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ON THE JOINT NUMERICAL RADIUS OF GENERALIZED SPHERICAL ALUTHGE TRANSFORMS OF OPERATORS

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Abstract. In this paper, we generalize and refine several operator inequalities involving the joint numerical radius and the joint operator norm of spherical Aluthge transform to generalized spherical Aluthge transforms. Moreover, we investigate the link between nontrivial joint invariant subspaces of the generalized spherical Aluthge transform and the original commuting d-tuples of bounded operators.

1. Introduction

Let $\mathscr{H},\mathscr{K}$ denote two Hilbert spaces, and let $\mathscr{B}(\mathscr{H},\mathscr{K})$ be the Banach space of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$. In the case $\mathscr{H}=\mathscr{K}$, the linear space $\mathscr{B}(\mathscr{H},\mathscr{K})$ is simply denoted $\mathscr{B}(\mathscr{H})$ and is a Banach algebra under the usual operator norm. We write $\langle .,.\rangle$ to denote the inner product on $\mathscr{H}$ and for $h\in\mathscr{H}$, the norm of h is given by $\|h\|=\sqrt{\langle h,h\rangle}$. For $T\in\mathscr{B}(\mathscr{H})$, we denote by $\mathscr{R}(T)$, $\ker(T)$ and T^* the range of T, the null space of T and the adjoint operator of T respectively. The polar decomposition of $T\in\mathscr{B}(\mathscr{H})$ is given by $T=U|T|$, where $|T|=\sqrt{T^*T}$ and U is a partial isometry satisfying $\ker U=\ker T$ and $\ker U^*=\ker T^*$.

The Aluthge transform of T, defined by the expression $\hat{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, was introduced by A. Aluthge in [2] to extend some operator inequalities related to hyponormal operators. The generalized Aluthge transform of T is introduced by M. Cho in [9] for $0\leqslant s\leqslant 1$ by the expression $\hat{T}^s=|T|^sU|T|^{1-s}$. Over the last two decades, the Aluthge transform and the generalized Aluthge transform attracted considerable attention because of wide range of possible applications. The guiding principle in work related to these transformations is to identify the links between an operator and its Aluthge transform. In particular it has been shown that $\hat{T}$ and $\hat{T}^s$ share several spectral properties with T. See for instance, [14, 18, 21, 28] for further details and additional information.

Due to the importance multi-variable operator theory, the interest in studying extensions of the Aluthge transform for d-tuples of operators has grown considerably in the recent few years. The spherical generalized Aluthge transform of $\mathbf{T}=(T_1,\dots,T_d)$,

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defined below, where $d \geqslant 2$ is an integer, has been considered recently as a field of investigations on d-tuples by several authors. We refer to [1, 4, 6, 7, 8, 10, 11, 15, 22, 29] and the references therein.

In the sequel, we will focus on the spherical generalized Aluthge transform of d-tuples of operators on $\mathscr{H}$. We are essentially concerned with generalisations of numerical range operator inequalities for single operators. Our goal is to provide appropriate extensions in the setting of d-tuples of numerical range inequalities of an operator and its Aluthge transform satisfied by single operators.

We start with some definitions and notations. A d-tuple $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H}, \mathscr{H}^d)$ of bounded operators on a Hilbert space $\mathscr{H}$ is said to be a commuting d-tuple of operators if $T_n T_m = T_m T_n$ for every $1 \leqslant n, m \leqslant d$. The joint norm $\|\mathbf{T}\|$ of $\mathbf{T}$, the spectral radius $r(\mathbf{T})$ of $\mathbf{T}$, the joint numerical range $W(\mathbf{T})$ of $\mathbf{T}$, and the numerical radius $w(\mathbf{T})$ of $\mathbf{T}$, are defined as follows

$$\begin{aligned}
\|\mathbf{T}\| &= \sup\left\{ \left(\sum_{k=1}^{d} \|T_k x\|^2 \right)^{\frac{1}{2}} : x \in \mathscr{H}, \|x\| = 1 \right\}; \\
r(\mathbf{T}) &= \sup\{\|\lambda\| : \lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_T(\mathbf{T})\}; \\
W(\mathbf{T}) &= \{(\langle T_1 x, x\rangle, \ldots, \langle T_d x, x\rangle) : x \in \mathscr{H}, \|x\| = 1\}; \\
w(\mathbf{T}) &= \sup\{\|\lambda\| : \lambda = (\lambda_1, \ldots, \lambda_d) \in W(\mathbf{T})\}.
\end{aligned}$$

Where $\sigma_T(\mathbf{T})$ denotes the usual Taylor joint spectrum of $\mathbf{T}$, defined for commuting d-tuples. For a detailed study on joint spectral theory of commuting d-tuples, we refer to [26].

