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A METHOD FOR PROVING REFINEMENTS OF INEQUALITIES RELATED TO CONVEX FUNCTIONS ON INTERVALS

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Abstract. In this paper, using the results of a recent paper by the author, we give a new method for proving refinements of inequalities related to convex functions on intervals. In many cases, the proof is simpler and more transparent than using the usual techniques, and the essence of the refinement is clearer. This is illustrated by two refinements of the Jensen's inequality and one refinement of the Lah-Ribarič inequality. As an application we generalize a recent result for strongly convex functions.

1. Introduction

Let $(X,\mathscr{A})$ be a measurable space ($\mathscr{A}$ always means a σ-algebra of subsets of X). If μ is either a measure or a signed measure on $\mathscr{A}$, then the real vector space of μ-integrable real functions on X is denoted by $L(\mu)$. The integrable functions are considered to be measurable.

To start with, we introduce some special function sets: Let $C\subset\mathbb{R}$ be an interval with nonempty interior, and let $(X,\mathscr{A},\mu)$ and $(Y,\mathscr{B},\nu)$ be measure spaces, where μ and ν are finite signed measures.

Let denote F_C the set of all convex functions on C. Furthermore, if $\varphi: X\rightarrow C$, $\psi: Y\rightarrow C$ are functions such that $\varphi\in L(\mu)$ and $\psi\in L(\nu)$, then we define $F_C(\varphi,\mu;\psi,\nu)$ as the set of all functions $f\in F_C$ for which $f\circ\varphi\in L(\mu)$ and $f\circ\psi\in L(\nu)$.

If $(X,\mathscr{A},\mu)=(Y,\mathscr{B},\nu)$ and $\varphi=\psi$, the shorter notation $F_C(\varphi,\mu)$ is used.

Let $C\subset\mathbb{R}$ be an interval with nonempty interior. The following notations are introduced for some special functions defined on C:

$$id_C(t):=t,\quad p_{C,w}(t):=(t-w)^+,\quad n_{C,w}(t):=(t-w)^-\quad t,w\in C,$$

where a^+ and a^- mean the positive and negative parts of $a\in\mathbb{R}$, respectively.

The following statement can be found in a more general form in paper [3].

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THEOREM 1. *Let* $(X,\mathscr{A},\mu)$ *and* $(Y,\mathscr{B},\nu)$ *be measure spaces, where* μ *and* ν *are finite signed measures. Let* $C \subset \mathbb{R}$ *be an interval with nonempty interior, and let* $\varphi : X \to C$, $\psi : Y \to C$ *be functions such that* $\varphi \in L(\mu)$ *and* $\psi \in L(\nu)$. *Then for every* $f \in F_C(\varphi,\mu;\psi,\nu)$ *inequality*

$$\int_X f \circ \varphi d\mu \leqslant \int_Y f \circ \psi d\nu$$

holds if and only if

$$\mu(X) = \nu(Y), \quad \int_X \varphi d\mu = \int_Y \psi d\nu$$

and it is satisfied in the following special cases: the function f *is* $p_{C,w}$ $(w \in C^\circ)$.

This result suggests a new method for proving refinements of inequalities for convex functions on intervals. We illustrate this with refinements of the integral Jensen inequality.

THEOREM 2. (Integral Jensen inequality, see [5]) *Let* $(X,\mathscr{A},\mu)$ *be a probability space, let* $C \subset \mathbb{R}$ *be an interval with nonempty interior, and let* $\varphi : X \to C$ *be a* μ*-integrable function. If* $f \in F_C(\varphi,\mu)$, *then*

$$f\left(\int_X \varphi d\mu\right) \leqslant \int_X f \circ \varphi d\mu.$$

The refinement of the Jensen's inequality means the following: an expression $R(f,\varphi,\mu,par)$ is created satisfying the inequality

$$f\left(\int_X \varphi d\mu\right) \leqslant R(f,\varphi,\mu,par) \leqslant \int_X f \circ \varphi d\mu, \tag{1}$$

where par denotes some auxiliary parameters.

There are many different types of refinements of the Jensen's inequality, but the essence of proving inequalities in (1) is the same in almost all cases, repeated applications of the Jensen's inequality. In this paper we present a new and effective method, based on Theorem 1, which does not use the Jensen's inequality and is generally simpler and more transparent. Of course, the expression $R(f,\varphi,\mu,par)$ still needs to be found, but the new method of the proof can help to generalize and make it more precise.

To achieve our goal, we first give an extension of Theorem 1 that is more applicable to refinements. We then demonstrate the new technique of proof on three known results, two refinements of the integral Jensen inequality and one refinement of the Lah–Ribarič inequality. Not only do we provide completely new proofs of these results, but we also extend them suggested by our method in all three cases. In the refinement from paper [1], we increase the number of parameters, while the refinement from paper [9] is formulated in a more general form that clearly explains the origin of the additional measure in the refinement. Finally, we formulate a refinement of the Lah–Ribarič inequality from paper [8] in measure spaces and study its sharpness. As an application we generalize a recent result for strongly convex functions in [7].

2. Preliminary results

We need the next lemma from paper [3].

LEMMA 1. *Let $(X,\mathscr{A},\mu)$ and $(Y,\mathscr{B},\nu)$ be measure spaces, where μ and ν are finite signed measures with $\mu(X)=\nu(Y)$. Assume $\varphi\in L(\mu)$ and $\psi\in L(\nu)$ such that $\int\limits_X \varphi d\mu=\int\limits_Y \psi d\nu$. Then for every $w\in\mathbb{R}$ the following two assertions are equivalent.*

(a)

$$\int\limits_X p_{\mathbb{R},w}\circ\varphi d\mu \leqslant \int\limits_Y p_{\mathbb{R},w}\circ\psi d\nu.$$

(b)

$$\int\limits_X n_{\mathbb{R},w}\circ\varphi d\mu \leqslant \int\limits_Y n_{\mathbb{R},w}\circ\psi d\nu.$$

We also need an approximation result from paper [2].

