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A LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION AND SEVERAL INEQUALITIES FOR q–MULTINOMINAL COEFFICIENTS AND q–MULTIVARIATE BETA FUNCTIONS

FENG QI

Dedicated to Dr. Professor Ravi Prakash Agarwal at Texas A&M University-Kingsville

(*Communicated by I. Perić*)

Abstract. In the paper, the author proves the logarithmically complete monotonicity of a function involving q-gamma functions and apply the logarithmically complete monotonicity to derive some inequalities for q-multinomial coefficients and q-multivariate beta functions. These conclusions generalize corresponding ones for gamma functions, multinomial coefficients, and multivariate beta functions, respectively.

1. Introduction and preliminaries

Suppose that $h : I \to [0,\infty)$ is an infinite differentiable function on an interval I. If the inequality $0 \leqslant (-1)^{m-1}h^{(m-1)}(u) < \infty$ is valid for $m \in \mathbb{N}$, then we call h a completely monotonic function; if the function $h : I \to (0,\infty)$ is positive and the inequality $(-1)^m[\ln h(u)]^{(m)} \geqslant 0$ is valid for $m \in \mathbb{N}$, then we call h a logarithmically completely monotonic function; if the function $h : (0,\infty) \to [0,\infty)$ can be written in the form $h(u) = \frac{a}{u} + b + \int_0^\infty \frac{1}{s+u} \mathrm{d}\mu(s)$, where a,b are non-negative constants and μ is a measure on $(0,\infty)$ such that $\int_0^\infty \frac{1}{1+s} \mathrm{d}\mu(s) < \infty$, then we call h a Stieltjes transform. For more details on definitions and properties of these kinds of functions, please see [15, Chapter XIII], [40, Chapter 1], and [41, Chapter IV]. Among these three kinds of functions, there are the following relations:

(i) a function h is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform, i.e., there is a positive measure μ on $[0,\infty)$ such that $h(u) = \int_0^\infty \mathrm{e}^{-us} \mathrm{d}\mu(s)$;

(ii) the set of all logarithmically completely monotonic functions is a strict subset of all completely monotonic functions;

(iii) the set of all Stieltjes transforms is a strict subset of all logarithmically completely monotonic functions on $(0,\infty)$.

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For details on these relations, please refer to [4, 7, 28, 30, 40] and closely related references therein. For information on new developments of these kinds of functions, please refer to [6, 8, 26, 27, 31, 34, 38, 39, 40] and closely related references therein.

The classical Euler's gamma function $\Gamma(w)$ can be defined [25] by

$$\Gamma(w)=\int_0^\infty u^{w-1}\mathrm{e}^{-u}\,\mathrm{d}u,\quad \Re(w)>0$$

or by

$$\Gamma(w)=\lim_{m\to\infty}\frac{m!m^w}{\prod_{j=0}^m(w+j)},\quad w\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

The logarithmic derivative of $\Gamma(w)$, denoted by $\psi(w)=\frac{\Gamma'(w)}{\Gamma(w)}$, and the derivatives $\psi^{(j)}(w)$ for $j\in\mathbb{N}$ are respectively called the digamma (or psi) function and polygamma functions. See [1, Chapter 6] and [16, Chapter 5].

The q-analogue $\Gamma_q(t)$ of the gamma function $\Gamma(t)$ for $q>0$ and $t>0$, called the q-gamma function, can be defined [3, pp. 493–496] and [5, Section 1.10] by

$$\Gamma_q(t)=\begin{cases}(1-q)^{1-t}\displaystyle\prod_{j=0}^\infty\frac{1-q^{j+1}}{1-q^{j+t}}, & 0<q<1;\\ (q-1)^{1-t}q^{\binom{t}{2}}\displaystyle\prod_{j=0}^\infty\frac{1-q^{-(j+1)}}{1-q^{-(j+t)}}, & q>1;\\ \Gamma(t), & q=1.\end{cases}\tag{1}$$

They satisfy

$$\lim_{q\to1^+}\Gamma_q(t)=\lim_{q\to1^-}\Gamma_q(t)=\Gamma(t)\quad\text{and}\quad \Gamma_q(t)=q^{\binom{t-1}{2}}\Gamma_{1/q}(t).$$

