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THE CONTINUITY OF PSEUDO–DIFFERENTIAL OPERATORS ON LOCAL VARIABLE HARDY SPACES AND THEIR DUAL SPACES

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Abstract. In this paper we establish the continuity of pseudo-differential operators with symbols in $S^{-\alpha}_{1,\delta}$ on local variable Hardy spaces and their dual spaces. Precisely, we show that the pseudo-differential operator maps continuously one local variable Hardy space into another one, maps continuously one local variable Carleson measure spaces into another one and maps continuously one variable Lebesgue spaces into one local variable Carleson measure spaces.

1. Introduction and statement of main results

The main purpose of this paper is to generalize the results due to Goldberg in [11] to a large class of pseudo-differential operators with symbols in $S^{-\alpha}_{1,\delta}$ and obtain the boundedness for such pseudo-differential operators on the local variable Hardy spaces and their dual spaces.

The real-variable theory of Hardy spaces in $\mathbb{R}^n$ was initiated by Stein and Weiss [29] and systematically developed by Fefferman and Stein [9]. The Hardy space $H^p(0<p\leqslant 1)$ is a suitable substitute of the Lebesgue space L^p when studying the boundedness of some classical operators. However, while they are well suited as functional spaces for their applications to PDE's with constant coefficients, the Hardy spaces are not stable under multiplication by Schwartz class, a fact that seriously hinders their role when it comes to PDE's with variable coefficient. Thus, the theory of local Hardy space h^p plays an important role in various fields of analysis and partial differential equations. In particular, pseudo-differential operators of order zero are bounded on local Hardy spaces h^p for $0<p<1$, but they are not bounded on Hardy spaces H^p for $0<p<1$ (see [11]). Now we come to the variable exponent counterpart. The variable Hardy spaces theory was established by Nakai and Sawano ([22]), Cruz-Uribe and Wang ([5]). The results concerning the boundedness of many classical operators on variable Hardy spaces have been obtained in recent years (see, for instance, [5, 16, 22, 26, 38]). For more results on the variable Hardy spaces theory, we also refer

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to [30, 33, 35, 36, 37, 39]. Recently, the atomic decomposition of the local version of variable Hardy spaces $h^{p(\cdot)}$ was studied in [31, 34]. Meanwhile, some boundedness results of linear and multi-linear pseudo-differential operators of order zero on the local variable Hardy spaces were also established in the work.

Before we state the main results of this paper, first we recall the definition of variable Lebesgue spaces. Note that the variable exponent function spaces, such as the variable Lebesgue spaces and the variable Sobolev spaces, were studied by a substantial number of researchers (see, for instance, [2, 3, 8, 20]). For any Lebesgue measurable function $p(\cdot):\mathbb{R}^n \to (0,\infty]$ and for any measurable subset $E \subset \mathbb{R}^n$, we denote $p^-(E) = \inf_{x\in E} p(x)$ and $p^+(E) = \sup_{x\in E} p(x)$. Especially, we denote $p^- = p^-(\mathbb{R}^n)$, $p^+ = p^+(\mathbb{R}^n)$ and $p_- = \min\{p^-,1\}$. Let $p(\cdot)\colon \mathbb{R}^n \to (0,\infty)$ be a measurable function with $0 < p^- \leqslant p^+ < \infty$ and $\mathscr{P}^0$ be the set of all these $p(\cdot)$. Let $\mathscr{P}$ denote the set of all measurable functions $p(\cdot):\mathbb{R}^n \to [1,\infty)$ such that $1 < p^- \leqslant p^+ < \infty$.

DEFINITION 1. Let $p(\cdot):\mathbb{R}^n \to (0,\infty]$ be a Lebesgue measurable function. The variable Lebesgue space $L^{p(\cdot)}$ consisits of all Lebesgue measurable functions f, for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)}dx$ is finite for some $\varepsilon > 0$ and

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

The variable Lebesgue space was first introduced by Orlicz [24] in 1931. Two decades later, Nakano [23] first systematically studied modular function spaces which include the variable Lebesgue spaces as specific examples. The modern development, however, started with the paper [20] of Kováčik and Rákosník in 1991. As a special case of the theory of Nakano and Luxemberg, we see that $L^{p(\cdot)}$ is a quasi-normed space. Especially, when $p^- \geqslant 1$, $L^{p(\cdot)}$ is a Banach space. Recall the following class of exponent function in [6]. Let $\mathscr{B}$ be the set of $p(\cdot) \in \mathscr{P}$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}$. An important subset of $\mathscr{B}$ is LH condition. In the study of variable exponent function spaces it is common to assume that the exponent function $p(\cdot)$ satisfies LH condition. We say that $p(\cdot) \in LH$, if $p(\cdot)$ satisfies

$$|p(x)-p(y)| \leqslant \frac{C}{\log(e+1/|x-y|)}, \quad |x-y| < 1/2$$

and there exists $p_\infty \in \mathbb{R}$ so that

$$|p(x)-p_\infty| \leqslant \frac{C}{\log(|x|+e)}, \quad \forall x \in \mathbb{R}^n.$$

It is well known that $p(\cdot) \in \mathscr{B}$ if $p(\cdot) \in \mathscr{P} \cap LH$. Moreover, example shows that the above LH conditions are necessary in certain sense, see Pick and Růžička ([25]) for more details. Next we also recall the definition of local Hardy spaces with variable exponents $h^{p(\cdot)}$ as follows.

