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ON THE MINIMUM RANK OF DISTANCE MATRICES

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Abstract. Let $X = \{x_1,\dots,x_n\}$ be a finite set endowed with a metric d. The matrix $A = (d(x_i,x_j))_{n\times n}$ is called a distance matrix. In this paper we discuss about the minimum rank that can be achieved by an $n\times n$ distance matrix and prove that the rank of every 5×5 and 6×6 distance matrix is not less than 4.

1. Introduction

The purpose of this article is discussing this important question: what is the minimum rank of $n\times n$ distance matrices and what are the applications of this problem? Let's begin the discussion with the important topic of multidimensional scaling (MDS). MDS is a statistical technique used to analyze and visualize the dissimilarities between objects or entities. It aims to represent the relationships between items in a lower-dimensional space, typically two or three dimensions, while preserving the original pairwise distances or dissimilarities as much as possible.

MDS is commonly used in various fields, such as psychology, marketing, geography, and data visualization. It helps in understanding the underlying structure or patterns in data, identifying clusters or groups of similar items, and providing a visual representation of complex relationships or proximity between objects. A matrix $A = (a_{ij})_{n\times n}$ is Euclidean distance matrix (EDM), if there exists some positive number k and some points $X_1,\dots,X_n \in \mathbb{R}^k$ such that $a_{ij} = \|X_i - X_j\|_2^2$, where $\|.\|_2$ is Euclidean norm. The smallest k with the stated property is called the embedding dimension of A and is displayed by $E(A)$. Gower in [2] proved that for every EDM matrix A, we have $E(A) = \operatorname{rank}(A) - 1$ or $E(A) = \operatorname{rank}(A) - 2$ (see [4, Corollary 2.3] for a simple proof).

In general, let A be an $n\times n$ distance matrix, and $\mathbb{R}^k$ be equipped with a norm $\|.\|$. The smallest value of k for which there exist points $y_1,y_2,\dots,y_n \in \mathbb{R}^k$ satisfying $d(x_i,x_j) = \|y_i - y_j\|$ is referred to as the embedding dimension of A with respect to the norm $\|.\|$. This embedding dimension is denoted as $E_{\|.\|}(A)$. Determining the precise value of $E_{\|.\|}(A)$ or establishing a limited range for it is of significant importance, particularly for $\|.\|_p$ with $p = 1,2,\infty$ [1, 3, 5, 6].

If $n \geqslant 4$, for every $n\times n$ distance matrix A the inequality $E_{\|.\|_\infty}(A) \leqslant n-2$ holds [3]. In this case, we face with the two following important questions that as far as the authors are aware, there is no proof or counterexample available for them:

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(i) Is it true that the inequality $E_{\|.\|_\infty}(A) \leqslant \text{rank}(A)$ holds for all distance matrices in $\mathbb{R}^{n\times n}$?

(ii) What is the minimum rank of distance matrices in $\mathbb{R}^{n\times n}$?

Here $\mathbb{R}^{n\times n}$ denotes the algebra of all $n \times n$ matrices with real entries. The answer to the second question determines the amount by which the upper bound of $E_{\|.\|_\infty}(A)$ in the first question can be reduced.

In this paper, we prove that when n is either 5 or 6, then the answer of the second question is 4.

2. Main result

In this section, we discuss about the values of d_n for $n = 2, 3, \ldots$, where

$$d_n = \min\{rank(D) : D \text{ is a distance, matrix in } \mathbb{R}^{n\times n}\}$$

It is easy to see that $\{d_n\}$ is an increasing sequence and by Cauchy's interlacing theorem for eigenvalues of Hermitian matrices, $d_{n+1} - d_n$ takes one of the values 0,1 or 2. Given the fact that the off-diagonal entries of distance matrices are positive, it becomes apparent that $d_2 = 2$ and $d_3 = 3$. There exist 4×4 distance matrices with rank of 3, such as:

$$\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}.$$

Hence the equality $d_4 = 3$ holds. In the following we will prove that $d_5 = d_6 = 4$.