DEFINITION 1. Let $C\subset\mathbb{R}$ be an interval with nonempty interior. A function $f:C\to\mathbb{R}$ is called piecewise linear if it is continuous and there exists finite points $w_1<w_2<\ldots<w_k$ in the interior of C such that the restriction of f to each interval $C\bigcap]-\infty,w_1]$, $[w_1,w_2]$, $\ldots$, $C\bigcap[w_k,\infty[$ is an affine function.

THEOREM 3. (see [2]) *If $C\subset\mathbb{R}$ be an interval with nonempty interior and $f:C\to\mathbb{R}$ is a continuous convex function, then f is the pointwise limit of an increasing sequence of piecewise linear convex functions on C.*

REMARK 1. Let $C\subset\mathbb{R}$ be an interval with nonempty interior. If a function $f:C\to\mathbb{R}$ is piecewise linear, then it has the form

$$f(t)=\alpha t+\beta+\sum_{i=1}^{k}\gamma_i\left((t-w_i)^+ + (t-w_i)^-\right),\quad t\in C$$

for suitable points $w_1<w_2<\ldots<w_k$ in the interior of C, α, $\beta\in\mathbb{R}$ and $\gamma_i>0$ $(i=1,\ldots,k)$.

To refine inequalities, the following statement will be useful.

THEOREM 4. *Let $(X,\mathscr{A},\mu)$, $(Y,\mathscr{B},\nu)$ and $(Z,\mathscr{C},\xi)$ be measure spaces, where μ, ν and ξ are finite signed measures and $\mu(X)=\nu(Y)=\xi(Z)$. Let $C\subset\mathbb{R}$ be* *an interval with nonempty interior, and let* $\varphi : X \to C$, $\psi : Y \to C$ *and* $\chi : Z \to C$ *be functions such that* $\varphi \in L(\mu)$, $\psi \in L(\nu)$ *and* $\chi \in L(\xi)$*. Assume inequalities*

$$\int_X f \circ \varphi d\mu \leqslant \int_Z f \circ \chi d\xi \leqslant \int_Y f \circ \psi d\nu \tag{2}$$

are satisfied in the following special cases: the function f *is either* id_C *or* $-id_C$ *or* $p_{C,w}$ $(w \in C^\circ)$*. Then* $F_C(\varphi,\mu;\psi,\nu) \subset F_C(\chi,\xi)$ *and for every* $f \in F_C(\varphi,\mu;\psi,\nu)$ *inequalities (2) hold.*

Proof. Since $\varphi \in L(\mu)$, $\psi \in L(\nu)$ and $\chi \in L(\xi)$, and μ, ν and ξ are finite, it follows from Remark 1 that all piecewise linear convex functions belong to $F_C(\varphi,\mu;\psi,\nu) \bigcap F_C(\chi,\xi)$.

Since inequalities (2) are satisfied when f is either id_C or $-id_C$ or $p_{C,w}$ $(w \in C^\circ)$, Lemma 1 and Remark 1 show that inequalities (2) also hold for every piecewise linear convex function on C.

(i) Assume that f is continuous. By Theorem 3, there exists an increasing sequence $(f_n)_{n=1}^\infty$ of piecewise linear convex functions on C such that $f_n \to f$ pointwise on C.

Then the sequence $(f_n \circ \varphi)$ is also increasing, and $f_n \circ \varphi \to f \circ \varphi$ pointwise on X. Since $f_n \circ \varphi \in L(\mu)$, B. Levi's theorem yields that

$$\lim_{n\to\infty} \int_X f_n \circ \varphi d\mu = \int_X f \circ \varphi d\mu.$$

Similarly, we can confirm that

$$\lim_{n\to\infty} \int_Y f_n \circ \psi d\nu = \int_Y f \circ \psi d\nu, \quad \lim_{n\to\infty} \int_Z f_n \circ \chi d\xi = \int_Z f \circ \chi d\xi.$$

Since inequalities (2) hold for every piecewise linear convex function on C, we thus obtain that inequalities also hold for every $f \in F_C(\varphi,\mu;\psi,\nu)$, and this implies that $f \in F_C(\chi,\xi)$.

(ii) If f is not continuous, then it is not hard to think that there exists a decreasing sequence $(f_n)_{n=1}^\infty$ from $F_C(\varphi,\mu;\psi,\nu)$ such that f_n is continuous $(n \in \mathbb{N}_+)$ and (f_n) converges pointwise to f on C, and therefore the result follows from part (i) of the proof and B. Levi's theorem.

The proof is complete. $\square$

REMARK 2. Of course, that the inequalities in (2) are satisfied for the functions id_C and $-id$ is equivalent to

$$\int_X \varphi d\mu = \int_Z \chi d\xi = \int_Y \psi d\nu.$$

3. Extensions and new proofs of some refinements

Before formulating our first result, we introduce some further concepts and notations from measure and integration theory.