It is well known that the numerical radius defines an equivalent operator norm on $\mathscr{B}(\mathscr{H}, \mathscr{H}^d)$ for every $d \geqslant 1$ and that for $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H}, \mathscr{H}^d)$, we have

$$\frac{1}{2\sqrt{d}} \|\mathbf{T}\| \leqslant w(\mathbf{T}) \leqslant \|\mathbf{T}\|.$$

See [23] for example.

As extension of the canonical polar decomposition of a single operator, the spherical polar decomposition of a d-tuple $\mathbf{T}$ of operators on $\mathscr{H}$ is given by the expression

$$\mathbf{T} = (T_1, \ldots, T_d) = (V_1, \ldots, V_d)P = (V_1 P, \ldots, V_d P),$$

where $P = \sqrt{\mathbf{T}^*.\mathbf{T}} = \sqrt{T_1^* T_1 + \ldots + T_d^* T_d}$ and $\mathbf{V} = (V_1, \ldots, V_d)$ is a joint partial isometry on $\mathscr{H}$ subject to the condition $\ker \mathbf{V} = \bigcap_{i=1}^{d} \ker V_i = \bigcap_{i=1}^{d} \ker T_i = \ker P$. In particular, $\mathbf{V}^*.\mathbf{V} = \sum\limits_{k=1}^{d} V_k^* V_k$ is the orthogonal projection onto the initial space $(\ker \mathbf{V})^{\perp} = \mathscr{R}(P)$.

Analogously, for $0 \leqslant s \leqslant 1$, the generalized spherical Aluthge transform $\widehat{\mathbf{T}}^s$ of $\mathbf{T}$ is defined by

$$\widehat{\mathbf{T}}^s = (P^s V_1 P^{1-s}, \ldots, P^s V_d P^{1-s}) = P^s (V_1, \ldots, V_d) P^{1-s}.$$

When $s = \frac{1}{2}$, we have $\widehat{\mathbf{T}}^{\frac{1}{2}} = \widehat{\mathbf{T}} = (\sqrt{P}V_1\sqrt{P}, \ldots, \sqrt{P}V_d\sqrt{P})$ is the classical spherical Aluthge transform of $\mathbf{T}$. Also under the convention $P^0 = I$, we get $\widehat{\mathbf{T}}^0 = \mathbf{T}$ and $\widehat{\mathbf{T}}^1 = \widehat{\mathbf{T}}^D = P\mathbf{V} = (PV_1, \ldots, PV_d)$ is the usual spherical Duggal transform of $\mathbf{T}$.

Our investigations hereafter are mostly devoted to the numerical range of generalized spherical Aluthge transforms. We focus primarily on generalizations of several known operator inequalities for single operators which have been obtained recently for spherical Aluthge transforms to the case of generalized spherical Aluthge transforms of d-tuples.

This paper is organized as follows. In Section 2, we prove inequalities involving the joint operator norm and the joint numerical radius in the case of the generalized spherical Aluthge transform. Mainly, we extend two recent results of K. Feki and T. Yamazaki [13, Theorem 2.1 and Theorem 2.2] by showing that for an arbitrary d-tuple $\mathbf{T}$ and $0 \leqslant s \leqslant 1$, we have

$$\|\widehat{\mathbf{T}}^s\| \leqslant \|\mathbf{T}\| \quad \text{and} \quad w(\widehat{\mathbf{T}}^s + \widehat{\mathbf{T}}^{1-s}) \leqslant w(\mathbf{T}) + w(\mathbf{T}^D) \leqslant 2w(\mathbf{T}).$$

In addition, we provide a simple proof of the recent result, [1, Theorem 2.6] that improves [13, Theorem 3.1]. More precisely, if $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^d$, then

$$w(\mathbf{T}) \leqslant \inf_{0 \leqslant s \leqslant 1} \left(\frac{1}{4}(\|\mathbf{T}\|^{2s} + \|\mathbf{T}\|^{2(1-s)}) + \frac{1}{2}w(\widehat{\mathbf{T}}^s) \right).$$

We use the previous extension to obtain further results extending known ones as for Corollary 2.10, Corollary 2.11 and Proposition 2.13 below. As an application of our calculations, we retrieve some recent results from [13, 28] by using alternative proofs. Section 3 is devoted to the link between the existence of nontrivial joint invariant subspaces for the generalized Aluthge transform and for the original d-tuple of bounded operators.

2. Numerical range and generalized spherical Aluthge transforms

In a proof relying on classical operator inqualities, K. Feki and T. Yamazaki show in [13, Theorem 2.1], that for an arbitrary d-tuple of operators $\mathbf{T} = (T_1, \ldots, T_d)$, the next inequality holds

$$\|\widehat{\mathbf{T}}\| \leqslant \|\mathbf{T}\|. \tag{1}$$

We extend (1) to generalized spherical Aluthge transforms, using a short proof.