THEOREM 1. *Let*

$$D = \begin{bmatrix} 0 & d_{12} & d_{13} & d_{14} & d_{15} \\ d_{21} & 0 & d_{23} & d_{24} & d_{25} \\ d_{31} & d_{32} & 0 & d_{34} & d_{35} \\ d_{41} & d_{42} & d_{43} & 0 & d_{45} \\ d_{51} & d_{52} & d_{53} & d_{54} & 0 \end{bmatrix},$$

be a distance matrix. Then $rank(D) \geqslant 4$.

Proof. It is easy to see that the rank of matrix D is neither 1 nor 2. Suppose that $\text{rank}(D) = 3$. Without loss of generality, we can assume that the fourth and fifth columns of matrix D are linear combinations of the other columns. This assumption implies the existence of nonzero real numbers $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ that satisfy the following equations (1)–(10):

$$\beta d_{12} + \gamma d_{13} = d_{14}, \tag{1}$$
$$\beta' d_{12} + \gamma' d_{13} = d_{15}, \tag{2}$$
$$\alpha d_{12} + \gamma d_{23} = d_{24}, \tag{3}$$
$$\alpha' d_{12} + \gamma' d_{23} = d_{25}, \tag{4}$$

$$\alpha d_{13}+\beta d_{23}=d_{34}, \tag{5}$$
$$\alpha' d_{13}+\beta' d_{23}=d_{35}, \tag{6}$$
$$\alpha d_{14}+\beta d_{24}+\gamma d_{34}=0, \tag{7}$$
$$\alpha' d_{14}+\beta' d_{24}+\gamma' d_{34}=d_{45}, \tag{8}$$
$$\alpha d_{15}+\beta d_{25}+\gamma d_{35}=d_{45}, \tag{9}$$
$$\alpha' d_{15}+\beta' d_{25}+\gamma' d_{35}=0. \tag{10}$$

Considering the positivity of off-diagonal entries of D and equations (1)–(10), it can be deduced that among α,β,γ (α',β',γ'), one of them is negative while the other two are positive. Due to the symmetry of the problem, we can focus on the following case:

$$\gamma<0<\beta\leqslant\alpha, \qquad \gamma'<0<\beta'\leqslant\alpha'.$$

For the continuation of our proof, we require the following lemma:

LEMMA 1. *The following relations are valid:*

i. $\alpha\beta d_{12}+\alpha\gamma d_{13}+\beta\gamma d_{23}=0.$

ii. $\beta d_{12}\leqslant d_{34}.$

iii. $\alpha+\beta+\gamma\geqslant 1.$

iv. $\alpha-\beta+\gamma\geqslant -1.$

v. $\gamma,\gamma'\geqslant -1.$

vi. $\dfrac{\alpha\beta'\gamma'}{\alpha'\beta\gamma}=\dfrac{d_{15}}{d_{14}},\ \dfrac{\alpha'\beta\gamma'}{\alpha\beta'\gamma}=\dfrac{d_{25}}{d_{24}},\ \dfrac{\alpha\beta\gamma'}{\alpha'\beta'\gamma}=\dfrac{d_{34}}{d_{35}}.$

Proof.

i. Applying the equalities (1), (3), (5) in (7), we get the desired equality.

ii. Since $\alpha\geqslant\beta$, (5) imply that

$$d_{34}=\alpha d_{13}+\beta d_{23}\geqslant\beta(d_{13}+d_{23})\geqslant\beta d_{12}.$$

iii. The inequality $d_{14}+d_{24}\geqslant d_{12}$, (1), (3) and negativity of γ imply that

$$d_{12}\leqslant\beta d_{12}+\gamma d_{13}+\alpha d_{12}+\gamma d_{23}=(\beta+\alpha)d_{12}+\gamma(d_{13}+d_{23})\leqslant(\alpha+\beta+\gamma)d_{12}.$$

Therefore $\alpha+\beta+\gamma\geqslant 1$.