Let $(X,\mathscr{A},\mu)$ be a measure space, and let z be a nonnegative and $\mathscr{A}$-measurable function on X. Then the measure ν defined on $\mathscr{A}$ by

$$\nu(A) := \int_A z d\mu$$

is called the measure having density z with respect to μ. It will be denoted by $\nu = z\mu$. An $\mathscr{A}$-measurable function $\psi : X \to \mathbb{R}$ is ν-integrable if and only if ψz is μ-integrable, and the relationship between the ν- and μ-integrals is

$$\int_X \psi d\nu = \int_X \psi z d\mu.$$

Let $(X,\mathscr{A},\mu)$ and $(Y,\mathscr{B},\nu)$ be σ-finite measure spaces. The σ-algebra in $X \times Y$ generated by the projection mappings $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$

$$pr_1(x,y) := x, \quad pr_2(x,y) := y$$

is denoted by $\mathscr{A} \otimes \mathscr{B}$. The product measure $\mu \times \nu$ on $\mathscr{A} \otimes \mathscr{B}$ is defined in the usual way: this measure is uniquely specified by

$$(\mu \times \nu)(A \times B) := \mu(A)\nu(B), \quad A \in \mathscr{A}, \quad B \in \mathscr{B}.$$

If $(X,\mathscr{A},\mu) = (Y,\mathscr{B},\nu)$, then the product space will be denoted by $\left(X^2,\mathscr{A}^2,\mu^2\right)$.

The product of finitely many σ-finite measure spaces can be defined in a similar way, and notations can be generalized in a natural way.

We say that the numbers $(p_i)_{i=1}^k$ represent a (positive) discrete probability distribution if $(p_i > 0)$ $p_i \geqslant 0$ $(i = 1,\ldots,k)$ and $\sum\limits_{i=1}^k p_i = 1$.

Our result below provides an essentially new proof for Theorem 2 in [1], and by analyzing the proof, we can obtain a new refinement.

THEOREM 5. *Let $(X,\mathscr{A},\mu)$ be a measure space such that $0 < \mu(X) < \infty$, and let z be a positive function on X such that $\int_X z d\mu = 1$. Assume k is a positive integer. Let $(p_i)_{i=1}^k$ represent a positive discrete probability distribution, and let $z_1,\ldots,z_k$ be positive functions on X such that $\sum\limits_{j=1}^k p_j z_j = z$. If $\psi : X \to \mathbb{R}$ is a $z\mu$-integrable function taking values in an interval $C \subset \mathbb{R}$ with nonempty interior, then for every $f \in F_C(\psi, z\mu)$ we have*

(a)

$$f\left(\int_X \psi z d\mu\right)$$

$$\leqslant \frac{1}{\mu(X)^{k-1}}\int_{X^k} f\left(\frac{\sum_{j=1}^{k} p_j z_j(x_j)\,\psi(x_j)}{\sum_{j=1}^{k} p_j z_j(x_j)}\right)\sum_{j=1}^{k} p_j z_j(x_j)\,d\mu^k(x_1,\ldots,x_k)$$

$$\leqslant \int_X (f\circ\psi)\,z d\mu.$$

(b) For $z_j = z$ *and* $p_j = \frac{1}{k}$ $(j=1,\ldots,k)$

$$N_{k+1} \leqslant N_k, \quad k \geqslant 1,$$

where

$$N_k := \frac{1}{k\mu(X)^{k-1}}\int_{X^k} f\left(\frac{\sum_{j=1}^{k} z(x_j)\,\psi(x_j)}{\sum_{j=1}^{k} z(x_j)}\right)\sum_{j=1}^{k} z(x_j)\,d\mu^k(x_1,\ldots,x_k), \quad k \geqslant 1.$$

Proof. (a) Introduce the constant function

$$\varphi : X \to \mathbb{R}, \quad \varphi(x) := \int_X \psi z d\mu \tag{3}$$

and the other two functions

$$\psi_k : X^k \to \mathbb{R}, \quad \psi_k(x_1,\ldots,x_k) := \frac{\sum_{j=1}^{k} p_j z_j(x_j)\,\psi(x_j)}{\sum_{j=1}^{k} p_j z_j(x_j)}$$

and

$$z_k : X^k \to \mathbb{R}, \quad z_k(x_1,\ldots,x_k) := \frac{1}{\mu(X)^{k-1}} p_j z_j(x_j).$$

Obviously, $\psi_1 = \psi$, $z_1 = z$ and $z_k \in L\left(\mu^k\right)$.

For simplicity, let

$$\nu := z\mu, \quad \nu_k := z_k\mu^k.$$

It is easy to think that, $\psi_k \in L(\nu_k)$, and if $f \in L_C(\psi,\nu)$, then $f \in L_C(\psi_k,\nu_k)$.

With these notations we have to prove that for all $k \geqslant 1$ inequalities

$$\int_X f \circ \varphi d\nu \leqslant \int_{X^k} f \circ \psi_k d\nu_k \leqslant \int_X f \circ \psi d\nu \tag{4}$$

are satisfied.

By Theorem 1, it is enough to show that

$$\nu(X) = \nu_k\left(X^k\right), \tag{5}$$

$$\int_X \varphi d\nu = \int_{X^k} \psi_k d\nu_k = \int_X \psi d\nu, \tag{6}$$

and

$$\int_X p_{C,w} \circ \varphi d\nu \leqslant \int_{X^k} p_{C,w} \circ \psi_k d\nu_k \leqslant \int_X p_{C,w} \circ \psi d\nu, \quad w \in C^\circ. \tag{7}$$

The equalities (5) and (6) can be easily checked.