DEFINITION 2. ([22]) Let $f \in \mathscr{S}'$, $p(\cdot) \in \mathscr{P}^0 \cap LH$ and $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$, $x \in \mathbb{R}^n$. Denote by $\mathscr{M}$ the grand maximal operator given by $\mathscr{M}_{loc}f(x) = \sup\{|\varphi_t * f(x)| : 0 < t < 1, \varphi \in \mathscr{F}_N\}$ for any fixed large integer N, where $\mathscr{F}_N = \{\varphi \in \mathscr{S} : \int \varphi(x)dx = 1, \sum_{|\alpha| \leqslant N} \sup(1+|x|)^N |\partial^\alpha \varphi(x)| \leqslant 1\}$. The local Hardy space with variable exponent $h^{p(\cdot)}$ is the set of all $f \in \mathscr{S}'$ for which the quantity

$$\|f\|_{h^{p(\cdot)}} = \|\mathscr{M}_{loc}f\|_{L^{p(\cdot)}} < \infty.$$

We recall the Hörmander class of pseudo-differential operators [18]. Suppose that $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$. Let T_σ be a classical pseudo-differential operator of the form

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi, \qquad f \in \mathscr{S},$$

where $\sigma \in S^m_{\rho,\sigma}$, that is, $\sigma(x,\xi)$ is a smooth function for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leqslant C(1+|\xi|)^{m-\rho|\beta|+\sigma|\alpha|}. \tag{1}$$

We say that a cube $Q \subset \mathbb{R}^n$ is dyadic if $Q = Q_{j\mathbf{k}} = \{x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n : 2^{-j}k_i \leqslant x_i < 2^{-j}(k_i+1), i = 1,2,\ldots,n\}$ for some $j \in \mathbb{Z}$, some fixed positive large integer N and $\mathbf{k} = (k_1,k_2,\ldots,k_n) \in \mathbb{Z}^n$. Denote by $\ell(Q) = 2^{-j}$ the side length of $Q = Q_{j\mathbf{k}}$. Denote $\mathscr{D}_j = \{Q : Q = Q_{j\mathbf{k}}\}$ and $\mathscr{D} = \cup_{j\in\mathbb{N}}\mathscr{D}_j$. Denote by $z_Q = 2^{-j}\mathbf{k}$ the left lower corner of Q and by x_Q is any point in Q when $Q = Q_{j\mathbf{k}}$. For any function ψ defined on $\mathbb{R}^n$, $j \in \mathbb{Z}$, and $Q = Q_{j\mathbf{k}}$, set

$$\psi_j(x) = 2^{jn}\psi(2^j x), \quad \psi_Q(x) = |Q|^{1/2}\psi_j(x - z_Q).$$

DEFINITION 3. Let $0 < p^- \leqslant p^+ \leqslant 1$ and $\psi_0, \psi \in \mathscr{S}(\mathbb{R}^n)$ with

$$\mathrm{supp}\widehat{\psi_0} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\};\ \widehat{\psi_0}(\xi) = 1,\ \text{if}\ |\xi| \leqslant 1; \tag{2}$$

$$\mathrm{supp}\widehat{\psi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leqslant |\xi| \leqslant 2\}; \tag{3}$$

and

$$|\widehat{\psi_0}(\xi)|^2 + \sum_{j=1}^{\infty} |\widehat{\psi}(2^{-j}\xi)|^2 = 1,\ \text{ for all}\ \xi \in \mathbb{R}^n. \tag{4}$$

The local variable Carleson measure space $cmo^{p(\cdot)}(\mathbb{R}^n)$ is the collection of all $f \in \mathscr{S}'$ fulfilling

$$\|f\|_{cmo^{p(\cdot)}} := \sup_{P\in\mathscr{D}} \left\{ \frac{|P|}{\|\chi_P\|^2_{p(\cdot)}} \int_P \sum_{j\in\mathbb{N}} \sum_{Q\in\mathscr{D}_j,\, Q\subset P} |Q|^{-1} |\langle f, \psi_Q\rangle|^2 \chi_Q(x)dx \right\}^{1/2} < \infty.$$

The main goal of this paper is to prove the following results:

THEOREM 1. *Let* $\alpha \in [0,n)$, $0 \leqslant \delta < 1$ *and* $p(\cdot) \in LH \cap \mathscr{P}^0$ *and* $p(\cdot) \in \mathscr{P}^0$ *be Lebesgue measure functions satisfying*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \qquad x \in \mathbb{R}^n. \tag{5}$$

Then T_σ *with* $\sigma \in S^{-\alpha}_{1,\delta}$ *maps continuously* $h^{p(\cdot)}$ *to* $h^{q(\cdot)}$.

THEOREM 2. *Let* $\alpha \in [0,n)$, $0 \leqslant \delta < 1$ *and* $p(\cdot) \in LH \cap \mathscr{P}^0$ *and* $p(\cdot) \in \mathscr{P}^0$ *be Lebesgue measure functions satisfying*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \qquad x \in \mathbb{R}^n. \tag{6}$$

For $0 < q^- \leqslant q^+ \leqslant 1$, *then* T_σ *with* $\sigma \in S^{-\alpha}_{1,\delta}$ *maps continuously* $cmo^{q(\cdot)}$ *to* $cmo^{p(\cdot)}$. *For* $p^+ \leqslant 1 < q^- \leqslant q^+ < \infty$, *then* T_σ *with* $\sigma \in S^{-\alpha}_{1,\delta}$ *maps continuously* $L^{q(\cdot)'}$ *to* $cmo^{p(\cdot)}$.

REMARK 1. It follows from the (1) that $S^{-\alpha_1}_{1,\delta} \subset S^{-\alpha}_{1,\delta}$ when $\alpha_1 \geqslant \alpha$. Thus, it is obviously that under the above assumptions any T_σ with $S^{-\alpha_1}_{1,\delta}(\alpha_1 \geqslant \alpha)$ is also $(h^{p(\cdot)}, h^{q(\cdot)})$-bounded, $(cmo^{q(\cdot)}, cmo^{p(\cdot)})$-bounded or $(L^{q(\cdot)'}, cmo^{p(\cdot)})$-bounded. Moreover, the result is optimal. To see this, even $p(\cdot)$ and $q(\cdot)$ are constant functions, we know that from [17, Proposition 1.1] if $\alpha_1 < \alpha$ there exists T_σ with $\sigma \in S^{-\alpha}_{1,\sigma}$ which is not continuous from h^p to h^q.