PROPOSITION 2.1. *Let $\mathbf{T} = (T_1, \ldots, T_d)$ be a d-tuple of operators. Then, for every $0 \leqslant s \leqslant 1$, we have*

$$\|\widehat{\mathbf{T}}^s\| \leqslant \|\mathbf{T}\|.$$

Proof. We have $\|\mathbf{V}\| = \sup_{\|x\|=1} (\sum_{k=1}^{d} \|V_k x\|^2)^{\frac{1}{2}} \leqslant 1$ and since P is selfadjoint, we obtain $\|P^a\| = \|P\|^a = \|\mathbf{T}\|^a$ for every $a > 0$. Thus

$$\|\widehat{\mathbf{T}}^s\| = \|P^s\mathbf{V}P^{1-s}\| \leqslant \|P^s\|\|\mathbf{V}\|\|P^{1-s}\| \leqslant \|P^s\|\|P^{1-s}\| = \|\mathbf{T}\|.$$

We also mention the next inequality stated in [13, Theorem 2.2]. For $\mathbf{T}\in\mathscr{B}(\mathscr{H})^d$, we have

$$w(\widehat{\mathbf{T}})\leqslant\frac{1}{2}(w(\mathbf{T})+w(\mathbf{T}^D)). \tag{2}$$

We have the next extension of (2). $\square$

THEOREM 2.2. *Let* $\mathbf{T}=(T_1,\ldots,T_d)\in\mathscr{B}(\mathscr{H})^d$ *be a* d*-tuple of operators and* $0\leqslant s\leqslant 1$*. Then*

$$w(\widehat{\mathbf{T}}^s+\widehat{\mathbf{T}}^{1-s})\leqslant w(\mathbf{T})+w(\mathbf{T}^D).$$

The following observation from [27] is to be used in our proof. For $T\in\mathscr{B}(\mathscr{H})$, we have

$$w(T)=\sup_{\theta\in\mathbb{R}}\|\Re(e^{i\theta}T)\|,\ \text{ where }\ \Re(T)=\frac{T+T^*}{2}. \tag{3}$$

Also, we will use two known results.

THEOREM 2.3. [25, Theorem 4] *Let* $T\in\mathscr{B}(\mathscr{H})$*. Then*

$$\overline{W(T)}=\bigcap_{\mu\in\mathbb{C}}\{z:|z-\mu|\leqslant\|T-\mu I\|\},$$

where $\overline{W(T)}$ *means the closure of the numerical range of* T*.*

LEMMA 2.4. [16, Theorem 3.12.1] *Let* S_1 *and* S_2 *be positive operators and* $Q\in\mathscr{B}(\mathscr{H})$*. For every* $0\leqslant\alpha\leqslant 1$*, we have*

$$\|S_1^{\alpha}QS_2^{1-\alpha}+S_1^{1-\alpha}QS_2^{\alpha}\|\leqslant\|S_1Q+QS_2\|.$$

Proof of Theorem 2.2. We notice first that since $P\geqslant 0$, we have $P+\varepsilon I$ is invertible for every $\varepsilon>0$. Let us then denote $V_{i_\varepsilon}=V_i-\mu(P+\varepsilon I)^{-1}$ for $i=1,\ldots,d$. From Lemma 2.4, we derive

$$\|P^sV_{i_\varepsilon}P^{1-s}+P^{1-s}V_{i_\varepsilon}P^s\|\leqslant\|PV_{i\varepsilon}+V_{i_\varepsilon}P\|.$$

Moreover,

$$P^sV_{i_\varepsilon}P^{1-s}=P^sV_iP^{1-s}-\mu P^s(P+\varepsilon I)^{-1}P^{1-s}=P^sV_iP^{1-s}-\mu P(P+\varepsilon I)^{-1}$$

and similarly

$$P^{1-s}V_{i_\varepsilon}P^s=P^{1-s}V_iP^s-\mu P(P+\varepsilon I)^{-1}.$$

Now, by using Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{\varepsilon\to 0}P^s(P+\varepsilon I)^{-1}P^{1-s}=\lim_{\varepsilon\to 0}P^{1-s}(P+\varepsilon I)^{-1}P^s=\lim_{\varepsilon\to 0}P(P+\varepsilon I)^{-1}=I.$$

This yields

$$\|P^sV_iP^{1-s}+P^{1-s}V_iP^s-2\mu I\|\leqslant\|PV_i+V_iP-2\mu I\|.$$

Therefore, from Theorem 2.3, it follows that

$$\overline{W(P^sV_iP^{1-s}+P^{1-s}V_iP^s)} \subset \overline{W(PV_i+V_iP)} \subset \{\, \overline{W(PV_i)}+\overline{W(V_iP)}\}.$$