For every $w \in C^\circ$ let

$$A_{w,k} := \left\{ (x_1, \ldots, x_k) \in X^k \mid \sum_{j=1}^k p_j z_j(x_j)(\psi(x_j) - w) \geqslant 0 \right\}$$

and

$$B_{w,k,j} := \left\{ (x_1, \ldots, x_k) \in X^k \mid \psi(x_j) - w \geqslant 0 \right\}, \quad j = 1, \ldots, k.$$

Then for every $w \in C^\circ$ we have that $A_{w,k}$, $B_{w,k,j} \in \mathscr{A}^k$ $(j = 1, \ldots, k)$.

Using the definition of the set $A_{w,k}$ and Fubini's theorem, we obtain that

$$\int_{X^k} p_{C,w} \circ \psi_k d\nu_k$$

$$= \frac{1}{\mu(X)^{k-1}} \int_{A_{w,k}} \sum_{j=1}^k p_j z_j(x_j)(\psi(x_j) - w) d\mu^k(x_1, \ldots, x_k)$$

$$\geqslant \max\left(0, \frac{1}{\mu(X)^{k-1}} \int_{X^k} \sum_{j=1}^k p_j z_j(x_j)(\psi(x_j) - w) d\mu^k(x_1, \ldots, x_k)\right)$$

$$= \max\left(0, \int_X (\psi - w) z d\mu\right) = \int_X p_{C,w} \circ \varphi d\nu,$$

which yields the first inequality in (7).

To prove the second inequality in (7), an obvious calculation shows that

$$\int_{X^k} p_{C,w} \circ \psi_k d\nu_k$$

$$= \frac{1}{\mu(X)^{k-1}} \sum_{j=1}^{k} \int_{A_{w,k}} p_j z_j(x_j)(\psi(x_j)-w)\, d\mu^k(x_1,\ldots,x_k)$$

$$\leqslant \frac{1}{\mu(X)^{k-1}} \sum_{j=1}^{k} \int_{A_{w,k}\bigcap B_{w,k,j}} p_j z_j(x_j)(\psi(x_j)-w)\, d\mu^k(x_1,\ldots,x_k)$$

$$\leqslant \frac{1}{\mu(X)^{k-1}} \sum_{j=1}^{k} \int_{B_{w,k,j}} p_j z_j(x_j)(\psi(x_j)-w)\, d\mu^k(x_1,\ldots,x_k)$$

$$= \int_{\{\psi \geqslant w\}} z(\psi - w)\, d\mu = \int_X p_{C,w} \circ \psi d\nu, \quad w \in C^{\circ}.$$

(b) Since $z_j = z$ and $p_j = \frac{1}{k}$ $(j=1,\ldots,k)$, the function ψ_k, the measure ν_k and the set $A_{w,k}$ are defined for all $k \geqslant 1$.

Again using Theorem 1, we can see that it is sufficient to prove inequality

$$\int_{X^{k+1}} p_{C,w} \circ \psi_{k+1} d\nu_{k+1} \leqslant \int_{X^k} p_{C,w} \circ \psi_k d\nu_k, \quad w \in C^{\circ}. \tag{8}$$

Let

$$C^i_{w,k+1} := \Big\{ (x_1,\ldots,x_{k+1}) \in X^{k+1} \mid$$
$$(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{k+1}) \in A_{w,k} \Big\}, \quad i = 1,\ldots,k+1.$$

Then $C^i_{w,k+1} \in \mathscr{A}^{k+1}$ $(i=1,\ldots,k+1)$.

By elementary calculation we obtain that

$$\int_{X^{k+1}} p_{C,w} \circ \psi_{k+1} d\nu_{k+1}$$

$$= \int_{A_{w,k+1}} \left(\frac{\sum_{j=1}^{k+1} z(x_j)\,\psi(x_j)}{\sum_{j=1}^{k+1} z(x_j)} - w \right) d\nu_{k+1}(x_1,\ldots,x_{k+1})$$

$$= \frac{1}{k(k+1)\mu(X)^k} \sum_{i=1}^{k+1} \int_{A_{w,k+1}} \left(\sum_{\substack{j=1 \\ j \neq i}}^{k} z(x_j)(\psi(x_j)-w) \right) d\mu^{k+1}(x_1,\ldots,x_{k+1})$$

$$= \frac{1}{k(k+1)\mu(X)^k} \sum_{i=1}^{k+1} \left(\int\limits_{A_{w,k+1} \cap C^i_{w,k+1}} + \int\limits_{A_{w,k+1} \setminus C^i_{w,k+1}} \right)$$

$$\left(\sum_{\substack{j=1 \\ j \neq i}}^{k} z(x_j)(\psi(x_j) - w) \right) d\mu^{k+1}(x_1, \ldots, x_{k+1}). \tag{9}$$

Using the properties of the sets $C^i_{w,k+1}$, and then applying Fubini's theorem, we find that the expression (9)

$$\leqslant \frac{1}{k(k+1)\mu(X)^k} \sum_{i=1}^{k+1} \int\limits_{C^i_{w,k+1}} \left(\sum_{\substack{j=1 \\ j \neq i}}^{k} z(x_j)(\psi(x_j) - w) \right) d\mu^{k+1}(x_1, \ldots, x_{k+1})$$

$$= \frac{1}{k(k+1)\mu(X)^k} \sum_{i=1}^{k+1} \mu(X) \int\limits_{A_{w,k}} \sum_{j=1}^{k} z(x_j)(\psi(x_j) - w) \, d\mu^k(x_1, \ldots, x_k)$$

$$= \frac{1}{k\mu(X)^{k-1}} \int\limits_{A_{w,k}} \sum_{j=1}^{k} z(x_j)(\psi(x_j) - w) \, d\mu^k(x_1, \ldots, x_k)$$

$$= \int\limits_{X^k} p_{C,w} \circ \psi_k d\nu_k, \quad w \in C^\circ,$$

which gives exactly inequality (8).