Throughout this paper, C or c will denote a positive constant that may vary at each occurrence but is independent to the essential variables, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the essential variables such that $C_1 B \leqslant A \leqslant C_2 B$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function. For a cube Q, let Q^* denote with the same center and $2\sqrt{n}$ its side length. The symbols $\mathscr{S}$ and $\mathscr{S}'$ denote the class of Schwartz functions and tempered functions, respectively. As usual, for a function ψ on $\mathbb{R}^n$ and $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. We also use the notations $j \wedge j' = \min\{j, j'\}$ and $j \vee j' = \max\{j, j'\}$. We write $\mathbb{N} = \{0, 1, 2, \cdots\}$.

2. Proof of Theorem 1

In this section, we will show that the continuity of pseudo-differential operators in the class $OpS^{-\alpha}_{1,\delta}$ for $0 \leqslant \delta < 1$ on local Hardy spaces with variable exponents by applying the atomic decomposition theory. Atomic decomposition is a significant tool in harmonic analysis and wavelet analysis for the study of function spaces and the operators acting on these spaces. In 1979, Goldberg ([11]) introduced the atomic decomposition of local Hardy spaces. The atomic decomposition of variable Hardy spaces was established independently in [5, 22]. By using local grand maximal characterization we recall the new atomic decompositions for local Hardy spaces with variable exponents $h^{p(\cdot)}$ in [34]. In what follows, we recall the definitions of local $(p(\cdot),q)$-atom and $(p(\cdot),q)$-block for $h^{p(\cdot)}$.

DEFINITION 4. Let $p(\cdot):\mathbb{R}^n\to(0,\infty)$, $p(\cdot)\in\mathscr{P}^0$ and $1<q\leqslant\infty$. Fix an integer $d\geqslant d_{p(\cdot)}\equiv\min\{d\in\mathbb{N}: p^-(n+d+1)>n\}$. Define a local $(p(\cdot),q)$-atom of $h^{p(\cdot)}$ to be a function a of compact support which has the additional properties that $\|a\|_{L^q}\leqslant\frac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$ and $\int_{\mathbb{R}^n}a(x)x^\alpha dx=0$ for all $|\alpha|\leqslant d$ and $|Q|\leqslant 1$, where Q is the smallest cube containing the support of a.

DEFINITION 5. Let $p(\cdot):\mathbb{R}^n\to(0,\infty)$, $p(\cdot)\in\mathscr{P}^0$ and $1<q\leqslant\infty$. Define a $(p(\cdot),q)$-block of $h^{p(\cdot)}$ to be a function b of compact support which has the additional properties that $\|b\|_{L^q}\leqslant\frac{|P|^{1/q}}{\|\chi_P\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$ and $|P|>1$, where P is the smallest cube containing the support of b.

For convenience, the set of all such pairs (a,Q) will be denoted by $\mathscr{A}(p(\cdot),q)$ and the set of all such pairs (b,P) will be denoted by $\mathscr{B}(p(\cdot),q)$.

For sequences of scalars $\{\lambda_j\}$ and cubes $\{Q_j\}$, define that

$$\mathscr{A}_s(\{\lambda_j\}_{j=1}^\infty,\{Q_j\}_{j=1}^\infty)=\left\|\left\{\sum_j\left(\frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}}\right)^s\right\}^{\frac{1}{s}}\right\|_{L^{p(\cdot)}},$$

and for sequences of scalars $\{\kappa_j\}$ and cubes $\{P_j\}$,

$$\mathscr{B}_s(\{\kappa_j\}_{j=1}^\infty,\{P_j\}_{j=1}^\infty)=\left\|\left\{\sum_j\left(\frac{|\kappa_j|\chi_{P_j}}{\|\chi_{P_j}\|_{L^{p(\cdot)}}}\right)^s\right\}^{\frac{1}{s}}\right\|_{L^{p(\cdot)}}.$$

When $s=p^-$, we denote

$$\mathscr{A}_{p^-}(\{\lambda_j\}_{j=1}^\infty,\{Q_j\}_{j=1}^\infty)=\mathscr{A}(\{\lambda_j\}_{j=1}^\infty,\{Q_j\}_{j=1}^\infty)$$

and

$$\mathscr{B}_{p^-}(\{\kappa_j\}_{j=1}^\infty,\{P_j\}_{j=1}^\infty)=\mathscr{B}(\{\kappa_j\}_{j=1}^\infty,\{P_j\}_{j=1}^\infty).$$

Now we recall the definition of atomic local Hardy space with variable exponent $h^{p(\cdot),q}_{atom}$.

DEFINITION 6. Let $1<q\leqslant\infty$ and $p(\cdot)\in\mathscr{P}^0\cap LH$. The function space $h^{p(\cdot),q}_{atom}$ is defined to be the set of all distributions $f\in\mathscr{S}'$ which can be written as $f=\sum_j\lambda_j a_j+\sum_j\kappa_j b_j$ in $\mathscr{S}'$, where $\{a_j,Q_j\}\subset\mathscr{A}(p(\cdot),q)$ and $\{b_j,P_j\}\subset\mathscr{B}(p(\cdot),q)$ with the quantities

$$\mathscr{A}(\{\lambda_j\}_{j=1}^\infty,\{Q_j\}_{j=1}^\infty)+\mathscr{B}(\{\kappa_j\}_{j=1}^\infty,\{P_j\}_{j=1}^\infty)<\infty.$$

One define

$$\|f\|_{h^{p(\cdot),q}_{atom}} \equiv \mathscr{A}(\{\lambda_j\}_{j=1}^{\infty},\{Q_j\}_{j=1}^{\infty}) + \mathscr{B}(\{\kappa_j\}_{j=1}^{\infty},\{P_j\}_{j=1}^{\infty}).$$

The atomic decomposition for local variable Hardy spaces was established in [31, 34].