Taking the supremum, we deduce that

$$w(\widehat{\mathbf{T}}^s+\widehat{\mathbf{T}}^{1-s}) \leqslant w(\mathbf{T})+w(\mathbf{T}^D).$$

Recall that two d-tuples of operators $\mathbf{T}=(T_1,\ldots,T_d)$ and $\mathbf{S}=(S_1,\ldots,S_d)$, are said to be criss-cross commuting if, for every $1\leqslant i,j,k\leqslant d$ we have $T_iS_jT_k=T_kS_jT_i$ and $S_iT_jS_k=S_kT_jS_i$. It is not difficult to see that if $\mathbf{T}=(T_1,\ldots,T_d)$ and $\mathbf{S}=(S_1,\ldots,S_d)$, are criss-cross commuting, then $\mathbf{ST}=(S_1T_1,\ldots,S_dT_d)$ is a commuting d-tuple if and only if $\mathbf{TS}=(T_1S_1,\ldots,T_dS_d)$ is a commuting d-tuple. See [8] for further properties on criss-cross commuting d-tuples. $\square$

The next proposition written in [5, Lemma 2.6] for commuting pairs $(d=2)$ is valid for arbitrary commuting d-tuples.

PROPOSITION 2.5. *Let* $\mathbf{T}=(T_1,\ldots,T_d)=(V_1P,\cdots,V_dP)$ *be a commuting* d*-tuple of operators on* $\mathscr{H}$. *We have the following*

- $\mathbf{V}=(V_1,\cdots,V_d)$ *and* $\mathbf{P}=(P,\cdots,P)$ *are criss-cross commuting;*
- *For every* $0\leqslant s\leqslant 1$, *the generalized spherical Aluthge transform* $\widehat{\mathbf{T}}^s$ *is a commuting* d*-tuple.*

In the following, we recover [13, Theorem 2.3] in an alternative way.

PROPOSITION 2.6. *Let* $\mathbf{T}$ *be a* d*-commuting tuple. Then*

$$w(\mathbf{T}^D)\leqslant w(\mathbf{T}).$$

Proof. Since $\sum\limits_{k=1}^{d}(V_k^*V_k)$ is a projection onto $\overline{\mathscr{R}(P)}$, we have

$$\begin{aligned}
|\langle P\mathbf{V}x,x\rangle| &= |\langle \sum_{i=1}^{d} PV_ix,x\rangle| \\
&= |\langle \sum_{k=1}^{d}(\,V_k^*V_k\,)\sum_{i=1}^{d} PV_ix,x\rangle| \\
&= |\sum_{k=1}^{d}\sum_{i=1}^{d}\langle V_kPV_ix,V_kx\rangle| \\
&= |\sum_{k=1}^{d}\sum_{i=1}^{d}\langle V_iPV_kx,V_kx\rangle| \quad \text{(by proposition 2.5)} \\
&\leqslant \sum_{k=1}^{d}\|V_k\|^2|\sum_{i=1}^{d}\langle V_iPy_k,y_k\rangle| \quad \left(\text{where } y_k=\frac{V_kx}{\|V_k\|}\right) \\
&\leqslant w(\mathbf{V}P).
\end{aligned}$$

By taking the supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$, we get $w(\mathbf{T}^D) \leqslant w(\mathbf{T})$ as required. $\square$

The next extension of [13, Theorem 3.1] to the setting of generalized spherical Aluthge transforms can be found in [1, Theorem 2.6]. We provide a new short proof.

THEOREM 2.7. *Let* $\mathbf{T} = (T_1, \dots, T_d) \in \mathscr{B}(\mathscr{H})^d$ *and* $0 < s < 1$. *Then*

$$w(\mathbf{T}) \leqslant \inf_{0 \leqslant s \leqslant 1} \left(\frac{1}{4}(\|\mathbf{T}\|^{2s} + \|\mathbf{T}\|^{2(1-s)}) + \frac{1}{2} w(\widehat{\mathbf{T}}^s) \right).$$

We start by showing the next auxiliary result, needed in the proof of Theorem 2.7.