The proof is complete. $\square$

REMARK 3. Part (a) of the previous statement is a new refinement, and part (b) has a new proof.

In the following theorem, we give a generalization and a new proof of the main result in paper [9]. Here again, the new technique suggests the possibility of generalization.

Suppose that $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ are probability spaces. By a (separately) weight function on $X \times Y$ we mean an $\mathscr{A} \times \mathscr{B}$-measurable mapping $z : X \times Y \to [0, \infty[$ for which

$$\int\limits_X z(x,y) \, d\mu(x) = 1, \quad y \in Y \tag{10}$$

and

$$\int\limits_Y z(x,y) \, d\nu(y) = 1, \quad x \in X.$$

It follows from the Fubini's theorem that $z \in L(\mu \times \nu)$ and $\int\limits_{X \times Y} z d\mu \times \nu = 1$.

THEOREM 6. *Let* $(X,\mathscr{A},\mu)$ *and* $(Y,\mathscr{B},\nu)$ *be probability spaces, let* $z : X\times Y \to [0,\infty[$ *be a weight function, and let* $C\subset\mathbb{R}$ *be an interval with nonempty interior. If* $\psi : X\times Y\to C$ *is* $z(\mu\times\nu)$*-integrable and* $f\in F_C(\psi,z(\mu\times\nu))$*, then*

(a) The function

$$y\to\int_X \psi(x,y)\,z(x,y)\,d\mu(x)$$

is defined ν*-almost everywhere on* Y*. Denoting it with* χ*, it is* ν*-integrable.*

(b) The function χ *takes values in* C*.*

(c) Inequalities

$$f\left(\int_{X\times Y}\psi(x,y)\,z(x,y)\,d\mu(x)\,d\nu(y)\right)\leqslant\int_Y f\left(\int_X\psi(x,y)\,z(x,y)\,d\mu(x)\right)d\nu(y) \tag{11}$$

$$\leqslant\int_{X\times Y} f(\psi(x,y))\,z(x,y)\,d\mu(x)\,d\nu(y)$$

hold.

Proof. (a) It follows immediately from the Fubini's theorem.

(b) Assume $a\in\mathbb{R}$ is the left-hand endpoint of C.

If $\psi\geqslant a$, then the definition of χ and (10) imply that $\chi\geqslant a$.

Now assume that $\psi>a$ and there exists an $y\in Y$ such that χ is defined at y and $\chi(y)=a$. Then

$$0=\chi(y)-a=\int_X(\psi(x,y)-a)\,z(x,y)\,d\mu(x).$$

Since $\psi>a$ and z is nonnegative, this yields that $z(\cdot,y)=0$ μ-almost everywhere on X, which contradicts to (10). Therefore, where χ is defined, it is greater than a.

The other case (C is bounded from above) can be treated similarly.

(c) Let $\omega:=z(\mu\times\nu)$. We have to show that

$$f\left(\int_{X\times Y}\psi\,d\omega\right)\leqslant\int_Y f\circ\chi\,d\nu\leqslant\int_{X\times Y} f\circ\psi\,d\omega. \tag{12}$$

It is obvious that

$$\omega(X\times Y)=\nu(Y)=1 \tag{13}$$

and

$$\int_{X\times Y}\psi\,d\omega=\int_Y\chi\,d\nu. \tag{14}$$

The conditions $\psi\in L(\omega)$ and $f\in F_C(\psi,\omega)$, the equality (13), and parts (a) and (b) show that the conditions of Theorem 4 are satisfied, and therefore, by using (14), it is enough to prove (12) for $f=p_{C,w}$ $(w\in C^\circ)$.

Consider the first inequality in (12): for every $w \in C^{\circ}$ we have

$$\int_Y p_{C,w} \circ \chi d\nu = \int_{\{\chi \geqslant w\}} (\chi - w) d\nu \geqslant \max\left(0, \int_Y (\chi - w) d\nu\right)$$
$$= \max\left(0, \int_Y \left(\int_X (\psi(x,y) - w) z(x,y) d\mu(x)\right) d\nu(y)\right)$$
$$= \max\left(0, \int_{X\times Y} \psi d\omega - w\right) = p_{C,w}\left(\int_{X\times Y} \psi d\omega\right).$$

Now we prove the second inequality in (12). Fix $w \in C^{\circ}$. We obtain by using the Fubini's theorem that

$$\int_Y p_{C,w} \circ \chi d\nu = \int_{\{\chi \geqslant w\}} (\chi - w) d\nu$$
$$= \int_X \left(\int_{\{\chi \geqslant w\}} (\psi(x,y) - w) z(x,y) d\nu(y)\right) d\mu(x), \tag{15}$$

where the function, denoted by ς,

$$x \longrightarrow \int_{\{\chi \geqslant w\}} (\psi(x,y) - w) z(x,y) d\nu(y)$$

is defined μ-almost everywhere on X and it is μ-integrable. Estimating the expression (15) from above, we have

$$\int_Y p_{C,w} \circ \chi d\nu \leqslant \int_{\{\varsigma \geqslant 0\}} \left(\int_{\{\chi \geqslant w\}} (\psi(x,y) - w) z(x,y) d\nu(y)\right) d\mu(x)$$
$$= \int_{\{\varsigma \geqslant 0\}\times\{\chi \geqslant w\}} (\psi(x,y) - w) z(x,y) d\mu(x) d\nu(y)$$
$$\leqslant \int_{\{\psi \geqslant w\}} (\psi(x,y) - w) z(x,y) d\mu(x) d\nu(y) = \int_Y p_{C,w} \circ \psi d\mu.$$

The proof is complete. □

REMARK 4. In the main result of paper [9] the following special case of inequalities (11) are studied

$$f\left(\int_X \psi d\mu\right) \leqslant \int_Y f\left(\int_X \psi(x) z(x,y) d\mu(x)\right) d\nu(y) \leqslant \int_X f \circ \psi d\mu.$$

Our result is, first, more general than Rooin's result, second, it gives a new proof, and third, it explains the role of the probability measure ν in the refinement.

