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JOHN–NIRENBERG INEQUALITY FOR LIPSCHITZ MARTINGALE SPACES

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Abstract. In this article, John-Nirenberg inequality for Lipschitz martingale spaces is established. We further prove that Lipschitz martingale spaces $\Lambda_p(\alpha)$ are equivalent for $0<p<\infty$, which generalizes an important result in classical martingale H_p theory.

1. Introduction

John-Nirenberg inequality plays an important role in harmonic analysis. A well-known immediate consequence of the John-Nirenberg inequality is the famous John-Nirenberg theorem, which implied the equivalence of BMO spaces (the function spaces of bounded mean oscillation) and BMO_p for $1 \leqslant p < \infty$.

In martingale setting, John-Nirenberg inequality and John-Nirenberg theorem for BMO martingale spaces are two basic and important results in classical martingale H_p theory (see Theorem 4.1.2 in [9]). John-Nirenberg theorem for BMO martingale spaces states that martingale spaces BMO_p are equivalent for $1 \leqslant p < \infty$. In the past decade, more attention has been paid to John-Nirenberg type theorems on various BMO type martingale spaces. Miyamoto, Nakai and Sadasue obtained a John-Nirenberg type inequality for generalized martingale Campanato spaces when the stochastic basis is regular ([10]). Hong and Mei considered the John-Nirenberg inequality for noncommutative martingales in [2]. John-Nirenberg inequalities on generalized BMO martingale spaces were studied by Jiao et al. in [3]. For other related work, one can refer to [4, 5, 6, 7, 8, 12, 13].

As we know, BMO space is a special case of Lipschitz space. What we are interested in is whether John-Nirenberg type inequality holds for Lipschitz martingale spaces, and naturally consider the problem of the equivalence of Lipschitz martingale spaces, that is, John-Nirenberg theorem for Lipschitz martingale spaces. In this article, John-Nirenberg inequality for Lipschitz martingale spaces is established. By using this inequality, we show that Lipschitz martingale spaces $\Lambda_p(\alpha)$ are equivalent for $0<p<\infty$, and thus the conclusion on BMO martingale spaces is extended to Lipschitz martingale spaces. In particular, we obtain that martingale spaces BMO_p are equivalent

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for $0<p<\infty$, which generalizes John-Nirenberg theorem in classical martingale H_p theory.

This paper is divided into two further sections. In Section 2 some notions and notations appeared in this article are given. Main results and its proofs are given in the final section.

2. Preliminaries

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space, and $(\mathscr{F}_n)_{n\geqslant 0}$ a nondecreasing sequence of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F}=\sigma(\bigcup_n \mathscr{F}_n)$. The conditional expectation operator relative to $\mathscr{F}_n$ is denoted by $\mathbb{E}_n$. A martingale $f=(f_n)_{n\geqslant 0}$ relative to $(\Omega,\mathscr{F},\mathbb{P};(\mathscr{F}_n)_{n\geqslant 0})$ is an integrable sequence, which is adapted and $\mathbb{E}_n f_m = f_n$ for all $n\leqslant m$. A stopping time relative to $(\mathscr{F}_n)_{n\geqslant 0}$ is a map $\tau:\Omega\to\mathbf{N}\bigcup\{\infty\}$, which satisfies $\{\omega\in\Omega:\tau(\omega)=n\}\in\mathscr{F}_n$ for each $n\geqslant 0$.

Let $0<p<\infty$, $\alpha\geqslant 0$, Lipschitz martingale spaces and BMO martingale spaces are defined as follows:

$$\Lambda_p(\alpha)=\left\{f=(f_n)_{n\geqslant 0}:\|f\|_{\Lambda_p(\alpha)}=\sup_{n\in\mathbf{N}}\sup_{A\in\mathscr{F}_n}\frac{(\int_A|f-f_{n-1}|^p\,d\mathbb{P})^{\frac{1}{p}}}{\mathbb{P}(A)^{\frac{1}{p}+\alpha}}<\infty\right\};$$

$$BMO_p=\left\{f=(f_n)_{n\geqslant 0}:\|f\|_{BMO_p}=\sup_{n\geqslant 0}\|(\mathbb{E}_n|f-f_{n-1}|^p)^{\frac{1}{p}}\|_\infty<\infty\right\}.$$

We recall that $\Lambda_p(0)=BMO_p$. For more information on the theory of Lipschitz martingale spaces and BMO martingale spaces, one can refer to [9] and [11].