PROPOSITION 1. *Let* $1 < q \leqslant \infty$ *and* $p(\cdot) \in \mathscr{P}^0 \cap LH$. *Then*

$$h^{p(\cdot)} = h^{p(\cdot),\infty}_{atom}.$$

Especially, if $f \in h^{p(\cdot)}$ *and* $0 < s < \infty$, *there are* $\{a_j, Q_j\} \subset \mathscr{A}(p(\cdot,q))$ *and* $\{b_j, P_j\} \subset \mathscr{B}(p(\cdot,q))$ *with*

$$\mathscr{A}_s(\{\lambda_j\}_{j=1}^{\infty},\{Q_j\}_{j=1}^{\infty}) + \mathscr{B}_s(\{\kappa_j\}_{j=1}^{\infty},\{P_j\}_{j=1}^{\infty}) \leqslant C\|f\|_{h^{p(\cdot)}},$$

such that $f = \sum_j \lambda_j a_j + \sum_j \kappa_j b_j$, *where the series converges to* f *in both* $h^{p(\cdot)}$ *and* L^q *norms.*

To prove Theorem 1, we also need the Fefferman–Stein vector-valued fractional maximal inequality.

PROPOSITION 2. [3] *Given* $0 \leqslant \alpha < n$, $1 < r < \infty$ *and* $p(\cdot) \in LH \cap \mathscr{P}^0$ *with* $1 < p^- \leqslant p^+ < \frac{n}{\alpha}$. *Define* $q(x)$ *by* $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$, *for any* $x \in \mathbb{R}^n$. *Then for* $f = \{f_i\}_{i\in\mathbb{Z}}$ *and* $\mathbb{M}_\alpha(f) = \{M_\alpha(f_i)\}_{i\in\mathbb{Z}}$,

$$\|\|\mathbb{M}_\alpha f)\|_{l^q}\|_{L^{q(\cdot)}} \leqslant C\|\|f\|_{l^q}\|_{L^{p(\cdot)}}.$$

The following result provides the (L^p, L^q)-boundedness of pseudo-differential operators with $\sigma \in S^{-\alpha}_{1,\delta}$.

PROPOSITION 3. [1] *Let* T_σ *is a pseudo-differential operator with* $\sigma \in S^{-\alpha}_{1,\delta}$ *and* $0 \leqslant \delta < 1$. *Then* T_σ *maps continuously* L^p *into* L^q *for* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $1 < p \leqslant q < \infty$.

The following propositions also play a key role in the proof of the main result.

PROPOSITION 4. [4, 34] *Let* $p(\cdot) \in LH \cap \mathscr{P}^0$. *Suppose that we are given a sequence of cubes* $\{Q_j\}_{j=1}^{\infty}$ *and a sequence of non-negative functions* $\{F_j\}_{j=1}^{\infty}$. *Then for any* q *such that* $p^+ < q < \infty$ *we have*

$$\left\|\sum_{j=1}^{\infty} \chi_{Q_j} F_j\right\|_{L^{p(\cdot)}} \leqslant C \left\|\sum_{j=1}^{\infty} \left(\frac{1}{|Q_j|}\int_{Q_j} F_j^q(y)dy\right)^{\frac{1}{q}} \chi_{Q_j}\right\|_{L^{p(\cdot)}}.$$

PROPOSITION 5. [26] *Given* $0 \leqslant \alpha < n$, *suppose* $(p\cdot) \in LH \cap \mathscr{P}^0$. *Define* $q(\cdot)$ *by* $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n}$. *Then for any countable collection of cubes* Q_k *and* $\lambda_k > 0$,

$$\left\|\sum_j \lambda_j |Q_j|^{\frac{\alpha}{n}} \chi_{Q_j}\right\|_{L^{q(\cdot)}} \leqslant \left\|\sum_j \lambda_j \chi_{Q_j}\right\|_{L^{p(\cdot)}}.$$

We are now ready to show Theorem 1.

Proof of Theorem 1. By applying the atomic decomposition of local Hardy space $h^{p(\cdot)}$ in Proposition 1, for each $f \in h^{p(\cdot)}$, $0 < s < \infty$, f admits an atomic decomposition: There exists a sequence of nonnegative numbers η_j, κ_j, cubes Q_j satisfying

$$\mathscr{A}_s(\{\eta_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) + \mathscr{B}_s(\{\kappa_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) \leqslant C\|f\|_{h^{p(\cdot)}},$$

for $\{a_j, Q_j\} \subset \mathscr{A}(p(\cdot,q))$ and $\{b_j, Q_j\} \subset \mathscr{B}(p(\cdot,q))$, and f can be decomposed as

$$f = \sum_{j\in\mathbb{N}} \eta_j a_j + \sum_{j\in\mathbb{N}} \kappa_j b_j =: \sum_{j\in\mathbb{N}} \lambda_j c_j \quad \text{in} \quad h^{p(\cdot)} \cap L^{p^+ + 1},$$

where $\lambda_j = \eta_j$ and $c_j = a_j$ for $|Q_j| \leqslant 1$, and $\lambda_j = \kappa_j$ and $c_j = b_j$ for $|Q_j| > 1$.

Since T_σ maps continuously L^{p_0} into L^{q_0}, where $1 < p_0 \leqslant q_0 \leqslant 2$ and $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$. Then we can obtain

$$|\mathscr{M}_{loc} T_\sigma(f)(x)| \leqslant \sum_j |\lambda_j| |\mathscr{M}_{loc} T_\sigma(c_j)(x)|. \tag{7}$$

For $x \in \mathbb{R}^n$, we can split (7) into two terms, that is,

$$\begin{aligned}|\mathscr{M}_{loc} T_\sigma(f)(x)| &\leqslant \sum_j |\lambda_j| \mathscr{M}_{loc} T_\sigma(c_j)(x)|\chi_{Q_j^*} + \sum_j |\lambda_j| |\mathscr{M}_{loc} T_\sigma(c_j)(x)|\chi_{Q_j^{*,c}}(x) \\ &=: I + II.\end{aligned}$$