Our last result relates to the integral form of the Lah–Ribarič inequality which is given in the next theorem.

THEOREM 7. (see [4]) *Let $[a,b] \subset \mathbb{R}$ be an interval, let $p : [a,b] \to \mathbb{R}$ be a nonnegative and integrable function such that $P := \int_a^b p > 0$, let $\psi : [a,b] \to \mathbb{R}$ be a measurable function taking values in an interval $[m,M]$ with $m < M$, and let $f : [m,M] \to \mathbb{R}$ be a convex function. If $p\psi$ and $p(f \circ \psi)$ are integrable, then*

$$\frac{1}{P}\int_a^b p(f\circ\psi) \leqslant \frac{M - \frac{1}{P}\int_a^b p\psi}{M-m}\psi(m) + \frac{\frac{1}{P}\int_a^b p\psi - M}{M-m}\psi(M).$$

In paper [8] the authors obtain a refinement of the previous inequality. We first generalize this result to measure spaces with a new proof.

If $(X,\mathscr{A})$ is a measurable space, the unit mass at $x \in X$ (the Dirac measure at x) is denoted by ε_x.

The set of positive integers will be denoted by $\mathbb{N}_+$.

THEOREM 8. *Let $(X,\mathscr{A},\mu)$ be a probability space. Let the set I denote either $\{1,\ldots,n\}$ for some $n \geqslant 1$ or $\mathbb{N}_+$, and assume we are given a sequence $(A_i)_{i\in I}$ of pairwise disjoint sets $A_i \in \mathscr{A}$ with $\mu(A_i) > 0$ for all $i \in I$ and $\bigcup_{i\in I} A_i = X$. Furthermore, we assume that $\psi \in L(\mu)$, $\psi(x) \in [m_i, M_i]$ for all $x \in A_i$, where $m_i < M_i$ $(i \in I)$, and*

$$m := \inf_{i\in I} m_i \in \mathbb{R} \quad \text{and} \quad M := \sup_{i\in I} M_i \in \mathbb{R}.$$

Then for every $f \in F_{[m,M]}(\psi,\mu)$ we have

$$\int_X f\circ\psi\, d\mu$$

$$\leqslant \sum_{i\in I}\mu(A_i)\left(\frac{M_i - \frac{1}{\mu(A_i)}\int_{A_i}\psi\, d\mu}{M_i - m_i} f(m_i) + \frac{\frac{1}{\mu(A_i)}\int_{A_i}\psi\, d\mu - m_i}{M_i - m_i} f(M_i)\right) \tag{16}$$

$$\leqslant \frac{M - \int_X \psi\, d\mu}{M-m} f(m) + \frac{\int_X \psi\, d\mu - m}{M-m} f(M). \tag{17}$$

Proof. If we introduce the discrete probability measures

$$\nu := \frac{M - \int_X \psi\, d\mu}{M-m}\varepsilon_m + \frac{\int_X \psi\, d\mu - m}{M-m}\varepsilon_M,$$

and

$$\xi := \sum_{i\in I} \frac{M_i\mu(A_i) - \int\limits_{A_i} \psi d\mu}{M_i - m_i}\varepsilon_{m_i} + \frac{\int\limits_{A_i} \psi d\mu - m_i\mu(A_i)}{M_i - m_i}\varepsilon_{M_i}$$

on the Borel subsets of $[m,M]$, then inequalities (16–17) can be written in the following form

$$\int\limits_X f\circ\psi d\mu \leqslant \int\limits_{[m,M]} f d\xi \leqslant \int\limits_{[m,M]} f d\nu. \tag{18}$$

By Theorem 1, it is enough to prove (18) for $f = p_{[m,M],w}$ $(w \in]m,M[)$. In this case (18) is equivalent to

$$\int\limits_{\{\psi\geqslant w\}} (\psi - w)\, d\mu \leqslant \sum_{\{i\in I|m_i\geqslant w\}} (m_i - w)\frac{M_i\mu(A_i) - \int\limits_{A_i}\psi d\mu}{M_i - m_i}$$

$$+\sum_{\{i\in I|M_i\geqslant w\}} (M_i - w)\frac{\int\limits_{A_i}\psi d\mu - m_i\mu(A_i)}{M_i - m_i} \leqslant (M - w)\frac{\int\limits_X \psi d\mu - m}{M - m}. \tag{19}$$

To prove the first inequality in (19), we show that for every $i \in I$

$$\int\limits_{A_i\bigcap\{\psi\geqslant w\}} (\psi - w)\, d\mu \leqslant (M_i - w)\frac{\int\limits_{A_i}\psi d\mu - m_i\mu(A_i)}{M_i - m_i}. \tag{20}$$

By some easy calculations, this inequality can be rewritten as

$$\int\limits_{A_i\bigcap\{\psi\geqslant w\}} (\psi - w)\, d\mu \leqslant \frac{M_i - w}{w - m_i}\left(\int\limits_{A_i\bigcap\{\psi< w\}} (\psi - w)\, d\mu + (w - m_i)\,\mu(A_i)\right).$$