The q-analogue $\psi_q(t)$ of the digamma function $\psi(t)$ for $q>0$ and $t>0$, called the q-digamma function, is defined by

$$\psi_q(t)=\begin{cases}\dfrac{\Gamma_q'(t)}{\Gamma_q(t)}, & q\ne1;\\ \psi(t), & q=1.\end{cases}$$

The q-analogues $\psi_q^{(k)}(t)$ of the polygamma functions $\psi^{(k)}(t)$ for $k\in\mathbb{N}$ are called the q-polygamma functions. From the definition (1), we obtain that,

(i) when $0<q<1$ and $t\in(0,\infty)$,

$$\psi_q(t)=-\ln(1-q)+(\ln q)\sum_{j=0}^\infty\frac{q^{j+t}}{1-q^{j+t}}=-\ln(1-q)+(\ln q)\sum_{j=1}^\infty\frac{q^{jt}}{1-q^j},$$

(ii) when $q>1$ and $t\in(0,\infty)$,

$$\psi_q(t)=-\ln(q-1)+(\ln q)\left(t-\frac12-\sum_{j=1}^\infty\frac{q^{-jt}}{1-q^{-j}}\right).$$

The formula (1.11) in the paper [10] and its corrected version [11] reads that

$$\psi_q(t) = -\ln(1-q) - \int_0^\infty \frac{e^{-tu}}{1-e^{-u}} \, d\gamma_q(u) \tag{2}$$

for $0 < q < 1$ and $t > 0$, where

$$\gamma_q(u) = \begin{cases} -\ln q \sum_{j=1}^{\infty} \delta(u + j\ln q), & 0 < q < 1 \\ u, & q = 1 \end{cases} \tag{3}$$

and $\delta(u)$ represents the Dirac delta function, that is, $d\gamma_q(u)$ is a discrete measure with positive masses $|\ln q|$ at the positive points $k|\ln q|$ for $k \in \mathbb{N}$. Accordingly, we obtain

$$\int_0^\infty e^{-tu} \, d\gamma_q(u) = -\frac{q^t \ln q}{1-q^t} \quad \text{and} \quad \int_0^\infty u e^{-tu} \, d\gamma_q(u) = \frac{q^t (\ln q)^2}{(1-q^t)^2}$$

for $0 < q < 1$ and $t > 0$. Differentiating with respect to t on both sides of (2) yields

$$\psi_q^{(j)}(t) = (-1)^{j+1} \int_0^\infty \frac{u^j e^{-tu}}{1-e^{-u}} \, d\gamma_q(u), \quad 0 < q < 1, \quad j \in \mathbb{N}. \tag{4}$$

In [9, p. 1245, Theorem 4.4, (4.15)], [19, Lemma 2.3 and Remark 2.1], [21, Theorem 7.2, (7.5)], and [36, p. 152, Theorem 4.22, (4.20)], it was presented that, when $0 < q < 1$, the identity

$$\psi_q^{(j-1)}(t) - \psi_q^{(j-1)}(t+1) = (\ln q) \frac{d^{j-1}}{dt^{j-1}} \left(\frac{q^t}{1-q^t} \right)$$

is valid for $t \in (0, \infty)$ and $k \in \mathbb{N}$. One can also find these knowledge in [12, 19, 21, 22, 24, 35] and closely related references.

2. Motivations

Let $\boldsymbol{b} = (b_1, b_2, \dots, b_m)$ with $b_j > 0$ for $1 \leqslant j \leqslant m$, let $\boldsymbol{w} = (w_1, w_2, \dots, w_m)$ with $\sum_{j=1}^m w_j = 1$ and $w_j \in (0,1)$ for $1 \leqslant j \leqslant m$, let

$$\binom{\sum_{j=1}^m b_j}{b_1, b_2, \dots, b_m} = \frac{\Gamma\left(1 + \sum_{j=1}^m b_j\right)}{\Gamma(1+b_1)\Gamma(1+b_2)\cdots\Gamma(1+b_m)}$$

denote the multinomial coefficient, and let

$$B(b_1, b_2, \dots, b_m) = \frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_m)}{\Gamma(b_1 + b_2 + \cdots + b_m)}$$

be the multivariate beta function. In [17, 37], the authors considered the function