Let $0<p<\infty$, we use $L_p(\mathscr{F}_n)$ ($L_p(\mathscr{F})$) to denote the collection of p-th integrable and $\mathscr{F}_n$-measurable ($\mathscr{F}$-measurable) functions on $(\Omega,\mathscr{F}_n,\mathbb{P})$ ($(\Omega,\mathscr{F},\mathbb{P})$).

Let X and Y be two quasi-normed spaces, we say that X and Y are equivalent (denoted by $X=Y$), if X and Y are isomorphic and their quasi-norms are equivalent, i.e. there exist positive constants c and C such that for all $x\in X$

$$c\,\|x\|_X\leqslant\|x\|_Y\leqslant C\,\|x\|_X\,.$$

Throughout this paper, we use $\mathbf{N}$ and $\mathbf{Z}^+$ to denote the set of nonnegative integers and the set of positive integers, respectively. We use C and c to denote positive constants and may be different from one line to another.

3. Main results and its proofs

We first establish John-Nirenberg inequality for Lipschitz martingale spaces.

THEOREM 1. *Let $f=(f_n)_{n\geqslant 0}\in\Lambda_1(\alpha)$. Then for any $n\in\mathbf{Z}^+$ and $A\in\mathscr{F}_n$, we have*

$$\mathbb{P}\Big(\Big\{\omega\in A:\frac{|f-f_{n-1}|}{\mathbb{P}(A)^\alpha}>\lambda\Big\}\Big)\leqslant\mathbb{P}(A)e^2e^{-\frac{\lambda}{e\|f\|_{\Lambda_1(\alpha)}}},\ \forall\lambda>0.$$

Proof. We first assume that $\|f\|_{\Lambda_1(\alpha)} \leqslant 1$, and for any fixed $A \in \mathscr{F}_n$, set

$$g_m = \frac{f_m - f_{n-1}}{\mathbb{P}(A)^\alpha}, \quad m \geqslant n-1, \quad g^* = \sup_{m \geqslant n-1} |\, g_m \,| .$$

For $\lambda, \mu > 0$, define stopping times

$$T = \inf\{m : |\, g_m \,| > \lambda\},\ \tau = \inf\{m : |\, g_m \,| > \lambda + \mu\}.$$

Obviously, $T > 0$ and $T \leqslant \tau$. Define

$$T_A = \begin{cases} T, & \omega \in A; \\ \infty, & \omega \notin A, \end{cases} \qquad \tau_A = \begin{cases} \tau, & \omega \in A; \\ \infty, & \omega \notin A, \end{cases}$$

it is easy to see that $T_A \leqslant \tau_A$ and $|\, g_{T_A - 1}\chi_A \,| \leqslant \lambda$, we then have

$$\begin{aligned} \sigma_A(\lambda+\mu) &:= \mathbb{P}(\{g^* > \lambda+\mu\} \bigcap A) = \mathbb{P}(\{\tau_A < \infty\}) \\ &\leqslant \mathbb{P}\Big(\Big\{T_A < \infty, |\, g_{\tau_A} - g_{T_A-1} \,| = \frac{|\, f_{\tau_A} - f_{T_A-1} \,|}{\mathbb{P}(A)^\alpha} \geqslant \mu\Big\}\Big) \\ &\leqslant \frac{1}{\mu} \int_{\{T_A<\infty\}} \frac{|\, f_{\tau_A} - f_{T_A-1} \,|}{\mathbb{P}(A)^\alpha} d\mathbb{P} \\ &\leqslant \frac{1}{\mu} \int_{\{T_A<\infty\}} \frac{|\, \mathbb{E}((f - f_{T_A-1})\chi_A \mid \mathscr{F}_{\tau_A}) \,|}{\mathbb{P}(A)^\alpha} d\mathbb{P} \\ &\leqslant \frac{1}{\mu} \mathbb{P}(\{T_A < \infty\}) \left\| \frac{\mathbb{E}((f - f_{T_A-1})\chi_A \mid \mathscr{F}_{\tau_A})}{\mathbb{P}(A)^\alpha} \right\|_\infty . \end{aligned}$$