First we will show that

$$\|I\|_{L^{q(\cdot)}} \leqslant C\|f\|_{h^{p(\cdot)}}. \tag{8}$$

Now fix atoms c_j supported in cubes Q_j. By Proposition 3, we get that T_σ maps continuously L^{p_0} into L^{q_0}, where $(\frac{n}{n-\alpha} \vee q^+) \leqslant q_0 < \infty$ and $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$. Meanwhile, $\mathscr{M}_{loc}$ is bounded on L^s for all $1 < s \leqslant \infty$. Then we have

$$\begin{aligned}\left(\frac{1}{|Q_j|}\int_{Q_j} |\mathscr{M}_{loc} T_\sigma(c_j)(x)|^{q_0} dx\right)^{1/q_0} &\leqslant \frac{1}{|Q_j|^{1/q_0}} \|\mathscr{M}_{loc} T_\sigma(c_j)\|_{L^{q_0}} \\ \leqslant C\frac{1}{|Q_j|^{1/q_0}} \|c_j\|_{L^{p_0}} \leqslant C|Q_j|^{\frac{\alpha}{n}} \frac{1}{\|\chi_{Q_j}\|_{p(\cdot)}} \\ \leqslant C\frac{1}{|Q_j|^{1/q_0}} \|c_j\|_{L^{p_0}} \leqslant C|Q_j^*|^{\frac{\alpha}{n}} \frac{1}{\|\chi_{Q_j}\|_{p(\cdot)}}.\end{aligned} \tag{9}$$

By (9), Proposition 4 and Proposition 5, we get that

$$
\begin{aligned}
\|I\|_{L^{q(\cdot)}} &\leqslant \Big\| \sum_j |\lambda_j| |\mathscr{M}_{loc} T_\sigma(c_j)| \chi_{Q_j^*} \Big\|_{L^{q(\cdot)}} \\
&\leqslant C \Big\| \sum_j |\lambda_j| \left(\frac{1}{|Q_j|} \int_{Q_j} |\mathscr{M}_{loc} T_\sigma(c_j)(x)|^{q_0} dx \right)^{1/q_0} \chi_{Q_j^*} \Big\|_{L^{q(\cdot)}} \\
&\leqslant C \Big\| \sum_j |\lambda_j| |Q_j^*|^{\frac{\alpha}{n}} \frac{1}{\|\chi_{Q_j}\|_{p(\cdot)}} \chi_{Q_j^*} \Big\|_{L^{q(\cdot)}} \\
&\leqslant C \Big\| \sum_j \frac{|\lambda_j|}{\|\chi_{Q_j}\|_{p(\cdot)}} \chi_{Q_j^*} \Big\|_{L^{p(\cdot)}}.
\end{aligned}
$$

We just need to consider the case $p^- \leqslant 1$. The other case $p^- > 1$ is obvious due to $h^{p(\cdot)} \sim L^{p(\cdot)}$ $(p^- > 1)$. Applying Proposition 2 yields that

$$
\begin{aligned}
&\Big\| \sum_j |\lambda_j| \|\chi_{Q_j}\|_{L^{p(\cdot)}}^{-1} \chi_{Q_j^*} \Big\|_{L^{p(\cdot)}} \\
&\leqslant C \Big\| \Big(\sum_j |\lambda_j|^{p^-} \|\chi_{Q_j}\|_{L^{p(\cdot)}}^{-p^-} M \chi_{Q_j} \Big)^{1/p^-} \Big\|_{L^{p(\cdot)}} \\
&\leqslant C \Big\| \Big(\sum_j \Big(|\lambda_j| \frac{\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \Big)^{p^-} \Big)^{\frac{1}{p^-}} \Big\|_{L^{p(\cdot)}} \\
&\leqslant C \|f\|_{h^{p(\cdot)}}.
\end{aligned}
$$

Now we estimate the term II. We divide in two case. When $|Q_j| \leqslant 1$, $c_j = a_j$ is an $(p(\cdot), q)$-atom. Then c_j has zero vanishing moment up to the order d. We denote T_σ^ε the composition operator $a \to \varphi_\varepsilon * T_\sigma(c_j)$ with the kernel K_ε for some $\varphi \in \mathscr{S}$. By Remark 3.1 in [17], if $M \in \mathbb{N}$ and $M - \alpha + n > 0$ then for the multi-index α, β the kernel K_ε satisfies

$$
\sup_{|\alpha|+|\beta|=M} |\partial_x^\alpha \partial_y^\beta K_\varepsilon(x,y)| \leqslant C \frac{1}{|x-y|^{M-\alpha+n}}, \quad x \neq y. \tag{10}
$$

Furthermore, there exists $L_0 \in \mathbb{Z}_+$

$$
\sup_{|x-y|\geqslant 1/2} |x-y|^L |\partial_x^\alpha \partial_y^\beta K_\varepsilon(x,y)| \leqslant C \tag{11}
$$

for each $L > L_0$. By using the Taylor expansion we get

$$\begin{aligned}
T_\sigma^\varepsilon(c_j)(x) &= \int_{Q_j} K_\varepsilon(x,y)c_j(y)dy \\
&= \int_{Q_j} [K_\varepsilon(x,y) - P_{z_j}^d(x,y)]c_j(y)dy \\
&= \int_{Q_j} \sum_{|\gamma|=d+1} (\partial_y^\gamma K_\varepsilon)(x,\xi)\frac{(y-z_j)^\gamma}{\gamma!}c_j(y)dy
\end{aligned}$$

for some ξ on the line segment joining y to z_j, where $P_{z_j}^d(x,y)$ is the Taylor polynomial of $K(x,y)$. Since $x \in (Q_j^*)^c$, we get that $|x-\xi| \geqslant \frac{1}{2}|x-z_j|$ and $|y-z_j| \leqslant \ell(Q)$.

Applying the estimate for the kernel K_ε in (10) and the size condition for the $(p(\cdot),q)$-atoms, we have

$$\begin{aligned}
&\int_{Q_j} \sum_{|\gamma|=d+1} |(\partial_y^\gamma K_\varepsilon)(x,\xi)|\frac{|y-z_j|^\gamma}{\gamma!}|c_j(y)|dy \\
&\leqslant C\int_{Q_j} \frac{|y-z_j|^{d+1}}{(|x-\xi|)^{n+d+1-\alpha}} \\
&\leqslant C\|\chi_{Q_j}\|_{L^{p(\cdot)}}^{-1} \frac{|Q_j|^{(d+1)/n+1}}{|x-z_j|^{n+d+1-\alpha}} \\
&\leqslant C\frac{|Q_j|^{1+\frac{d+1}{n}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}(|x-z_j|)^{n+d+1-\alpha}} \\
&\leqslant C\frac{|Q_j|^{1+\frac{d+1}{n}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}(|x-z_j|+\ell(Q_j))^{n+d+1-\alpha}}
\end{aligned}$$

uniformly in ε and $x \in Q_j^{*,c}$.