$$\begin{aligned} \mathscr{Q}(t) = \mathscr{Q}_{\boldsymbol{b},\boldsymbol{w};m}(t) &= \frac{\Gamma\left(1 + t\sum_{j=1}^m b_j\right)}{\prod_{j=1}^m \Gamma(1+b_j t)} \prod_{j=1}^m w_j^{b_j t} \\ &= \binom{t\sum_{j=1}^m b_j}{tb_1, tb_2, \dots, tb_m} \prod_{j=1}^m w_j^{b_j t} = \frac{\sum_{j=1}^m b_j}{\prod_{j=1}^m b_j} \frac{\prod_{j=1}^m w_j^{b_j t}}{t^{m-1} B(b_1 t, b_2 t, \dots, b_m t)} \end{aligned} \tag{5}$$

for $t \in (0,\infty)$ and $m \geqslant 2$, and, among other things, proved the following theorem.

THEOREM 1. ([17, Theorem 2.1] and [37, Thereom 2.2]) *The function* $\mathscr{Q}(t) = \mathscr{Q}_{\boldsymbol{b},\boldsymbol{w};m}(t)$ *defined in* (5) *is logarithmically completely monotonic in* $x \in (0,\infty)$.

For more information on the backgrounds, history, motivations, origins, and applications of the function $\mathscr{Q}(t) = \mathscr{Q}_{\boldsymbol{b},\boldsymbol{w};m}(t)$, please refer to [2, 13, 17, 37] and closely related references therein.

In this paper, we will consider the q-analogue of the function $\mathscr{Q}(t) = \mathscr{Q}_{\boldsymbol{b},\boldsymbol{w};m}(t)$, investigate logarithmically complete monotonicity of $\mathscr{Q}(t)$, and apply the logarithmically complete monotonicity of $\mathscr{Q}(t)$ to derive some inequalities for q-multinomial coefficients and q-multivariate beta functions.

3. A logarithmically completely monotonic function involving q-gamma functions

Let $\boldsymbol{b} = (b_1, b_2, \dots, b_m)$ with $b_j > 0$ for $1 \leqslant j \leqslant m$ and let $\boldsymbol{w} = (w_1, w_2, \dots, w_m)$ with $\sum_{j=1}^{m} w_j = 1$ and $w_j \in (0,1)$ for $1 \leqslant j \leqslant m$. Define

$$\mathscr{Q}_q(t) = \mathscr{Q}_{q;\boldsymbol{b},\boldsymbol{w};m}(t) = \frac{\Gamma_q\left(1 + t\sum_{j=1}^{m} b_j\right)}{\prod_{j=1}^{m} \Gamma_q(1 + b_j t)} \prod_{j=1}^{m} w_j^{b_j t} \tag{6}$$

for $q \in (0,1)$, $t \in (0,\infty)$, and $m \geqslant 2$. It is clear that

$$\lim_{q \to 1^-} \mathscr{Q}_q(t) = \lim_{q \to 1^-} \mathscr{Q}_{q;\boldsymbol{b},\boldsymbol{w};m}(t) = \mathscr{Q}(t) = \mathscr{Q}_{\boldsymbol{b},\boldsymbol{w};m}(t).$$

THEOREM 2. *The function* $\mathscr{Q}_q(t) = \mathscr{Q}_{q;\boldsymbol{b},\boldsymbol{w};m}(t)$ *defined in* (6) *is logarithmically completely monotonic on the infinite interval* $(0,\infty)$.

Proof. From (3) for $0 < q < 1$ and the well-known property $\delta(at) = \frac{\delta(t)}{|a|}$ for $a \neq 0$, it follows that

$$\gamma_q\left(\frac{t}{\tau}\right) = -\ln q \sum_{j=1}^{\infty} \delta\left(\frac{t + j\tau \ln q}{\tau}\right) = -\tau \ln q \sum_{j=1}^{\infty} \delta(t + j\tau \ln q)$$

for $\tau > 0$. This implies that

$$\mathrm{d}\gamma_q\left(\frac{t}{\tau}\right) = \tau\, \mathrm{d}\gamma_q(t). \tag{7}$$