iv. Using the inequality $d_{24}+d_{23}\geqslant d_{34}$, (3) and (5), we have

$$\alpha d_{13}+\beta d_{23}\leqslant\alpha d_{12}+\gamma d_{23}+d_{23}.$$

Hence

$$(\beta-\gamma-1)d_{23}\leqslant\alpha(d_{12}-d_{13})\leqslant\alpha d_{23}. \tag{11}$$

Therefore

$$\alpha-\beta+\gamma\geqslant -1.$$

v. By (ii) and (1), we have

$$d_{14} = \beta d_{12} + \gamma d_{13} \leqslant d_{34} + \gamma d_{13}.$$

This implies that

$$-\gamma d_{13} \leqslant d_{34} - d_{14} \leqslant d_{13}.$$

Therefore $-\gamma \leqslant 1$. Similarity, (2) and (6) imply that $\gamma' \geqslant -1$.

vi. By (1) we have $\alpha\beta d_{12} + \alpha\gamma d_{13} = \alpha d_{14}$. But (i) implies that $\alpha d_{14} + \beta\gamma d_{23} = 0$. Hence

$$\frac{\alpha}{-\beta\gamma} = \frac{d_{23}}{d_{14}}. \tag{12}$$

Similarity, (5) and (7) yield

$$\frac{\beta}{-\alpha\gamma} = \frac{d_{13}}{d_{24}}, \tag{13}$$

$$\frac{\gamma}{-\alpha\beta} = \frac{d_{12}}{d_{34}}. \tag{14}$$

Now repeating the above argument for α', β' and γ', we will have

$$\frac{\alpha'}{-\beta'\gamma'} = \frac{d_{23}}{d_{15}}, \tag{15}$$

$$\frac{\beta'}{-\alpha'\gamma'} = \frac{d_{13}}{d_{25}}, \tag{16}$$

$$\frac{\gamma'}{-\alpha'\beta'} = \frac{d_{12}}{d_{35}}. \tag{17}$$

The pairings of two equalities (12) and (15), (13) and (16) and (14) and (17) yield the following equalities:

$$\frac{\alpha\beta'\gamma'}{\alpha'\beta\gamma} = \frac{d_{15}}{d_{14}},$$

$$\frac{\alpha'\beta\gamma'}{\alpha\beta'\gamma} = \frac{d_{25}}{d_{24}},$$

$$\frac{\alpha\beta\gamma'}{\alpha'\beta'\gamma} = \frac{d_{34}}{d_{35}}. \quad \square$$

Now we return to the proof of the main theorem.

If $\alpha \geqslant 1$, then by (i) of Lemma 1 and (5), we have

$$\gamma = \frac{-\alpha\beta d_{12}}{\alpha d_{13} + \beta d_{23}} = \frac{-\alpha\beta d_{12}}{d_{34}}.$$

By (1), it follows

$$\beta d_{12} + \left(\frac{-\alpha\beta d_{12}}{d_{34}}\right) d_{13} = d_{14}.$$

Therefore

$$\beta d_{12}(d_{34} - \alpha d_{13}) = d_{14}d_{34}.$$

Since $\alpha \geqslant 1$, we have

$$d_{14}d_{34} = \beta d_{12}(d_{34} - \alpha d_{13}) \leqslant \beta d_{12}(d_{34} - d_{13}) \leqslant \beta d_{12}d_{14}.$$

Hence by (ii) of Lemma 1 the equality $d_{34} = \beta d_{12}$ holds. Hence

$$\gamma = \frac{-\alpha\beta d_{12}}{\beta d_{12}} = -\alpha.$$

Therefore, (v) of Lemma 1 yields $\alpha \leqslant 1$ which implies that $\alpha = 1$. On the other hand, by (iii) and (iv) of Lemma 1 and $\gamma = -\alpha$, we have $\beta = 1$ and so $\alpha = \beta$=1.