Since $M_i - w \geqslant \psi - w \geqslant m_i - w$ on A_i, we have that

$$\frac{M_i - w}{w - m_i}\left(\int\limits_{A_i\bigcap\{\psi< w\}} (\psi - w)\, d\mu + (w - m_i)\,\mu(A_i)\right)$$

$$\geqslant \frac{M_i - w}{w - m_i}\left((m_i - w)\,\mu\left(A_i\bigcap\{\psi < w\}\right) + (w - m_i)\,\mu(A_i)\right)$$

$$= (M_i - w)\,\mu\left(A_i\bigcap\{\psi \geqslant w\}\right) \geqslant \int\limits_{A_i\bigcap\{\psi\geqslant w\}} (\psi - w)\, d\mu.$$

Now, by applying (20), we have that

$$\sum_{\{i\in I|m_i\geqslant w\}}(m_i-w)\frac{M_i\mu(A_i)-\int\limits_{A_i}\psi d\mu}{M_i-m_i}+\sum_{\{i\in I|M_i\geqslant w\}}(M_i-w)\frac{\int\limits_{A_i}\psi d\mu-m_i\mu(A_i)}{M_i-m_i}$$

$$\geqslant\sum_{\{i\in I|M_i\geqslant w\}}\int\limits_{A_i\bigcap\{\psi\geqslant w\}}(\psi-w)\,d\mu$$

$$=\int\limits_{\{\psi\geqslant w\}}(\psi-w)\,d\mu.$$

Now we prove the second inequality in (19). Using the easily verifiable inequalities

$$m_i-w\leqslant\frac{M-w}{M-m}(m_i-m),\quad M_i-w\leqslant\frac{M-w}{M-m}(M_i-m),\quad i\in I$$

and equality

$$(m_i-m)\frac{M_i\mu(A_i)-\int\limits_{A_i}\psi d\mu}{M_i-m_i}+(M_i-m)\frac{\int\limits_{A_i}\psi d\mu-m_i\mu(A_i)}{M_i-m_i}$$

$$=\int\limits_{A_i}\psi d\mu-m\mu(A_i),\quad i\in I,$$

we obtain that

$$\sum_{\{i\in I|m_i\geqslant w\}}(m_i-w)\frac{M_i\mu(A_i)-\int\limits_{A_i}\psi d\mu}{M_i-m_i}+\sum_{\{i\in I|M_i\geqslant w\}}(M_i-w)\frac{\int\limits_{A_i}\psi d\mu-m_i\mu(A_i)}{M_i-m_i}$$

$$\leqslant\frac{M-w}{M-m}\sum_{\{i\in I|M_i\geqslant w\}}\left(\int\limits_{A_i}\psi d\mu-m\mu(A_i)\right)$$

$$\leqslant(M-w)\frac{\int\limits_{X}\psi d\mu-m}{M-m}.$$

The proof is complete. □

REMARK 5. Our result contains Theorem 2.1 in paper [8] as a special case, the proof uses a different method, and extends Theorem 2.1 to countably infinite index set I.

In the next result, we investigate how the Lah–Ribarič inequality changes for more precise bounds on the function ψ.

THEOREM 9. *Let* $(X, \mathscr{A}, \mu)$ *be a probability space. Furthermore, we assume that* $\psi \in L(\mu)$, $\psi(x) \in [k, K]$ *for all* $x \in X$, *where* $k < K$, *and* $m \leqslant k < K \leqslant M$.

Then for every $f \in F_{[m,M]}(\psi, \mu)$ *we have*

$$\int_X f \circ \psi d\mu \leqslant \frac{K - \int_X \psi d\mu}{K - k} f(k) + \frac{\int_X \psi d\mu - k}{K - k} f(K)$$

$$\leqslant \frac{M - \int_X \psi d\mu}{M - k} f(k) + \frac{\int_X \psi d\mu - k}{M - k} f(M)$$

$$\leqslant \frac{M - \int_X \psi d\mu}{M - m} f(m) + \frac{\int_X \psi d\mu - m}{M - m} f(M).$$

Proof. We have to prove only the second and the third inequalities.

The third inequality will be shown, the second inequality can be handled in a similar way.

By introducing the discrete probability measures

$$\mu := \frac{M - \int_X \psi d\mu}{M - k} \varepsilon_k + \frac{\int_X \psi d\mu - k}{M - k} \varepsilon_M$$

and

$$\nu := \frac{M - \int_X \psi d\mu}{M - m} \varepsilon_m + \frac{\int_X \psi d\mu - m}{M - m} \varepsilon_M$$

on the Borel subsets of $[m, M]$, the third inequality can be written in the form

$$\int_{[m,M]} f d\mu \leqslant \int_{[m,M]} f d\nu. \tag{21}$$

By Theorem 1, it is enough to prove inequality (21) for $f = p_{[m,M],w}$ $(w \in]m, M[)$. Therefore, we have to show that

$$\frac{M - \int_X \psi d\mu}{M - m}(w - m) \geqslant \begin{cases} 0, \text{ if } m < w < k \\ \frac{M - \int_X \psi d\mu}{M - k}(w - k), \text{ if } k \leqslant w < M \end{cases},$$

and this can be verified by elementary calculation.

The proof is complete. $\square$

4. Application

The applicability of our results is illustrated by generalizing a recent result for strongly convex functions.

DEFINITION 2. Let $C \subset \mathbb{R}$ be an interval. A function $f : C \to \mathbb{R}$ is called a strongly convex function with modulus $c > 0$ if

$$f(\alpha s + (1-\alpha)t) \leqslant \alpha f(s) + (1-\alpha) f(t) - c\alpha(1-\alpha)(s-t)^2$$

for all $s,\ t \in C$ and $\alpha \in [0,1]$.