LEMMA 2.8. *Let* $\mathbf{T} = (T_1, \dots, T_d) = (V_1P, \dots, V_dP) \in \mathscr{B}(\mathscr{H})^d$. *For every* $\theta \in \mathbb{R}$, *we have*

$$\Re(\, e^{i\theta} V_i P \,) \leqslant \frac{1}{4}(e^{-i\theta} P^s + V_i P^{1-s})(e^{i\theta} P^s + P^{1-s} V_i^*).$$

Proof. For $i = 1, \dots, d$ and $\theta \in \mathbb{R}$

$$\begin{aligned}
&4\Re(e^{i\theta} V_i P) - (e^{-i\theta} P^s + V_i P^{1-s})(e^{i\theta} P^s + P^{1-s} V_i^*) \\
&= -P^{2s} + e^{-i\theta} P V_i^* + e^{i\theta} V_i P - V_i P^{2(1-s)} V_i^* \\
&= -(e^{-i\theta} P^s - V_i P^{1-s})(e^{i\theta} P^s - P^{1-s} V_i^*\,) \\
&\leqslant 0,
\end{aligned}$$

as desired. $\square$

Proof of Theorem 2.7.

$$\begin{aligned}
\|\Re(e^{i\theta}\mathbf{T})\| &\leqslant \frac{1}{4}\|\sum_{i=1}^{d}(e^{-i\theta} P^s + V_i P^{1-s})(e^{i\theta} P^s + P^{1-s} V_i^*)\| \quad \text{(Lemma 2.8)} \\
&= \frac{1}{4}\|\sum_{i=1}^{d}(e^{i\theta} P^s + P^{1-s} V_i^*)(e^{-i\theta} P^s + V_i P^{1-s})\| \quad (\|X^*X\| = \|XX^*\|) \\
&= \frac{1}{4}\|\sum_{i=1}^{d} P^{2s} + P^{1-s} V_i^* V_i P^{1-s} + 2\Re(e^{i\theta} P^s V_i P^{1-s})\| \\
&\leqslant \frac{1}{4}\|P^{2s}\| + \frac{1}{4}\|P^{2(1-s)}\|\|\sum_{i=1}^{d} V_i^* V_i\| + \frac{1}{2}\|\sum_{i=1}^{d}\Re(e^{i\theta} P^s V_i P^{1-s})\| \\
&= \frac{1}{4}\|T\|^{2s} + \frac{1}{4}\|T\|^{2(1-s)}\|\sum_{i=1}^{d} V_i^* V_i\| + \frac{1}{2}\|\sum_{i=1}^{d}\Re(e^{i\theta} P^s V_i P^{1-s})\| \\
&\leqslant \frac{1}{4}\|T\|^{2s} + \frac{1}{4}\|T\|^{2(1-s)} + \frac{1}{2} w(P^s V_i P^{1-s}).
\end{aligned}$$

We conclude by taking the supremum over all $\theta \in \mathbb{R}$, and by using Expression 3 in the above inequality. $\square$

REMARK 2.9. In Theorem 2.7, we denote $\phi(s) = \frac{1}{4}(\|\mathbf{T}\|^{2s} + \|\mathbf{T}\|^{2(1-s)}) + \frac{1}{2}w(\widehat{\mathbf{T}}^s)$. Is it true that $\phi(\frac{1}{2}) = \inf_{s\in[0,1]} \phi(s)$?

After the submission of this paper, the referee pointed to us the recent reference [29], where Example 3.2 provides a negative answer to the previous question. It is then natural to rephrase the previous question in the following sense. Let $s_0 \in [0,1]$ be such that $\phi(s_0) = \inf_{s\in[0,1]} \phi(s)$. What information adds s_0 to the spectral structure of T?

Using the same proof as in the single case provided in [27], it is not difficult to see that $w(\widehat{\mathbf{T}}) \leqslant \|\mathbf{T}^2\|^{\frac{1}{2}}$. We derive the next consequence of Theorem 2.7.

COROLLARY 2.10. *Let $\mathbf{T} = (T_1, \ldots, T_d)$ be a commuting d-tuple of operators on $\mathscr{H}$. For $s = \frac{1}{2}$, we have*

$$\begin{aligned} 1)\ w(\widehat{\mathbf{T}}) \leqslant w(\mathbf{T}) &\leqslant \tfrac{1}{2}(\|\mathbf{T}\| + w(\widehat{\mathbf{T}})) \quad \text{[[13], Theorem 3.1]} \\ &\leqslant \tfrac{1}{2}(\|\mathbf{T}\| + \|\mathbf{T}^2\|^{\frac{1}{2}}) \ \text{[[20], Theorem 1]} \\ &\leqslant \|\mathbf{T}\|. \end{aligned}$$

2) *If* $\widehat{\mathbf{T}} = 0$ *or* $\mathbf{T}^2 = 0$, *then* $w(\mathbf{T}) = \frac{1}{2}\|\mathbf{T}\|$.

Recall that an operator $T \in \mathscr{H}$ is said to be a normaloid if $\|T\| = r(T)$. It is known that T is a normaloid if and only if $\|T\| = w(T)$. For further results on normaloid operators, see [16] for example. The following corollary we deal with normaloid Aluthge transforms of commuting d-tuple of operators.