Direct calculation gives

$$\ln \mathscr{Q}_q(t) = \ln \Gamma_q\left(1 + t\sum_{j=1}^{m} b_j\right) - \sum_{j=1}^{m} \ln \Gamma_q(1 + b_j t) + t\sum_{j=1}^{m} b_j \ln w_j,$$

$$[\ln \mathscr{Q}_q(t)]' = \left(\sum_{j=1}^m b_j\right)\psi_q\left(1+t\sum_{j=1}^m b_j\right) - \sum_{j=1}^m b_j\psi_q(1+b_jt) + \sum_{j=1}^m b_j \ln w_j,$$

and

$$[\ln \mathscr{Q}_q(t)]'' = \left(\sum_{j=1}^m b_j\right)^2 \psi_q'\left(1+t\sum_{j=1}^m b_j\right) - \sum_{j=1}^m b_j^2\psi_q'(1+b_jt).$$

Making use of (4) for $j=1$ and the equality (7), it follows that

$$\begin{aligned}\psi_q'(\tau t+1) &= \int_0^\infty \frac{u}{1-\mathrm{e}^{-u}}\mathrm{e}^{-(\tau t+1)u}\,\mathrm{d}\gamma_q(u) = \int_0^\infty \frac{u}{\mathrm{e}^{u}-1}\mathrm{e}^{-\tau t u}\,\mathrm{d}\gamma_q(u)\\ &= \int_0^\infty h\left(\frac{v}{\tau}\right)\mathrm{e}^{-vt}\,\mathrm{d}\gamma_q\left(\frac{v}{\tau}\right) = \tau\int_0^\infty h\left(\frac{v}{\tau}\right)\mathrm{e}^{-vt}\,\mathrm{d}\gamma_q(v),\end{aligned}$$

where $\tau>0$ and $h(u)=\frac{u}{\mathrm{e}^u-1}$. Hence, we have

$$[\ln \mathscr{Q}_q(t)]'' = \int_0^\infty \left[\left(\sum_{j=1}^m b_j\right)^3 h\left(\frac{v}{\sum_{j=1}^m b_j}\right) - \sum_{j=1}^m b_j^3 h\left(\frac{v}{b_j}\right)\right]\mathrm{e}^{-tv}\,\mathrm{d}\gamma_q(v). \qquad (8)$$

By calculus, we have

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[u^3 h\left(\frac{1}{u}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{u^2}{\mathrm{e}^{1/u}-1}\right) = \frac{\mathrm{e}^{1/u}(2u+1)-2u}{(\mathrm{e}^{1/u}-1)^2},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left[u^3 h\left(\frac{1}{u}\right)\right] = \frac{2u^2-\mathrm{e}^{1/u}(4u^2+2u-1)+\mathrm{e}^{2/u}(2u^2+2u+1)}{(\mathrm{e}^{1/u}-1)^3u^2} \to 0,$$

$$\frac{\mathrm{d}^3}{\mathrm{d}u^3}\left[u^3 h\left(\frac{1}{u}\right)\right] = \frac{\mathrm{e}^{1/u}(\mathrm{e}^{2/u}+4\mathrm{e}^{1/u}+1)}{(\mathrm{e}^{1/u}-1)^4u^4} > 0$$

as $u\to0^+$ on $(0,\infty)$. As a result, the function $u^3h\left(\frac{1}{u}\right)$ is strictly convex on $(0,\infty)$. Recall from [14, p. 650] that

(i) a function $\Phi:[0,\infty)\to\mathbb{R}$ is said to be star-shaped if $\Phi(\alpha s)\leqslant\alpha\Phi(s)$ for all $\alpha\in[0,1]$ and $s\geqslant0$;

(ii) a real function Φ defined on a set $S\subset\mathbb{R}^n$ is said to be super-additive if $s,t\in S$ implies $s+t\in S$ and $\Phi(s+t)\geqslant\Phi(s)+\Phi(t)$;

(iii) if Φ is a real function defined on $[0,\infty)$, $\Phi(0)\leqslant0$, and Φ is convex, then Φ is star-shaped, but convexity is not a property of all star-shaped functions;

(iv) if $\Phi:[0,\infty)\to\mathbb{R}$ is star-shaped, then Φ is super-additive.