If $\alpha \leqslant 1$, by (i) of Lemma 1, (1) and (3), it follows

$$\alpha = \frac{-\beta\gamma d_{23}}{\beta d_{12} + \gamma d_{13}} = \frac{\beta(\alpha d_{12} - d_{24})}{d_{14}} \leqslant \frac{\beta(d_{12} - d_{24})}{d_{14}} \leqslant \beta.$$

Hence again $\alpha = \beta$, since $\alpha \geqslant \beta$.

Consequently $\alpha = \beta$ for every $\alpha > 0$. Similarly, we can prove $\alpha' = \beta'$. Then (vi) of Lemma 1 implies

$$\frac{d_{15}}{d_{14}} = \frac{\alpha\beta'\gamma'}{\alpha'\beta\gamma} = \frac{\gamma'}{\gamma},$$
$$\frac{d_{25}}{d_{24}} = \frac{\alpha'\beta\gamma'}{\alpha\beta'\gamma} = \frac{\gamma'}{\gamma},$$
$$\frac{d_{34}}{d_{35}} = \frac{\alpha\beta\gamma'}{\alpha'\beta'\gamma} = \frac{\gamma'\alpha^2}{\gamma\alpha'^2}.$$

On the other hand, by (5), (6), $\alpha' = \beta'$ and $\alpha = \beta$, we have $\frac{\alpha}{\alpha'} = \frac{d_{34}}{d_{35}}$, so $\frac{\gamma'}{\gamma} = \frac{d_{35}}{d_{34}}$. Now we have three the following cases:

- $\dfrac{\gamma'}{\gamma} = 1$: In this case,

$$d_{15} = d_{14}\ ,\ d_{25} = d_{24}\ ,\ d_{34} = d_{35}.$$

Therefore $\alpha = \alpha'$, $\beta = \beta'$. So by (7) and (8), $d_{45} = 0$ which is a contradiction.

- $\dfrac{\gamma'}{\gamma} < 1$. In this case,

$$d_{15} < d_{14}\ ,\ d_{25} < d_{24}\ ,\ d_{35} < d_{34}.$$

But (7) and (9) imply that

$$\begin{aligned} d_{45} &= \alpha d_{15} + \beta d_{25} + \gamma d_{35} < \alpha d_{14} + \beta d_{24} + \gamma d_{35} \\ &= -\gamma d_{34} + \gamma d_{35} = -\gamma(d_{34} - d_{35}) \leqslant -\gamma d_{45} \leqslant d_{45}. \end{aligned}$$

which is contradiction.

- $\frac{\gamma'}{\gamma} > 1$. In this case,

$$d_{15} > d_{14}\,,\; d_{25} > d_{24}\,,\; d_{35} > d_{34}.$$

But (8) and (10) imply that

$$\begin{aligned} d_{45} &= \alpha' d_{14} + \beta' d_{24} + \gamma' d_{34} < \alpha' d_{15} + \beta' d_{25} + \gamma' d_{34} \\ &= -\gamma' d_{35} + \gamma' d_{34} = -\gamma'(d_{35} - d_{34}) \leqslant d_{45}. \end{aligned}$$

which is contradiction.

Therefore, we have $rank(D) \neq 3$. But the distance matrix

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 3 & 2 \\ 1 & 2 & 3 & 0 & 1 \\ 2 & 1 & 2 & 1 & 0 \end{bmatrix},$$

has rank 4 and so the minimum rank of 5×5 distance matrices is equal to 4. □

COROLLARY 1. *The minimum rank of* 6×6 *distance matrices is equal to 4.*

Proof. The rank of the following 6×6 distance matrix is equal to 4. Therefore, by Theorem 1,we get the desired result.

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 3 & 2 & 1 \\ 1 & 2 & 3 & 0 & 1 & 2 \\ 2 & 1 & 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 & 1 & 0 \end{bmatrix}. \quad \square$$

The simulations confirm the following values of d_n for $n = 7, 8, \ldots, 15$:

n	7	8	9	10	11	12	13	14	15
d_n	5	5	6	7	7	7	8	9	9

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