If we set $M = \left\| \frac{\mathbb{E}((f - f_{T_A-1})\chi_A | \mathscr{F}_{\tau_A})}{\mathbb{P}(A)^\alpha} \right\|_\infty$, and for arbitrary $\varepsilon > 0$, define

$$A_1 = \Big\{\omega \in A : \frac{\mathbb{E}((f - f_{T_A-1})\chi_A \mid \mathscr{F}_{\tau_A})}{\mathbb{P}(A)^\alpha} > M - \varepsilon\Big\},$$

then we have

$$M \leqslant \left\| \frac{(f - f_{T_A-1})\chi_A \chi_{A_1}}{\mathbb{P}(A)^\alpha \mathbb{P}(A_1)} \right\|_1 \leqslant \sup_{B \in \mathscr{F}_{T_A}} \left\| \frac{(f - f_{T_A-1})\chi_B}{\mathbb{P}(B)^{1+\alpha}} \right\|_1 \leqslant \|\, f \,\|_{\Lambda_1(\alpha)} .$$

Hence

$$\sigma_A(\lambda+\mu) \leqslant \frac{1}{\mu} \mathbb{P}(\{T_A < \infty\}) \,\|\, f \,\|_{\Lambda_1(\alpha)} \leqslant \frac{1}{\mu} \sigma_A(\lambda).$$

Now take $\mu = e$ and $\lambda = ke$, we get

$$\sigma_A((k+1)e) \leqslant \frac{1}{e} \sigma_A(ke) \leqslant \mathbb{P}(A) e^{-k}, \ \ k \in \mathbf{Z}^+.$$

Since $\sigma_A(\lambda)$ is decreasing, for $\lambda \in [e, +\infty)$ with $ke \leqslant \lambda < (k+1)e$ ($k \in \mathbf{Z}^+$), we have

$$\sigma_A(\lambda) \leqslant \sigma_A(ke) \leqslant \mathbb{P}(A) e^2 e^{-\frac{\lambda}{e}}.$$

The above inequality still holds for $0<\lambda<e$, since

$$\sigma_A(\lambda)\leqslant \mathbb{P}(A)e\leqslant \mathbb{P}(A)e^2e^{-\frac{\lambda}{e}}.$$

For any $f=(f_n)_{n\geqslant 0}\in \Lambda_1(\alpha)$, if we replace f above with $\frac{f}{\|f\|_{\Lambda_1(\alpha)}}$, then the desired inequality is obtained. $\square$

Now let us turn to consider the equivalence of Lipschitz martingale spaces, for which we need to provide several lemmas first.

Let A be a measurable set, we use χ_A to denote its characteristic function.

LEMMA 1. (1) *If $0<p\leqslant 1$, then for each $a\in L_p(\mathscr{F}_n)$ with $\| a \|_p\leqslant 1$, it can be approached by $\sum_{k=1}^m \lambda_k \frac{\chi_{A_k}}{\mathbb{P}(A_k)^{\frac{1}{p}}}$ in $L_p(\mathscr{F}_n)$, where $\sum_{k=1}^m \mid \lambda_k \mid^p\leqslant 1$.*

(2) *If $1\leqslant p<\infty$, then for each $a\in L_{p,1}(\mathscr{F}_n)$ with $\| a \|_{p,1}\leqslant 1$, it can be approached by $\frac{1}{p}\sum_{k=1}^m \lambda_k \frac{\chi_{A_k}}{\mathbb{P}(A_k)^{\frac{1}{p}}}$ in $L_{p,1}(\mathscr{F}_n)$, where $\sum_{k=1}^m \mid \lambda_k \mid\leqslant 1$.*

Proof. It is easy to obtain (1) from the approximation properties of integrable functions. Now we only give a proof for (2).

Here we only consider the case of nonnegative functions. In general, we can consider the positive part and the negative part respectively according to the general practice. Since simple functions are dense in $L_{p,1}(\mathscr{F}_n)$, then for each nonnegative function $a\in L_{p,1}(\mathscr{F}_n)$ with $\| a \|_{p,1}\leqslant 1$, it can be approached by positive simple functions $\sum_{k=1}^m \mu_k\chi_{B_k}$, and $\| \sum_{k=1}^m \mu_k\chi_{B_k} \|_{p,1}\leqslant 1$. Suppose that $\{\mu_1,\cdots,\mu_m\}$ is decreasing, it can be calculated that

$$\begin{aligned}\| \sum_{k=1}^m \mu_k\chi_{B_k} \|_{p,1} &= p\big((\mu_1-\mu_2)\mathbb{P}(B_1)^{\frac{1}{p}}+(\mu_2-\mu_3)(\mathbb{P}(B_1)+\mathbb{P}(B_2))^{\frac{1}{p}}\\ &\quad +\cdots+\mu_{k_m}(\mathbb{P}(B_1)+\cdots+\mathbb{P}(B_m))^{\frac{1}{p}}\big).\end{aligned}$$