Consequently, since $\lim_{u\to0^+}\frac{u^2}{\mathrm{e}^{1/u}-1}=0$, the function $u^3h\left(\frac{1}{u}\right)$ is star-shaped, and then super-additive, on $(0,\infty)$. As a result, it follows inductively that

$$\left(\sum_{j=1}^m \frac{b_j}{v}\right)^3 h\left(\frac{1}{\sum_{j=1}^m(b_j/v)}\right) \geqslant \sum_{j=1}^m\left(\frac{b_j}{v}\right)^3 h\left(\frac{1}{b_j/v}\right)$$

which can be rearranged as

$$\left(\sum_{j=1}^{m} b_j\right)^3 h\left(\frac{v}{\sum_{j=1}^{m} b_j}\right) \geqslant \sum_{j=1}^{m} b_j^3 h\left(\frac{v}{b_j}\right).$$

Combining this with (8) yields that the second derivative $[\ln \mathscr{Q}_q(t)]''$ is completely monotonic on the infinite interval $(0,\infty)$.

Complete monotonicity of $[\ln \mathscr{Q}_q(t)]''$ implies that the first derivative $[\ln \mathscr{Q}_q(t)]'$ is strictly increasing on $(0,\infty)$. Hence, from (2), it follows that

$$\begin{aligned}
[\ln \mathscr{Q}_q(t)]' &\leqslant \lim_{t\to\infty}\left[\left(\sum_{j=1}^{m} b_j\right)\psi_q\left(1+t\sum_{j=1}^{m} b_j\right) - \sum_{j=1}^{m} b_j\psi_q(1+b_jt)\right] + \sum_{j=1}^{m} b_j \ln w_j \\
&= \left[\left(\sum_{j=1}^{m} b_j\right)\lim_{t\to\infty}\psi_q\left(1+t\sum_{j=1}^{m} b_j\right) - \sum_{j=1}^{m} b_j \lim_{t\to\infty}\psi_q(1+b_jt)\right] + \sum_{j=1}^{m} b_j \ln w_j \\
&= \left[-\left(\sum_{j=1}^{m} b_j\right)\ln(1-q) + \sum_{j=1}^{m} b_j \ln(1-q)\right] + \sum_{j=1}^{m} b_j \ln w_j \\
&= \sum_{j=1}^{m} b_j \ln w_j < 0.
\end{aligned}$$

By definition, the function $\mathscr{Q}_q(t)$ is logarithmically completely monotonic on $(0,\infty)$. The proof of Theorem 2 is complete. □

4. Inequalities for q-multinomial coefficients

Let $\boldsymbol{b} = (b_1, b_2, \dots, b_m)$ with $b_j > 0$ for $1 \leqslant j \leqslant m$. Then q-multinomial coefficients for $q \in (0,1)$ is

$$\binom{\sum_{j=1}^{m} b_j}{b_1, b_2, \dots, b_m}_q = \frac{\Gamma_q\left(1+\sum_{j=1}^{m} b_j\right)}{\Gamma_q(1+b_1)\Gamma_q(1+b_2)\dots\Gamma_q(1+b_m)}. \tag{9}$$

With the aid of logarithmically complete monotonicity of $\mathscr{Q}_q(t) = \mathscr{Q}_{q;\boldsymbol{b},\boldsymbol{w};m}(t)$, we now present some inequalities for q-multinomial coefficients.

THEOREM 3. *Let $q \in (0,1)$, $\ell, m \geqslant 2$, and $\boldsymbol{b} = (b_1, b_2, \dots, b_m)$ with $b_j > 0$ for $1 \leqslant j \leqslant m$. If $y_j > 0$ for $1 \leqslant j \leqslant \ell$ and $\theta_j \in (0,1)$ with $\sum_{j=1}^{\ell} \theta_j = 1$, then*

$$\binom{\sum_{j=1}^{\ell} \theta_j y_j \sum_{j=1}^{m} b_j}{b_1 \sum_{j=1}^{\ell} \theta_j y_j, b_2 \sum_{j=1}^{\ell} \theta_j y_j, \dots, b_m \sum_{j=1}^{\ell} \theta_j y_j}_q \leqslant \prod_{j=1}^{\ell}\left[\binom{y_j \sum_{j=1}^{m} b_j}{y_j b_1, y_j b_2, \dots, y_j b_m}_q\right]^{\theta_j} \tag{10}$$

and the equality in (10) holds if and only if $y_1 = y_2 = \cdots = y_\ell$.

