n$;

(b) $\Delta_n(k+\frac{1}{2}) > 0$ for $k \geqslant n$, and crucially;

(c) $\Delta_n(k-\frac{1}{2}) + \Delta_n(k) > 0$ for $k > n$.

(a) and (b) are immediate and for (c), we have

$$\begin{aligned}\Delta_n\Big(k-\frac{1}{2}\Big)+\Delta_n(k) &= \frac{1}{2}\sum_{r=n}^{k-1}\log\Big(\frac{2r+1}{2r}\Big)-\frac{1}{2}\sum_{r=n+1}^{k}\log\Big(\frac{2r}{2r-1}\Big)\\ &= \frac{1}{2}\sum_{r=n}^{k-1}\log\Big(\frac{(2r+1)^2}{2r(2r+2)}\Big) > 0.\end{aligned}$$

Hence

$$\begin{aligned}\int_n^\infty \frac{\Delta_n(x)}{x^{s+1}}\,dx &= \sum_{k>n}\left(\int_{k-\frac{1}{2}}^{k}\frac{\Delta_n(x)}{x^{s+1}}\,dx+\int_k^{k+\frac{1}{2}}\frac{\Delta_n(x)}{x^{s+1}}\,dx\right)\\ &= \sum_{k>n}\left(\Delta_n\Big(k-\frac{1}{2}\Big)\int_{k-\frac{1}{2}}^{k}\frac{1}{x^{s+1}}\,dx+\Delta_n(k)\int_k^{k+\frac{1}{2}}\frac{1}{x^{s+1}}\,dx\right)\\ &> \sum_{k>n}\left(\Delta_n\Big(k-\frac{1}{2}\Big)+\Delta_n(k)\right)\int_{k-\frac{1}{2}}^{k}\frac{1}{x^{s+1}}\,dx > 0,\end{aligned}$$

by (a) and (c). This establishes Theorem 1. □

3. Some comments

(a) The cases $s = 0$ and $s = 1$. The inequalities in Theorem 1 and Corollary 1 say nothing directly about $s = 0, 1$. However, by taking limits we can see what they tell us.

First consider $s = 0$. The functions occurring in Theorem 1 and Corollary 1 are continuous at 0, so immediately we have

$$F_n(0) \leqslant F_{2n}(0) \quad \text{and} \quad F_n(0) \leqslant 1. \tag{3.1}$$

We have from (1.2)

$$\begin{aligned}F_n(0) &= \frac{1}{n}\Big(-\zeta'(0)-\log n!-(\zeta(0)-n)\log n\Big)\\ &= \frac{1}{n}\Big(n\log n-\log n!+\frac{1}{2}\log 2n\pi\Big)\\ &= \frac{1}{n}\log\left(\frac{n^n\sqrt{2n\pi}}{n!}\right),\end{aligned}$$

using the values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log 2\pi$. Thus (3.1) becomes, respectively, the known inequalities:

$$\frac{(2n)!}{(n!)^2} \leqslant \frac{2^{2n}}{\sqrt{n\pi}} \quad \text{and} \quad n! \geqslant \sqrt{2n\pi}\Big(\frac{n}{e}\Big)^n.$$

For the case $s = 1$, we cannot just let $s \to 1$. Instead consider $F_n(1+\varepsilon)$ with ε small. From (1.2) and using $\zeta(1+\varepsilon) = \frac{1}{\varepsilon} + \gamma - \gamma_1\varepsilon + O(\varepsilon^2)$[1] it follows directly that

$$F_n(1+\varepsilon) = \frac{1}{\varepsilon^2} - A_n + O(\varepsilon),$$

where, with $H_n = \sum_{r=1}^n \frac{1}{r}$ and $S_n = \sum_{r=1}^n \frac{\log r}{r}$,

$$A_n = (S_n - \tfrac{1}{2}(\log n)^2 - \gamma_1) - (H_n - \log n - \gamma)\log n.$$

Hence Theorem 1 and Corollary 1 imply $A_n \geqslant A_{2n} \geqslant 0$. Writing $H_n = \log n + \gamma + \mu_n$ and $S_n = \frac{1}{2}(\log n)^2 + \gamma_1 + \nu_n$, these become

$$\nu_n - \mu_n \log n \geqslant \nu_{2n} - \mu_{2n}\log 2n \geqslant 0.$$

(b) As noted in the introduction

$$F_{2n}(s) - F_n(s) = \frac{1}{2n}\sum_{m>2n}(-1)^{m-1}\frac{\log\frac{m}{2n}}{(\frac{m}{2n})^s},$$

which is an alternating series. If the terms decrease, Theorem 1 would follow immediately. However the terms do not decrease for with $m = 2n + k$ with n large and bounded k and s, the term is $\sim \frac{k}{2n}$; i.e. they increase in size for the initial part of the series.