Now set

$$\begin{aligned}\lambda_1 &= p(\mu_1-\mu_2)\mathbb{P}(B_1)^{\frac{1}{p}},\\ \lambda_2 &= p(\mu_2-\mu_3)(\mathbb{P}(B_1)+\mathbb{P}(B_2))^{\frac{1}{p}},\\ &\cdots\\ \lambda_{m-1} &= p(\mu_{m-1}-\mu_m)(\mathbb{P}(B_1)+\cdots+\mathbb{P}(B_m))^{\frac{1}{p}},\\ \lambda_m &= p\mu_m(\mathbb{P}(B_1)+\cdots+\mathbb{P}(B_m))^{\frac{1}{p}}\end{aligned}$$

and

$$A_k=B_1+\cdots+B_k,$$

it is clear that $\sum_{k=1}^{m} \mid \lambda_k \mid \leqslant 1$, and

$$\begin{aligned}\sum_{k=1}^{m} \mu_i \chi_{B_k} &= (\mu_1-\mu_2)\chi_{B_1} + (\mu_2-\mu_3)\chi_{B_1+B_2} \\ &\quad + \cdots + (\mu_{m-1}-\mu_m)\chi_{B_1+\cdots+B_{m-1}} + \mu_m \chi_{B_1+\cdots+B_m} \\ &= \frac{1}{p}\sum_{k=1}^{m} \lambda_k \frac{\chi_{A_k}}{\mathbb{P}(A_k)^{\frac{1}{p}}}. \quad \square\end{aligned}$$

LEMMA 2. (1) *Let* $0<p\leqslant 1,\ \alpha \geqslant 0,\ f=(f_n)_{n\geqslant 0}\in \Lambda_p(\alpha),\ T_n : L_{\frac{p}{p\alpha+1}}(\mathscr{F}_n)\to L_p(\mathscr{F})$ *be a linear operator with* $T_n a=(f-f_{n-1})a,\ n\in \mathbf{N}$, *then* $\sup_{n\in\mathbf{N}} \parallel T_n \parallel = \parallel f \parallel_{\Lambda_p(\alpha)}$.

(2) *Let* $1\leqslant p<\infty,\ \alpha\geqslant 0,\ \frac{p}{p\alpha+1}\leqslant 1,\ f=(f_n)_{n\geqslant 0}\in \Lambda_p(\alpha),\ T_n : L_{\frac{p}{p\alpha+1}}(\mathscr{F}_n)\to L_p(\mathscr{F})$ *be a linear operator with* $T_n a=(f-f_{n-1})a,\ n\in \mathbf{N}$, *then* $\sup_{n\in\mathbf{N}} \parallel T_n \parallel = \parallel f \parallel_{\Lambda_p(\alpha)}$.

(3) *Let* $1\leqslant p<\infty,\ \alpha\geqslant 0,\ \frac{p}{p\alpha+1}\geqslant 1,\ f=(f_n)_{n\geqslant 0}\in \Lambda_p(\alpha),\ T_n : L_{\frac{p}{p\alpha+1},1}(\mathscr{F}_n)\to L_p(\mathscr{F})$ *be a linear operator with* $T_n a=(f-f_{n-1})a,\ n\in \mathbf{N}$, *then there exist positive constants* c *and* C *such that*

$$c \parallel f \parallel_{\Lambda_p(\alpha)} \leqslant \sup_{n\in\mathbf{N}} \parallel T_n \parallel \leqslant C \parallel f \parallel_{\Lambda_p(\alpha)} .$$

Proof. We only show that (1) holds, (2) and (3) can be proved in a similar way.

It is easy to obtain that $\sup_{n\in\mathbf{N}} \parallel T_n \parallel \geqslant \parallel f \parallel_{\Lambda_p(\alpha)}$, if we take $a=\frac{\chi_A}{\mathbb{P}(A)^{\frac{1}{p}+\alpha}}$, $A\in \mathscr{F}_n$.