Proof. The logarithmically complete monotonicity in Theorem 2 indicates that the function $\mathscr{Q}_q(t)$ is logarithmically convex on $(0,\infty)$. Hence, one acquires

$$\mathscr{Q}_q\left(\sum_{j=1}^{\ell}\theta_j y_j\right) \leqslant \prod_{j=1}^{\ell}\mathscr{Q}_q^{\theta_j}(y_j).$$

Using the expression

$$\mathscr{Q}_q(t) = \binom{t\sum_{j=1}^{m} b_j}{tb_1, tb_2, \dots, tb_m}_q \prod_{j=1}^{m} w_j^{b_j t}$$

leads to

$$\binom{\sum_{j=1}^{\ell}\theta_j y_j \sum_{j=1}^{m} b_j}{b_1\sum_{j=1}^{\ell}\theta_j y_j, b_2\sum_{j=1}^{\ell}\theta_j y_j, \dots, b_m\sum_{j=1}^{\ell}\theta_j y_j}_q \prod_{j=1}^{m} w_j^{b_j\sum_{j=1}^{\ell}\theta_j y_j} \leqslant \prod_{j=1}^{\ell}\left[\binom{y_j\sum_{j=1}^{m} b_j}{y_j b_1, y_j b_2, \dots, y_j b_m}_q \prod_{j=1}^{m} w_j^{b_j y_j}\right]^{\theta_j}$$

which can be rearranged as (10). The proof of Theorem 3 is complete. □

THEOREM 4. *Let $q\in(0,1)$, $\ell,m\geqslant 2$, and $\boldsymbol{b}=(b_1,b_2,\dots,b_m)$ with $b_j>0$ for $1\leqslant j\leqslant m$. If $y_j>0$ for $1\leqslant j\leqslant \ell$, then*

$$\prod_{j=1}^{\ell}\binom{y_j\sum_{j=1}^{m} b_j}{y_j b_1, y_j b_2, \dots, y_j b_m}_q < \binom{\sum_{j=1}^{\ell} y_j \sum_{j=1}^{m} b_j}{b_1\sum_{j=1}^{\ell} y_j, b_2\sum_{j=1}^{\ell} y_j, \dots, b_m\sum_{j=1}^{\ell} y_j}_q. \tag{11}$$

Proof. In [2, Lemma 3], it was proved that, if $g:[0,\infty)\to(0,1]$ is differentiable and $\frac{g'(s)}{g(s)}$ is strictly increasing on $(0,\infty)$, then the inequality $g(s)g(t)<g(s+t)$ is valid for $s,t\in(0,\infty)$. As a result, we can inductively deduce

$$\prod_{j=1}^{\ell} g(y_j) < g\left(\sum_{j=1}^{\ell} y_j\right).$$

Applying this inequality to the logarithmic convexity of the function $\mathscr{Q}_q(t)$ reveals

$$\prod_{j=1}^{\ell}\left[\binom{y_j\sum_{j=1}^{m} b_j}{y_j b_1, y_j b_2, \dots, y_j b_m}_q \prod_{j=1}^{m} w_j^{b_j y_j}\right] < \binom{\sum_{j=1}^{\ell} y_j \sum_{j=1}^{m} b_j}{b_1\sum_{j=1}^{\ell} y_j, b_2\sum_{j=1}^{\ell} y_j, \dots, b_m\sum_{j=1}^{\ell} y_j}_q \prod_{j=1}^{m} w_j^{b_j\sum_{j=1}^{\ell} y_j}$$

which can be rewritten as (11). The proof of Theorem 4 is complete. □

THEOREM 5. *Let $q \in (0,1)$, $m \geqslant 2$, and $\boldsymbol{b} = (b_1, b_2, \dots, b_m)$ with $b_j > 0$ for $1 \leqslant j \leqslant m$. If $0 < a \leqslant c$ and $t > 0$, then*

$$\binom{(a+t)\sum_{j=1}^m b_j}{(a+t)b_1, (a+t)b_2, \dots, (a+t)b_m}_q \binom{c\sum_{j=1}^m b_j}{cb_1, cb_2, \dots, cb_m}_q \leqslant \binom{a\sum_{j=1}^m b_j}{ab_1, ab_2, \dots, ab_m}_q \binom{(c+t)\sum_{j=1}^m b_j}{(c+t)b_1, (c+t)b_2, \dots, (c+t)b_m}_q \tag{12}$$

and the equality in (12) *holds if and only if $a = c$.*

Proof. For $0 < a < c$, set

$$V(t) = \ln \mathscr{Q}_q(a+t) + \ln \mathscr{Q}_q(c) - \ln \mathscr{Q}_q(a) - \ln \mathscr{Q}_q(c+t).$$