COROLLARY 2.11. *Let $\mathbf{T} = (T_1, \ldots, T_d)$ be a commuting d-tuple of operators. We have*

- *$\mathbf{T}$ is a normaloid, $\iff$ $\widehat{\mathbf{T}}$ is normaloid.*
- *If $\mathbf{T}$ is a normaloid, then $w(\mathbf{T}) = w(\widehat{\mathbf{T}})$.*

F. Kittanneh established in [19] that the numerical range of a bounded operator satisfies $w^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|$. In [12] S. S. Dragomir proved that $w^2(T) \leqslant \frac{1}{2}(w(T^2) + \|T\|^2)$. Also the following sharper inequality was obtained recently by S. Bag, P. Bhunia and K. Paul in [3].

PROPOSITION 2.12. [3, Theorem 2.5] *Let $T \in \mathscr{B}(\mathscr{H})$. Then*

$$w^2(T) \leqslant \frac{1}{2}\|T\|\|\hat{T}\| + \frac{1}{4}\|T^*T + TT^*\| \leqslant \|T\|^2.$$

Our next result generalizes Proposition 2.12 for d-tuples.

PROPOSITION 2.13. *Let $\mathbf{T} = (T_1, \ldots, T_d) = (V_1P, \ldots, V_dP) \in \mathscr{B}(\mathscr{H})^d$. Then*

$$w^2(\mathbf{T}) \leqslant \frac{1}{2}w(\mathbf{T}^2) + \frac{1}{4}\|\mathbf{T}^*\mathbf{T} + \mathbf{T}\mathbf{T}^*\| \leqslant \frac{1}{2}\|\mathbf{T}\|\|\widehat{\mathbf{T}}^s\| + \frac{1}{4}\|\mathbf{T}^*\mathbf{T} + \mathbf{T}\mathbf{T}^*\| \leqslant \|\mathbf{T}\|^2. \quad (4)$$

Proof. By writing $\Re(e^{i\theta}\mathbf{T}) = \frac{1}{2}(e^{i\theta}\mathbf{T}+e^{-i\theta}\mathbf{T}^*)$, it follows that

$$\begin{aligned}4\|(\Re(e^{i\theta}\mathbf{T}))^2\| &= \|\sum_{k=1}^{d} e^{2i\theta}T_k^2+e^{-2i\theta}T_k^{*2}+T_k^*T_k+T_kT_k^*\| \\ &\leqslant \|2\Re(e^{i2\theta}\mathbf{T}^2)\|+\|\mathbf{T}^*\mathbf{T}+\mathbf{T}\mathbf{T}^*\| \\ &\leqslant 2w(\mathbf{T}^2)+\|\mathbf{T}^*\mathbf{T}+\mathbf{T}\mathbf{T}^*\|.\end{aligned}$$

Now, by taking the supremum over all $\theta\in\mathbb{R}$, and then using Expression (3), we obtain the first inequality.

For the second inequality, we have

$$\begin{aligned}w(\mathbf{T}^2)^2 = \sup_{\|x\|\leqslant 1}|\langle\mathbf{T}^2x,x\rangle|^2 &= \sup_{\|x\|\leqslant 1}\sum_{k=1}^{d}|\langle V_kP^{1-s}P^sV_kP^{1-s}P^sx,x\rangle|^2 \\ &= \sup_{\|x\|\leqslant 1}\sum_{k=1}^{d}|\langle P^sV_kP^{1-s}P^sx,P^{1-s}V_k^*x\rangle^2| \\ &\leqslant \sup_{\|x\|\leqslant 1}\sum_{k=1}^{d}(\|\widehat{\mathbf{T}}^s\|\|P^s\|\|P^{1-s}\|)^2\|V_k^*x\|^2 \\ &\leqslant \|\widehat{\mathbf{T}}^s\|^2\|\mathbf{T}\|^2 \quad (\text{since } \sum_{k=1}^{d}\|V_k^*x\|^2\leqslant 1).\end{aligned}$$

The last inequality is obvious and the proof is complete. □

In [1, Theorem 3.1], it is shown that $w^2(\mathbf{T})\leqslant\frac{1}{2}\|\mathbf{T}^*\mathbf{T}+\|\mathbf{T}\|^2I\|\leqslant\|\mathbf{T}\|^2$. Noticing that $\mathbf{T}^*\mathbf{T}+\mathbf{T}\mathbf{T}^*\leqslant\mathbf{T}^*\mathbf{T}+\|\mathbf{T}\|^2I$, we can recover the previous inequality from the next result

PROPOSITION 2.14. *Let $\mathbf{T}=(T_1,\dots,T_d)=(V_1P,\dots,V_dP)\in\mathscr{B}(\mathscr{H})^d$. Then*

$$w^2(\mathbf{T})\leqslant\frac{1}{2}\|\mathbf{T}^*\mathbf{T}+\mathbf{T}\mathbf{T}^*\|\leqslant\|\mathbf{T}\|^2. \tag{5}$$