When $|Q_j| \geqslant 1$, $c_j = b_j$ is an $(p(\cdot),q)$-block. In this case, we have $|x-y| \sim |x-z_j|$ and $|x-y| \geqslant 1/2$ where $x \in Q_j^{*,c}$ and $y \in Q_j$. By applying the size condition of K_ε in (11), for sufficiently large $L > L_0$ and $x \in Q_j^{*,c}$ we obtain that

$$\begin{aligned}
|T_\sigma^\varepsilon(c_j)(x)| &= \left|\int_{Q_j} K_\varepsilon(x,y)c_j(y)dy\right| \\
&\leqslant \int_{Q_j} |K_\varepsilon(x,y)||c_j(y)|dy \\
&\leqslant C\frac{|Q_j|}{\|\chi_{Q_j}\|_{p(\cdot)}|x-z_j|^L} \\
&\leqslant C\frac{|Q_j|^{1+\frac{d+1}{n}}}{\|\chi_{Q_j}\|_{p(\cdot)}(|x-z_j|+\ell(Q_j))^L}.
\end{aligned}$$

For convenience, we can choose $L = n + d + 1 - \alpha$. Then we obtain

$$\|II\|_{L^{q(\cdot)}} \leqslant C \left\| \sum_j |\lambda_j| \frac{|Q_j|^{1+\frac{d+1}{n}} \chi_{Q_j^{*,c}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}} (|x - z_j| + \ell(Q_j))^{n+d+1-\alpha}} \right\|_{L^{q(\cdot)}}.$$

Denote $\theta = \frac{n+d+1}{n}$ and choose d such that $\theta q^- > 1$. Thus,

$$\begin{aligned} \|II\|_{L^{q(\cdot)}} &\leqslant C \left\| \sum_j |\lambda_j| \frac{(M_\alpha \chi_{Q_j})^\theta}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right\|_{L^{q(\cdot)}} \\ &\leqslant C \left\| \left(\sum_j |\lambda_j| \frac{\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{\frac{1}{\theta}} \right\|_{L^{\theta p(\cdot)}}^{\theta} \leqslant C \|f\|_{h^{p(\cdot)}}. \end{aligned}$$

Therefore, we have completed the proof of Theorem 1. □

3. Proof of Theorem 2

In this section, we will discuss the boundedness of the pseudo-differential operators with symbols in $S^{-\alpha}_{1,\delta}$ on the duals of local variable Hardy spaces. Namely, we will show that the pseudo-differential operator maps continuously one local variable Carleson measure spaces into another one and maps continuously one variable Lebesgue spaces into one local variable Carleson measure spaces. To prove it, first we see that the local variable Carleson measure space $cmo^{p(\cdot)}$ is the dual space of the local variable Hardy space $h^{p(\cdot)}$. See [12, 13, 14, 15, 21] for more details on some classical constant Carleson measure spaces.

PROPOSITION 6. *Suppose that* $p(\cdot) \in LH$, $0 < p^- \leqslant p^+ \leqslant 1$. *The dual space of* $h^{p(\cdot)}$ *is* $cmo^{p(\cdot)}$ *in the following sense.*

(1) For $g \in cmo^{p(\cdot)}$*, the linear functional* l_g*, defined initially on* $\mathscr{S}$*, extends to a continuous linear functional on* $h^{p(\cdot)}$ *with* $\|l_g\| \leqslant C\|g\|_{cmo^{p(\cdot)}}$.

(2) Conversely, every continuous linear functional l *on* $h^{p(\cdot)}$ *satisfies* $l = l_g$ *for some* $g \in cmo^{p(\cdot)}$ *with* $\|g\|_{cmo^{p(\cdot)}} \leqslant C\|l\|$.

The proof of Proposition 6 follows a standard procedure. For the homogeneous case, see [32]. We only point out the difference when we consider the inhomogeneous analogues. The main difference is that instead of using all cubes $Q \in \mathbb{R}^n$, we only use the cubes Q with $\ell(Q) \leqslant 1$, and the functions ψ_0 in (2) corresponding to the cubes Q with $\ell(Q) = 1$ are slightly different. Then applying similar argument as the proof of [32, Theorem 3.1] with using the inhomogeneous Calderón identity and inhomogeneous sequence spaces introduced in [10, Section 12], we can obtain the desired result.

Secondly, the following proposition on the weak density property for $cmo^{p(\cdot)}$ plays an important role in the proof of the main results.

PROPOSITION 7. *Let $p(\cdot)\in LH$ and $0<p^-\leqslant p^+\leqslant 1$. $cmo^{p(\cdot)}\cap L^2$ is dense in $cmo^{p(\cdot)}$ in the sense of weak topology. More precisely, for any $f\in cmo^{p(\cdot)}$, there exist a sequence $\{f_m\}\in cmo^{p(\cdot)}\cap L^2$ such that*

$$\lim_{m\to\infty}\langle f_m,g\rangle=\langle f,g\rangle,\quad \textit{for}\quad g\in\mathscr{S}$$

and

$$\|f_m\|_{cmo^{p(\cdot)}}\leqslant C\|f\|_{cmo^{p(\cdot)}},\quad \textit{for}\quad f\in cmo^{p(\cdot)}.$$

Proof. Suppose that $f\in cmo^{p(\cdot)}$. Then by the inhomogeneous Calderón identity [31, Theorem 1.1],

$$f(x)=\sum_{j\in\mathbb{N}}\sum_{Q\in\mathscr{D}}|Q|(\psi_j*f)(x_Q)\psi_j(x-x_Q), \tag{12}$$

where the series converges in L^2, $\mathscr{S}$ and $\mathscr{S}'$.