For the converse, by Lemma 1, for each $a\in L_{\frac{p}{p\alpha+1}}(\mathscr{F}_n)$ with $\parallel a \parallel_{\frac{p}{p\alpha+1}} \leqslant 1$, it can be approached by $\sum_{k=1}^{k_m} \lambda_k^{(m)} \frac{\chi_{A_k^{(m)}}}{\mathbb{P}(A_k^{(m)})^{\alpha+\frac{1}{p}}}$ in $L_{\frac{p}{p\alpha+1}}(\mathscr{F}_n)$, where $\sum_{k=1}^{k_m} \mid \lambda_k^{(m)} \mid^{\frac{p}{p\alpha+1}} \leqslant 1$, $m\in \mathbf{Z}^+$. Hence there exists a subsequence that converges almost everywhere, denote it still by $\left\{\sum_{k=1}^{k_m} \lambda_k^{(m)} \frac{\chi_{A_k^{(m)}}}{\mathbb{P}(A_k^{(m)})^{\alpha+\frac{1}{p}}}\right\}_{m\geqslant 1}$. By Egoroff theorem and the property of absolute continuity of Lebesgue integral, we have

$$\begin{aligned}\parallel (f-f_{n-1})a \parallel_p^p &= \lim_{m\to\infty} \left\| \sum_{k=1}^{k_m} \lambda_k^{(m)} (f-f_{n-1}) \frac{\chi_{A_k^{(m)}}}{\mathbb{P}(A_k^{(m)})^{\alpha+\frac{1}{p}}} \right\|_p^p \\ &\leqslant \lim_{m\to\infty} \sum_{k=1}^{k_m} \mid \lambda_k^{(m)} \mid^p \frac{\parallel (f-f_{n-1})\chi_{A_k^{(m)}} \parallel_p^p}{\mathbb{P}(A_k^{(m)})^{p\alpha+1}} \\ &\leqslant \lim_{m\to\infty} \sum_{k=1}^{k_m} \mid \lambda_k^{(m)} \mid^{\frac{p}{p\alpha+1}} \frac{\parallel (f-f_{n-1})\chi_{A_k^{(m)}} \parallel_p^p}{\mathbb{P}(A_k^{(m)})^{p\alpha+1}} \\ &\leqslant \parallel f \parallel_{\Lambda_p(\alpha)}^p,\end{aligned}$$

from which we obtain

$$\sup_{n\in\mathbf{N}} \| T_n \| \leqslant \| f \|_{\Lambda_p(\alpha)} . \quad \square$$

LEMMA 3. ([1]) *Let* $0 < p \leqslant \infty$ *and* $0 < \theta < 1$, *if* (A_0, A_1) *and* (B_0, B_1) *are two couples of quasi-normed spaces, and if* T *is a linear operator*

$$T : A_0 \to B_0,\ T : A_1 \to B_1$$

with the quasi-norms M_0 *and* M_1 *respectively, then*

$$T : (A_0, A_1)_{\theta,p} \to (B_0, B_1)_{\theta,p}$$

with quasi-norm $M \leqslant M_0^{1-\theta} M_1^{\theta}$.

LEMMA 4. ([1]) *Suppose that* $0 < p_0 < p_1 \leqslant \infty$, $0 < q, q_0, q_1 \leqslant \infty$ *and* $0 < \theta < 1$, *then*

$$(L_{p_0,q_0}, L_{p_1,q_1})_{\theta,q} = L_{p,q},\ \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}.$$

THEOREM 2. *Let* $0 < p < \infty$, $\alpha \geqslant 0$, *then* $\Lambda_p(\alpha) = \Lambda_1(\alpha)$.

Proof. For $1 \leqslant p < \infty$, by Theorem 1 we have

$$\begin{aligned}
&\mathbb{P}(A)^{-\frac{1}{p}-\alpha} \| (f - f_{n-1})\chi_A \|_p \\
&= \mathbb{P}(A)^{-\frac{1}{p}} \left(p \int_0^{\infty} s^{p-1} \mathbb{P}\left(\left\{ \omega \in A : \frac{| f - f_{n-1} |}{\mathbb{P}(A)^{\alpha}} > s \right\} \right) ds \right)^{\frac{1}{p}} \\
&\leqslant \left(p \int_0^{\infty} s^{p-1} e^2 e^{-\frac{s}{e\|f\|_{\Lambda_1(\alpha)}}} ds \right)^{\frac{1}{p}} \\
&\leqslant C \| f \|_{\Lambda_1(\alpha)},
\end{aligned}$$