(c) Perhaps a few words to explain the reason for considering $F_n(s)$ in the first place. This stemmed from trying to prove an upper bound for a certain sequence of functions which appeared in a spectral theory problem. Let $\alpha > 1$ and put $c_1(n) = \frac{1}{n^\alpha}$ $(n \in \mathbb{N})$. Then for $r \geqslant 1$ define recursively

$$c_{r+1}(n) = \sum_{m>n} c_r(m)\Big(\frac{1}{n^\alpha} - \frac{1}{m^\alpha}\Big).$$

It is straightforward to show using $\sum_{n>x} n^{-\beta} \sim \frac{1}{\beta-1}x^{1-\beta}$ for $\beta > 1$ that for fixed r, one has

$$c_r(n) \sim \frac{\lambda_r}{n^{(\alpha-1)r+1}} \quad \text{as } n \to \infty, \tag{3.2}$$

where $\lambda_1 = 1$ and $\lambda_{r+1} = \lambda_r\big(\frac{1}{(\alpha-1)r} - \frac{1}{(\alpha-1)r+\alpha}\big)$. One can obtain a formula for λ_r in terms of the Gamma function. However a good upper bound was required

[1] Here $\gamma_1 = \lim_{n\to\infty}\sum_{r=1}^n \frac{\log r}{r} - \frac{1}{2}(\log n)^2$.

for the $c_r(n)$ valid for large r and it seemed conceivable that the asymptotic equality in (3.2) could be replaced by an inequality; i.e. that

$$c_r(n) \leqslant \frac{\lambda_r}{n^{(\alpha-1)r+1}} \quad (r,n \in \mathbb{N}). \tag{3.3}$$

This is trivially true for $r=1$, but for $r=2$ it involves

$$\sum_{m>n} \frac{1}{m^\alpha}\left(\frac{1}{n^\alpha} - \frac{1}{m^\alpha}\right) \leqslant \frac{\lambda_2}{n^{2\alpha-1}},$$

or equivalently, on multiplying through by $n^{2\alpha-1}$,

$$\frac{1}{n}\sum_{m>n}\left(\frac{n}{m}\right)^\alpha - \frac{1}{n}\sum_{m>n}\left(\frac{n}{m}\right)^{2\alpha} \leqslant \frac{1}{\alpha-1} - \frac{1}{2\alpha-1}. \tag{3.4}$$

Note that (3.4) follows if $\frac{1}{n}\sum_{m>n}(\frac{n}{m})^s - \frac{1}{s-1}$ increases as s increases. Taking derivatives, this follows if $F_n(s) \leqslant \frac{1}{(s-1)^2}$ – precisely what we proved in Corollary 1. An inductive argument shows that Corollary 1 is also sufficient to prove (3.3).

Initial attempts to prove Corollary 1 by showing that $\int_n^\infty \frac{E_n(x)}{x^{s+1}}dx > 0$ proved difficult as $E_n(x)$ does not have a convenient form. However it turned out that $E_n(x) - \frac{1}{2}E_{2n}(2x)$ is much easier to deal with, being a step function.

We suspect that in fact one has the stronger result $F_n(s) < F_{n+1}(s)$, but that the proof will not be as elegant.

(d) As the last few lines in the proof of Theorem 1 indicate, there is some scope for improvement. Noting that $\frac{1}{2}\int_{k-\frac{1}{2}}^{k} \frac{1}{x^{s+1}}\,dx > \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{1}{x^{s+1}}\,dx$, we see that

$$\begin{aligned}
\int_n^\infty \frac{\Delta_n(x)}{x^{s+1}}\,dx &> \sum_{k>n}\sum_{r=n}^{k-1} \log\left(\frac{(2r+1)^2}{2r(2r+2)}\right) \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{1}{x^{s+1}}\,dx \\
&= \sum_{r=n}^{\infty} \log\left(\frac{(2r+1)^2}{2r(2r+2)}\right) \sum_{k>r} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{1}{x^{s+1}}\,dx \\
&= -\sum_{r=n}^{\infty} \log\left(1 - \frac{1}{(2r+1)^2}\right) \int_{r+\frac{1}{2}}^{\infty} \frac{1}{x^{s+1}}\,dx \\
&> \frac{2^s}{s}\sum_{r=n}^{\infty} \frac{1}{(2r+1)^{s+2}}.
\end{aligned}$$

Hence (2.2) implies, for $s>0$,

$$F_{2n}(s) - F_n(s) > 2^s n^{s-1} \sum_{r=n}^{\infty} \frac{1}{(2r+1)^{s+2}}.$$

This readily improves Theorem 1.

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