The partial sum of the identity will be denoted by f_m and it is given by

$$f_m(x)=\sum_{0\leqslant j\leqslant m}\sum_{Q\in\mathscr{D}}|Q|(\psi_j*f)(x_Q)\psi_j(x-x_Q).$$

First we see that $f_m\in L^2$. In fact, we only need to observe that for any fixed j and any given integer $M>0$ we have

$$\Big|\sum_{Q}|Q|(\psi_j*f)(x_Q)(\psi_j)(x-x_Q)\Big|\leqslant C2^{-j}(1+|x|)^{-M}.$$

Next we need to prove that $f_m\in cmo^{p(\cdot)}$. That is, we need to prove that for any $P\in\mathscr{D}$,

$$\left\{\frac{|P|}{\|\chi_P\|_{p(\cdot)}^2}\int_P\sum_{j'\in\mathbb{N}}\sum_{Q'\in\mathscr{D}_{j'},\,Q'\subset P}|Q'|^{-1}|\langle f_m,\psi_{Q'}\rangle|^2\chi_{Q'}(x)dx\right\}^{1/2}\leqslant C\|f\|_{cmo^{p(\cdot)}}.$$

Now we recall the classical almost orthogonality estimates. For any given positive integers L_1 and L_2, we have

$$|\psi_j*\psi_{j'}(x)|\leqslant C\frac{2^{-|j-j'|L_1}\,2^{(j\wedge j')n}}{(1+2^{(j\wedge j')}|x|)^{L_2}}.$$

Then by almost orthogonality estimates and repeating the similar argument in the proof of [32, Theorem 2.7] (also see [7, Lemma 3.2]), we can obtain that

$$\|f_m\|_{cmo^{p(\cdot)}}\leqslant C\|f\|_{cmo^{p(\cdot)}}.$$

Thus, by duality for any $g \in \mathscr{S}$, we have

$$\begin{aligned}\left\langle f - f_m, g \right\rangle &= \left\langle \sum_{j>m} \sum_{Q \in \mathscr{D}_j} |Q|(\psi_j * f)(x_Q)\psi_j(x - x_Q), g \right\rangle \\ &= \left\langle f, \sum_{j>m} \sum_{Q \in \mathscr{D}_j} |Q|(\psi_j * g)(x_Q)\psi_j(x - x_Q) \right\rangle \\ &\leqslant C\|f\|_{cmo^{p(\cdot)}} \left\| \sum_{j>m} \sum_{Q \in \mathscr{D}_j} |Q|(\psi_j * g)(x_Q)\psi_j(x - x_Q) \right\|_{h^{p(\cdot)}} \\ &\leqslant C\|f\|_{cmo^{p(\cdot)}} \|g - g_m\|_{h^{p(\cdot)}},\end{aligned}$$

which implies that $\langle f - f_m, g \rangle$ tends to 0 as $m \to \infty$ for $g \in \mathscr{S}$. Then by the fact that $\mathscr{S}$ is dense in $h^{p(\cdot)}$. Thus, f_m converges to f in the sense of weak topology. We complete the proof of the Proposition 7. $\square$

We are now turning to the proof of Theorem 2.

Proof of Theorem 2. Given $f \in cmo^{q(\cdot)}$, by applying Proposition 7, there exists a sequence $\{f_m\} \subset cmo^{q(\cdot)} \cap L^2$ with

$$\|f_m\|_{cmo^{q(\cdot)}} \leqslant C\|f\|_{cmo^{q(\cdot)}}$$

such that f_m converges to f in the weak sense. Since $S^{-\alpha}_{1,\delta} \subset S^0_{1,\delta}$, we obtain that T_σ with $\sigma \in S^{-\alpha}_{1,\delta}$ is also a bounded operator from L^2 into L^2. We claim that $\langle T_\sigma f_m, g \rangle$ is convergent as m tends to infinity. To prove it, we have $\left\langle T_\sigma(f_i - f_j), g \right\rangle = \left\langle f_i - f_j, T^*_\sigma(g) \right\rangle$, where $f_i - f_j$ and g belong to $L^2(\mathbb{R}^n)$ and where T^*_σ is the disjoint of T. Note that $T^*_\sigma = T_{\sigma^*}$ satisfies the same conditions of T_σ. For more information, we refer to [27, Theorem 5.13], [19, Theorem 4.1] and [28, p. 259]. Hence, applying Theorem 1 yields that $T^*_\sigma g \in h^{q(\cdot)} \cap L^2$ and that

$$\left\langle T_\sigma(f_i - f_j), g \right\rangle = \left\langle f_i - f_j, T^*_\sigma(g) \right\rangle \to 0$$

as $i,\ j$ tend to infinity. Thus, for $f \in cmo^{q(\cdot)}$ and $g \in h^{p(\cdot)} \cap L^2$, we can define

$$\langle T_\sigma f, g \rangle = \lim_{m \to \infty} \langle T_\sigma f_m, g \rangle, \qquad f_m \in h^{q(\cdot)} \cap L^2,$$

which implies that $T_\sigma f$ is well defined on $cmo^{p(\cdot)}$ and

$$\langle T_\sigma f, g \rangle = \lim_{m \to \infty} \langle T_\sigma f_m, g \rangle$$

for any $g \in h^{p(\cdot)} \cap L^2$ and $f_m \in cmo^{q(\cdot)} \cap L^2$. Now we show that for $f \in cmo^{q(\cdot)} \cap L^2$, T_σ is a bounded operator from $cmo^{q(\cdot)}$ to $cmo^{p(\cdot)}$, when $0 < q^- \leqslant q^+ \leqslant 1$. The adjoint operator T^*_σ is defined by

$$\langle T^*_\sigma f, g \rangle = \langle f, T_\sigma g \rangle, \quad f, g \in \mathscr{S}.$$

From Theorem 1, we know that T_σ^* is also a bounded operator from $h^{p(\cdot)}$ to $h^{q(\cdot)}$. Then we get that

$$|\langle T_\sigma f, g\rangle| = |\langle f, T_\sigma^* g\rangle| \leqslant \|f\|_{cmo^{q(\cdot)}} \|T_\sigma^* g\|_{h^{q(\cdot)}} \leqslant C\|f\|_{cmo^{q(\cdot)}} \|g\|_{h^{p(\cdot)}}.$$