Because

$$V'(t) = \frac{\mathscr{Q}_q'(a+t)}{\mathscr{Q}_q(a+t)} - \frac{\mathscr{Q}_q'(c+t)}{\mathscr{Q}_q(c+t)}$$

and the logarithmically complete monotonicity of $\mathscr{Q}_q(t)$ hints that the function $\frac{\mathscr{Q}_q'(t)}{\mathscr{Q}_q(t)}$ is strictly increasing on $(0,\infty)$, we see that $V'(t) < 0$ and $V(t) < V(0) = 0$. Consequently, it follows that

$$\ln \mathscr{Q}_q(a+t) + \ln \mathscr{Q}_q(c) \leqslant \ln \mathscr{Q}_q(a) + \ln \mathscr{Q}_q(c+t)$$

which is equivalent to

$$\ln\left[\binom{(a+t)\sum_{j=1}^m b_j}{(a+t)b_1, (a+t)b_2, \dots, (a+t)b_m}_q \prod_{j=1}^m w_j^{b_j(a+t)}\right] + \ln\left[\binom{c\sum_{j=1}^m b_j}{cb_1, cb_2, \dots, cb_m}_q \prod_{j=1}^m w_j^{b_jc}\right] \leqslant \ln\left[\binom{a\sum_{j=1}^m b_j}{ab_1, ab_2, \dots, ab_m}_q \prod_{j=1}^m w_j^{b_ja}\right] + \ln\left[\binom{(c+t)\sum_{j=1}^m b_j}{(c+t)b_1, (c+t)b_2, \dots, (c+t)b_m}_q \prod_{j=1}^m w_j^{b_j(c+t)}\right].$$

This can be simplified as (12). The proof of Theorem 5 is complete. □

5. Inequalities for q-multivariate beta functions

Let $\boldsymbol{b} = (b_1, b_2, \dots, b_m)$ with $b_j > 0$ for $1 \leqslant j \leqslant m$. Then q-multivariate beta functions for $q \in (0,1)$ is

$$B_q(b_1, b_2, \dots, b_m) = \frac{\Gamma_q(b_1)\Gamma_q(b_2)\cdots\Gamma_q(b_m)}{\Gamma_q(b_1 + b_2 + \cdots + b_m)}.$$

In [3, p. 494 or p. 544], it was proved that $\Gamma_q(t+1)=\frac{1-q^t}{1-q}\Gamma_q(t)$ with $\Gamma_q(1)=1$ for $q\in(0,1)$. Applying this recurrence relation to (9) yields

$$\binom{\sum_{j=1}^m b_j}{b_1,b_2,\dots,b_m}_q=\frac{(1-q)^{m-1}\big(1-q^{\sum_{j=1}^m b_j}\big)}{\prod_{j=1}^m(1-q^{b_j})}\frac{1}{B_q(b_1,b_2,\dots,b_m)}. \tag{13}$$

Substituting (13) into those inequalities in Theorems 3 to 5, we can conclude several inequalities for q-multivariate beta functions $B_q(b_1,b_2,\dots,b_m)$. Equivalently speaking, Theorems (3) to (5) can be respectively reformulated as follows.

Let $q\in(0,1)$, $\ell,m\geqslant 2$, and $\boldsymbol{b}=(b_1,b_2,\dots,b_m)$ with $b_j>0$ for $1\leqslant j\leqslant m$. If $y_j>0$ for $1\leqslant j\leqslant \ell$ and $\theta_j\in(0,1)$ with $\sum_{j=1}^\ell\theta_j=1$, then

$$\begin{aligned}\frac{\prod_{j=1}^m\big(1-q^{b_j\sum_{j=1}^\ell\theta_jy_j}\big)}{1-q^{\sum_{j=1}^\ell\theta_jy_j\sum_{j=1}^m b_j}}B_q\left(b_1\sum_{j=1}^\ell\theta_jy_j,b_2\sum_{j=1}^\ell\theta_jy_j,\dots,b_m\sum_{j=1}^\ell\theta_jy_j\right)\\ \geqslant\prod_{j=1}^\ell\left[\frac{\prod_{j=1}^m(1-q^{y_jb_j})}{1-q^{y_j\sum_{j=1}^m b_j}}B_q(b_1y_j,b_2y_j,\dots,b_my_j)\right]^{\theta_j}\end{aligned} \tag{14}$$

and the equality in (14) holds if and only if $y_1=y_2=\dots=y_\ell$.