it follows that

$$\| f \|_{\Lambda_p(\alpha)} \leqslant C \| f \|_{\Lambda_1(\alpha)} .$$

For the converse, by Hölder inequality, we have

$$\begin{aligned}
\| f \|_{\Lambda_1(\alpha)} &= \sup_{n\in\mathbf{N}} \sup_{A\in\mathscr{F}_n} \frac{\| (f - f_{n-1})\chi_A \|_1}{\mathbb{P}(A)^{1+\alpha}} \\
&\leqslant \sup_{n\in\mathbf{N}} \sup_{A\in\mathscr{F}_n} \frac{\mathbb{P}(A)^{\frac{1}{q}} \| (f - f_{n-1})\chi_A \|_p}{\mathbb{P}(A)^{1+\alpha}} = \| f \|_{\Lambda_p(\alpha)},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For $0 < p < 1$, by Hölder inequality, we have

$$\| (f - f_{n-1})\chi_A \|_p \leqslant \mathbb{P}(A)^{\frac{1}{q}} \| (f - f_{n-1})\chi_A \|_1,$$

where $\frac{1}{p}=\frac{1}{q}+1$, from which we know that $\| f \|_{\Lambda_p(\alpha)} \leqslant \| f \|_{\Lambda_1(\alpha)}$. For the converse, we choose $p_0 \in (1,\infty)$ such that $\frac{1-\theta}{p}+\frac{\theta}{p_0}=1$ for some $\theta \in (0,1)$. Now we consider the linear operator $T_n a=(f-f_{n-1})a$, $f=(f_n)_{n\geqslant 0} \in \Lambda_1(\alpha)$ (note that here $f=(f_n)_{n\geqslant 0} \in \Lambda_p(\alpha)$ and $f=(f_n)_{n\geqslant 0} \in \Lambda_{p_0}(\alpha)$). If $\frac{p_0}{p_0\alpha+1} \leqslant 1$, then we consider T_n as $T_n : L_{\frac{p}{p\alpha+1}}(\mathscr{F}_n) \to L_p(\mathscr{F})$ and $T_n : L_{\frac{p_0}{p_0\alpha+1}}(\mathscr{F}_n) \to L_{p_0}(\mathscr{F})$, respectively. Note that $(L_p, L_{p_0})_{\theta,1}=L_1, (L_{\frac{p}{p\alpha+1}}, L_{\frac{p_0}{p_0\alpha+1}})_{\theta,1}=L_{\frac{1}{1+\alpha},1}$ by Lemma 4, it follows from Lemma 3 that

$$\| T_n \|_{L_{\frac{1}{1+\alpha},1}(\mathscr{F}_n)\to L_1(\mathscr{F})} \leqslant \| T_n \|^{1-\theta}_{L_{\frac{p}{p\alpha+1}}(\mathscr{F}_n)\to L_p(\mathscr{F})} \| T_n \|^{\theta}_{L_{\frac{p_0}{p_0\alpha+1}}(\mathscr{F}_n)\to L_{p_0}(\mathscr{F})} .$$

Hence by Lemma 3 we have

$$c \| f \|_{\Lambda_1(\alpha)} \leqslant \| f \|^{1-\theta}_{\Lambda_p(\alpha)} \| f \|^{\theta}_{\Lambda_{p_0}(\alpha)} \leqslant C \| f \|^{1-\theta}_{\Lambda_p(\alpha)} \| f \|^{\theta}_{\Lambda_1(\alpha)},$$

from which we obtain

$$\| f \|_{\Lambda_1(\alpha)} \leqslant C \| f \|_{\Lambda_p(\alpha)} . \tag{1}$$

If $\frac{p_0}{p_0\alpha+1}>1$, then we consider T_n as $T_n : L_{\frac{p}{p\alpha+1}}(\mathscr{F}_n) \to L_p(\mathscr{F})$ and $T_n : L_{\frac{p_0}{p_0\alpha+1},1}(\mathscr{F}_n) \to L_{p_0}(\mathscr{F})$, respectively. Similar to the above proof, we can still obtain (1). □

COROLLARY 1. *Let $0<p<\infty$, then $BMO_p = BMO_1$.*

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