Namely, for each $f \in cmo^{q(\cdot)} \cap L^2$, $l_f(g) = \langle T_\sigma f, g\rangle$ defines a continuous linear functional on $h^{p(\cdot)} \cap L^2$. By the fact that $h^{p(\cdot)} \cap L^2$ is dense in $h^{p(\cdot)}$ (see [31, Corollary 1.1]), l_f can be extended to a continuous linear functional on $h^{p(\cdot)}$ with $\|L_f\| \leqslant C\|f\|_{cmo^{q(\cdot)}}$. On the other hand, by Proposition 6, there exists $h \in cmo^{p(\cdot)}$ such that $\langle T_\sigma f, g\rangle = \langle h, g\rangle$ for $g \in h_{p(\cdot)} \cap L^2$ and $\|h\|_{cmo^{p(\cdot)}} \leqslant C\|L_f\|$. Then for all $f \in cmo^{p(\cdot)} \cap L^2$

$$\begin{aligned}
\|T_\sigma f\|_{cmo^{p(\cdot)}} &= \sup_{P\in\mathscr{D}} \left\{ \frac{|P|}{\|\chi_P\|_{p(\cdot)}^2} \int_P \sum_{j\in\mathbb{N}} \sum_{Q\in\mathscr{D}_j,\, Q\subset P} |Q|^{-1} |\langle T_\sigma f, \psi_Q\rangle|^2 \chi_Q(x)dx \right\}^{1/2} \\
&= \sup_{P\in\mathscr{D}} \left\{ \frac{|P|}{\|\chi_P\|_{p(\cdot)}^2} \int_P \sum_{j\in\mathbb{N}} \sum_{Q\in\mathscr{D}_j,\, Q\subset P} |Q|^{-1} |\langle h, \psi_Q\rangle|^2 \chi_Q(x)dx \right\}^{1/2} \\
&= \|h\|_{cmo^{p(\cdot)}} \leqslant C\|l_f\| \leqslant C\|f\|_{cmo^{q(\cdot)}}.
\end{aligned}$$

Moreover, by Fatou's lemma, for each dyadic cube $P \in \mathscr{D}$,

$$\begin{aligned}
&\left\{ \frac{|P|}{\|\chi_P\|_{p(\cdot)}^2} \int_P \sum_{j\in\mathbb{N}} \sum_{Q\in\mathscr{D}_j,\, Q\subset P} |Q|^{-1} |\langle T_\sigma f, \psi_Q\rangle|^2 \chi_Q(x)dx \right\}^{1/2} \\
&\leqslant \liminf_{m\to\infty} \left\{ \frac{|P|}{\|\chi_P\|_{p(\cdot)}^2} \int_P \sum_{j\in\mathbb{N}} \sum_{Q\in\mathscr{D}_j,\, Q\subset P} |Q|^{-1} |\langle T_\sigma f_m, \psi_Q\rangle|^2 \chi_Q(x)dx \right\}^{1/2}.
\end{aligned}$$

Therefore, for any $f \in cmo^{q(\cdot)}$ we get that

$$\|T_\sigma f\|_{cmo^{p(\cdot)}} \leqslant \liminf_{m\to\infty} \|T_\sigma f_m\|_{cmo^{p(\cdot)}} \leqslant C\|f_m\|_{cmo^{q(\cdot)}} \leqslant C\|f\|_{cmo^{q(\cdot)}}.$$

Thus, T_σ with $\sigma \in S_{1,\delta}^{-\alpha}$ maps continuously $cmo^{q(\cdot)}$ to $cmo^{p(\cdot)}$.

Next we prove the other part of Theorem 2. Since $q \in \mathscr{P}$, it is well-known that $h^{q(\cdot)} = L^{q(\cdot)}$, and that the dual of $L^{q(\cdot)}$ is $L^{q'(\cdot)}$. By repeating the similar argument, we obtain that T_σ can be extended to be a bounded operator from $L^{q'(\cdot)}$ to $cmo^{p(\cdot)}$. Here we only need to show the difference. First we observe that T_σ is a bounded operator from $L^{q'(\cdot)}$ to $cmo^{p(\cdot)}$ for $f \in L^{q'(\cdot)} \cap L^2$ for $1 < q^- \leqslant q^+ < \infty$. By duality, for $p^+ \leqslant 1 < q^- \leqslant q^+ < \infty$ we have

$$|\langle T_\sigma f, g\rangle| = |\langle f, T_\sigma^* g\rangle| \leqslant \|f\|_{L^{q'(\cdot)}} \|T_\sigma^* g\|_{L^{q(\cdot)}} \leqslant C\|f\|_{L^{q'(\cdot)}} \|g\|_{h^{p(\cdot)}},$$

which means that for each $f \in L^{q'(\cdot)} \cap L^2$, $l_f(g) = \langle T_\sigma f, g\rangle$ is a continuous linear functional on $h^{p(\cdot)} \cap L^2$. Similarly, l_f can be extended to a continuous linear functional on $h_{p(\cdot)}$ with

$$\|l_f\| \leqslant C\|f\|_{L^{q'(\cdot)}}.$$

Meanwhile, there exists $h \in cmo^{p(\cdot)}$ such that $\langle T_\sigma f, g\rangle = \langle h, g\rangle$ for $g \in h_{p(\cdot)} \cap L^2$ and $\|h\|_{cmo^{p(\cdot)}} \leqslant C\|l_f\|$. Then we see that

$$\|T_\sigma f\|_{cmo^{p(\cdot)}} = \|h\|_{cmo^{p(\cdot)}} \leqslant C\|l_f\| \leqslant C\|f\|_{L^{q'(\cdot)}}.$$

Hence, T_σ with $\sigma \in S^{-\alpha}_{1,\delta}$ maps continuously $L^{q(\cdot)'}(\mathbb{R}^n)$ to $cmo^{p(\cdot)}(\mathbb{R}^n)$, for $p^+ \leqslant 1 < q^- \leqslant q^+ < \infty$. We completed the proof of Theorem 2. □

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