Proof. Thanks to Proposition 2.13, it suffices to see that $w(\mathbf{T}^2)\leqslant\frac{1}{2}\|\mathbf{T}^*\mathbf{T}+\mathbf{T}\mathbf{T}^*\|$. Indeed, using polarisation formula for the inner product on $x\in\mathscr{H}$, we obtain

$$\langle\mathbf{T}^2x,x\rangle=\sum_i\langle T_ix,T_i^*x\rangle=\sum_i\frac{1}{4}[\|(T_i+T_i^*)x\|^2-\|(T_i-T_i^*)x\|^2],$$

for every $x\in\mathscr{H}$. In addition

$$\begin{aligned}\|(T_i+T_i^*)x\|^2-\|(T_i-T_i^*)x\|^2 &= \langle((T_i+T_i^*)^2-(T_i-T_i^*)^2)x,x\rangle \\ &= 2\langle(T_iT_i^*+T_i^*T_i)x,x\rangle.\end{aligned}$$

It follows that

$$\langle\mathbf{T}^2x,x\rangle=\frac{1}{2}\langle(T_iT_i^*+T_i^*T_i)x,x\rangle,$$

and the required result derives by taking the supremum for $\|x\|=1$. □

3. Joint invariant subspace

In [17], the authors prove that an operator $T \in \mathscr{B}(\mathscr{H})$ has a nontrivial invariant subspace if and only if $\hat{T}$ does. We extend in this section this result to the more general setting of commuting d-tuples of bounded operators and for generalized Aluthge transforms. Recall first the next definition.

DEFINITION 3.1. Let $\mathbf{T} = (T_1, \ldots, T_d)$ be a d-tuple of operators on a Hilbert space $\mathscr{H}$. A closed subspace M in $\mathscr{H}$ is said to be a joint invariant subspace of $\mathbf{T}$ if $T_i(M) \subset M$ for every $1 \leqslant i \leqslant d$. We denote by $JLat(\mathbf{T})$ the lattice of all joint invariant subspaces of $\mathbf{T}$.

We have the following extension of [17].

PROPOSITION 3.2. *Let $\mathbf{T} = (T_1, \ldots, T_d)$ be a commuting d-tuple of operators and $0 \leqslant s \leqslant 1$. Then*

$$JLat(\mathbf{T}) \neq \{0, \mathscr{H}\} \iff JLat(\widehat{\mathbf{T}}^s) \neq \{0, \mathscr{H}\},$$

where 0 in the trivial null space.

Proof. $(\Rightarrow)$ Let $\mathscr{M} \in JLat(\widehat{\mathbf{T}}^s)$ be nontrivial and denote $\mathscr{N} = span_j V_j P^{1-s}(\mathscr{M})$. First, if $\mathscr{N} = 0$, then $\mathscr{M} \in JLat(\mathbf{T})$. Otherwise, we will show that $\mathscr{N} \in JLat(\mathbf{T})$. From $P^s V_j P^{1-s}\mathscr{M} \subset \mathscr{M}$ for every $j = 1, \cdots, d$, we derive that

$$T_i \mathscr{N} = V_i P \mathscr{N} = V_i P span_j(V_j P^{1-s}\mathscr{M}) = V_i P^{1-s} span_j(P^s V_j P^{1-s}\mathscr{M}) \subset V_i P^{1-s}\mathscr{M} \subset \mathscr{N},$$

and then $\mathscr{N} \in JLat(\mathbf{T})$.

$(\Leftarrow)$ Let $\mathscr{M} \in JLat(\mathbf{T})$ be nontrivial and denote $\mathscr{N} = P^s(\mathscr{M})$. Again, if $\mathscr{N} = 0$, then $\mathscr{M} \in JLat(\widehat{\mathbf{T}}^s)$, and if $\mathscr{N} \neq 0$, we will get $\mathscr{N} \in JLat(\widehat{\mathbf{T}}^s)$. Indeed, for $j = 1, \cdots, d$, we have

$$P^s V_j P^{1-s}(\mathscr{N}) = P^s V_j P \mathscr{M} \subset P^s \mathscr{M} = \mathscr{N}.$$

Finally $JLat(\mathbf{T}) \neq \{0, \mathscr{H}\} \iff JLat(\widehat{\mathbf{T}}^s) \neq \{0, \mathscr{H}\}$. $\square$

From the proof of the previous result, if $\ker(\mathbf{T}) =: \cap_{i \leqslant d} \ker(T_i) = 0$, then the mappings

$$\phi : \mathscr{M} \in JLat(\mathbf{T}) \to \mathscr{N} = P^s(\mathscr{M}) \in JLat(\widehat{\mathbf{T}}^s), \text{ and}$$

$$\psi : \mathscr{N} \in JLat(\widehat{\mathbf{T}}^s) \to \mathscr{M} = span_j V_j P^{1-s}(\mathscr{N}) \in JLat(\mathbf{T})$$

are one to one. We are however not able to show that $JLat(\mathbf{T})$ and $JLat(\widehat{\mathbf{T}}^s)$ are isomorphic in general. It is also known that even in the single case, that there are operators T such that $Lat(T)$ and $Lat(\widehat{T})$ are not isomorphic.