The inequality (14) implies that the function

$$\frac{\prod_{j=1}^m(1-q^{b_jt})}{1-q^{t\sum_{j=1}^m b_j}}B_q(b_1t,b_2t,\dots,b_mt)$$

for $m\geqslant 2$ is logarithmically concave with respect to $t\in(0,\infty)$. Generally, we claim that the function

$$\frac{1-q^{t\sum_{j=1}^m b_j}}{\prod_{j=1}^m(1-q^{b_jt})}\frac{1}{B_q(b_1t,b_2t,\dots,b_mt)}$$

for $m\geqslant 2$ is logarithmically completely monotonic with respect to $t\in(0,\infty)$.

Let $q\in(0,1)$, $\ell,m\geqslant 2$, and $\boldsymbol{b}=(b_1,b_2,\dots,b_m)$ with $b_j>0$ for $1\leqslant j\leqslant m$. If $y_j>0$ for $1\leqslant j\leqslant \ell$, then

$$\begin{aligned}\frac{1}{(1-q)^{(\ell-1)(m-1)}}\prod_{j=1}^\ell\left[\frac{\prod_{j=1}^m(1-q^{b_jy_j})}{1-q^{y_j\sum_{j=1}^m b_j}}B_q(b_1y_j,b_2y_j,\dots,b_my_j)\right]\\ >\frac{\prod_{j=1}^m\big(1-q^{b_j\sum_{j=1}^\ell y_j}\big)}{1-q^{\sum_{j=1}^\ell y_j\sum_{j=1}^m b_j}}B_q\left(b_1\sum_{j=1}^\ell y_j,b_2\sum_{j=1}^\ell y_j,\dots,b_m\sum_{j=1}^\ell y_j\right).\end{aligned}$$

Let $q\in(0,1)$, $m\geqslant 2$, and $\boldsymbol{b}=(b_1,b_2,\dots,b_m)$ with $b_j>0$ for $1\leqslant j\leqslant m$. If $0<a\leqslant c$ and $t>0$, then

$$\begin{aligned}\frac{1-q^{(a+t)\sum_{j=1}^m b_j}}{1-q^{(c+t)\sum_{j=1}^m b_j}}\frac{\prod_{j=1}^m(1-q^{(c+t)b_j})}{\prod_{j=1}^m(1-q^{(a+t)b_j})}\frac{B_q((c+t)b_1,(c+t)b_2,\dots,(c+t)b_m)}{B_q((a+t)b_1,(a+t)b_2,\dots,(a+t)b_m)}\\ \leqslant\frac{1-q^{c\sum_{j=1}^m b_j}}{1-q^{a\sum_{j=1}^m b_j}}\frac{\prod_{j=1}^m(1-q^{ab_j})}{\prod_{j=1}^m(1-q^{cb_j})}\frac{B_q(cb_1,cb_2,\dots,cb_m)}{B_q(ab_1,ab_2,\dots,ab_m)}\end{aligned} \tag{15}$$

and the equality in (15) holds if and only if $a = c$.

REMARK 1. This paper is a revised version of the electronic preprint [20] and a companion of the papers [17, 18, 23, 29, 32, 33, 37] and closely related references therein.

6. Conclusions

The main contributions in this paper are as follows.

(a) In Section 3, we prove the logarithmically complete monotonicity of a function involving q-gamma functions.

(b) By applying the logarithmically complete monotonicity in Section 3, we derive some inequalities for q-multinomial coefficients and q-multivariate beta functions in Sections 4 and 5.

In summary, our new results generalize and extend corresponding ones for gamma functions, multinomial coefficients, and multivariate beta functions, respectively.

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Feng Qi
Institute of Mathematics
Henan Polytechnic University
Jiaozuo 454003, China
and
School of Mathematics and Physics
Hulunbuir University
Inner Mongolia 021008, China
and
Independent researcher
Dallas, TX 75252-8024, USA
e-mail: honest.john.china@gmail.com
https://qifeng618.wordpress.com
https://orcid.org/0000-0001-6239-2968