It is known (see [6]) that a function $f : C \to \mathbb{R}$ is strongly convex with modulus $c > 0$ if and only if the function $g : C \to \mathbb{R}$, $g(t) = f(t) - ct^2$ is convex.

One of the main results in paper [7] is the next:

THEOREM 10. *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be probability spaces, let $z : X \times Y \to [0, \infty[$ be a weight function, and let $C \subset \mathbb{R}$ be an interval with nonempty interior. If $f : C \to \mathbb{R}$ is a strongly convex function with modulus $c > 0$ and $\psi : X \to C$ is a function such that $\psi,\ \psi^2 \in L(\mu)$, then*

$$f\left(\int_X \psi d\mu\right) \leqslant \int_Y f\left(\int_X \psi(x) z(x,y)\, d\mu(x)\right) d\nu(y)$$
$$-c\int_Y \left(\int_X \psi(x) z(x,y)\, d\mu(x) - \int_X \psi d\mu\right)^2 d\nu(y)$$
$$\leqslant \int_X f \circ \psi d\mu - c\int_X \left(\psi - \int_X \psi d\mu\right)^2 d\mu.$$

The proof is based on Rooin's result in [9]. We have extended this result in Theorem 6 (c) and this gives the possibility to generalize Theorem 10.

THEOREM 11. *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be probability spaces, let $z : X \times Y \to [0, \infty[$ be a weight function, and let $C \subset \mathbb{R}$ be an interval with nonempty interior. If $f : C \to \mathbb{R}$ is a strongly convex function with modulus $c > 0$ and $\psi : X \times Y \to C$ is a function such that $\psi,\ \psi^2,\ f \circ \psi \in L(z(\mu \times \nu))$, then*

$$f\left(\int_{X \times Y} \psi(x,y) z(x,y)\, d\mu(x)\, d\nu(y)\right) \leqslant \int_Y f\left(\int_X \psi(x,y) z(x,y)\, d\mu(x)\right) d\nu(y) \quad (22)$$

$$-c\int_Y \left(\int_{X \times Y} \psi(x,y) z(x,y)\, d\mu(x)\, d\nu(y) - \int_X \psi(x,y) z(x,y)\, d\mu(x)\right)^2 d\nu(y) \quad (23)$$

$$\leqslant \int_{X\times Y} f\left(\psi(x,y)\right) z(x,y)\, d\mu(x)\, d\nu(y) \tag{24}$$

$$-c\int_{X\times Y}\left(\psi(x,y)-\int_{X\times Y}\psi(u,v)\,z(u,v)\,d\mu(u)\,d\nu(v)\right)^{2} z(x,y)\,d\mu(x)\,d\nu(y)\,. \tag{25}$$

Proof. Since the function $t \to f(t)-ct^{2}$ $(t\in C)$ is convex, Theorem 6 implies that

$$f\left(\int_{X\times Y}\psi(x,y)\,z(x,y)\,d\mu(x)\,d\nu(y)\right)-c\left(\int_{X\times Y}\psi(x,y)\,z(x,y)\,d\mu(x)\,d\nu(y)\right)^{2}$$

$$\leqslant \int_{Y} f\left(\int_{X}\psi(x,y)\,z(x,y)\,d\mu(x)\right)d\nu(y)-c\int_{Y}\left(\int_{X}\psi(x,y)\,z(x,y)\,d\mu(x)\right)^{2}d\nu(y)$$

$$\leqslant \int_{X\times Y} f\left(\psi(x,y)\right)z(x,y)\,d\mu(x)\,d\nu(y)-c\int_{X\times Y}\psi^{2}(x,y)\,z(x,y)\,d\mu(x)\,d\nu(y)\,.$$

Note that there is no integrability problem, since the functions f and $t\to t^{2}$ $(t\in C)$ are convex.

Now some easy considerations show that

$$\left(\int_{X\times Y}\psi(x,y)\,z(x,y)\,d\mu(x)\,d\nu(y)\right)^{2}-\int_{Y}\left(\int_{X}\psi(x,y)\,z(x,y)\,d\mu(x)\right)^{2}d\nu(y)$$

$$=-\int_{Y}\left(\int_{X\times Y}\psi(x,y)\,z(x,y)\,d\mu(x)\,d\nu(y)-\int_{X}\psi(x,y)\,z(x,y)\,d\mu(x)\right)^{2}d\nu(y)$$

and

$$\left(\int_{X\times Y}\psi(x,y)\,z(x,y)\,d\mu(x)\,d\nu(y)\right)^{2}-\int_{X\times Y}\psi^{2}(x,y)\,z(x,y)\,d\mu(x)\,d\nu(y)$$

$$=-\int_{X\times Y}\left(\psi(x,y)-\int_{X\times Y}\psi(u,v)\,z(u,v)\,d\mu(u)\,d\nu(v)\right)^{2}z(x,y)\,d\mu(x)\,d\nu(y)\,,$$

which give the result.

The proof is complete. $\square$

REMARK 6. (a) Theorem 10 is a special case of the previous result.

(b) Since f is strongly convex, it is also convex, and therefore inequalities in (22–25) obviously refine the Jensen's inequality

$$f\left(\int\limits_{X\times Y}\psi(x,y)\,z(x,y)\,d\mu(x)\,d\nu(y)\right)\leqslant\int\limits_{X\times Y}f(\psi(x,y))\,z(x,y)\,d\mu(x)\,d\nu(y).$$

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