It is then natural to ask.

Let $\mathbf{T}$ be a commuting d-tuple. Is there any relation between $JLat(\mathbf{T})$ and $JLat(\widehat{\mathbf{T}}^s)$?

Our second question concerns the lattice of joint hyper invariant subspaces. More precisely, M is said to be hyper invariant subspaces for T if it is invariant under all operators commuting with T. The lattice of hyper invariant subspaces of T is denoted $HLat(T)$. We also denote $JHLat(\mathbf{T})$ for the lattice of joint hyper invariant subspaces of $\mathbf{T}$. In contrast with the lattice of invariant subspaces, even for a single operator, it is not true that $HLat(T) \neq \{0,\mathscr{H}\} \iff HLat(\widehat{T}^s) \neq \{0,\mathscr{H}\}$ as shown in [17, Example 1.7]. In the other direction if T is a quasi-affine transformation (T is one to one and has closed range), then $HLat(T) \neq \{0,\mathscr{H}\} \iff HLat(\widehat{T}^s) \neq \{0,\mathscr{H}\}$. See [17, Theorem 2.5]. The next question arises naturally.

Let $\mathbf{T}$ be a d-tuple of quasi-affine transformations, do we have

$$JHLat(\mathbf{T}) \neq \{0,\mathscr{H}\} \iff JHLat(\widehat{\mathbf{T}}^s) \neq \{0,\mathscr{H}\}?$$

The answer is clearly affirmative if $\mathbf{T}$ is a commuting tuples of quasi-affine transformations since in this case we have $JHLat(\mathbf{T}) = HLat(T_i)$ for every $1 \leqslant i \leqslant d$.

4. Additional remarks and comments

After the submission of this paper, the referee mentioned to us the recent paper [1] where many of our results on upper bounds of numerical radius are stated in [1] with different proofs. The proofs we present in this paper seems to be simpler, Theorem 2.7 above and Theorem 2.6 in [1], for example. Also in [1], the Euclidian norm of $\mathbf{T}$ introduced by G. Popescu in [23] was central in many expressions, while for us it does not appear anywhere.

We have considered along this paper $\mathbf{T} = (T_1,\dots,T_d) = (V_1P,\cdots,V_dP)$ a d-tuple of operators on $\mathscr{H}$ and its generalized Aluthge transform.

If we denote $\mathbf{A} = (V_1,\cdots,V_d)P^{1-s}$ and $\mathbf{B} = (P^s,\cdots,P^s)$, we remark that $\mathbf{T} = \mathbf{AB}$ and $\widehat{\mathbf{T}}^s = \mathbf{BA}$. Now using general result on common spectral properties of $\mathbf{AB}$ and $\mathbf{BA}$, as in [6] for example, several common spectral properties between $\mathbf{T}$ and $\widehat{\mathbf{T}}^s$ will follow. However, for results involving relations between operator inequalities of $\mathbf{T}$ and $\widehat{\mathbf{T}}^s$, no similar approach based of this commutation property has been developed.

We end this section with a recent extension of generalized Aluthge transform that fits with the previous context. To be precise, M. Bakherad and K. Shebrawi introduced in [24] the (f,g)-Aluthge transform as $\Delta_{f,g}(T) = f(|T|)Ug(|T|)$ for $T \in \mathscr{B}(\mathscr{H})$, and where f and g are both non-negative functions such that $f(x)g(x) = x$ for all $x \geqslant 0$. It is clear that for $A = g(|T|)$ and $B = f(|T|)U$, we have $AB = T$ and $BA = \Delta_{f,g}(T)$. Thus the $AB-BA$ approach applies to find common spectral properties in this setting. Several interesting numerical range operator inequalities involving convex functions have been obtained in [24]. The notion of (f,g)-spherical Aluthge transform of $\mathbf{T} = (T_1,\dots,T_d) = (V_1P,\cdots,V_dP) \in \mathscr{B}(\mathscr{H})^d$ has been defined in [29], by

$$\Delta_{f,g}(\mathbf{T}) = f(P)(V_1,\cdots,V_d)g(P).$$

The authors in [29], extend various results on generalized Aluthge transform to the promising class consisting in (f,g)-spherical Aluthge transforms. This transformation is very large and has wide range of application, it will be useful to develop adequate techniques to tackle more problems on related inequalities.

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