Mathematical
Inequalities
& Applications
Volume 27, Number 1 (2024), 193–200 doi:10.7153/mia-2024-27-15

ON A CONJECTURE RELATED TO THE GEOMETRIC MEAN AND NORM INEQUALITIES

SHAIMA'A FREEWAN AND MOSTAFA HAYAJNEH*

(*Communicated by J.-C. Bourin*)

Abstract. A conjecture of Dinh, Ahsani, and Tam, was recently settled in [7]. In this note, we give a refinement to that result, namely if A_i and B_i are positive definite matrices and $Z=[Z_{ij}]$ is the block matrix such that $Z_{ij}=B_i^{\frac{1}{2}}\left(\sum_{k=1}^{m}A_k\right)B_j^{\frac{1}{2}}$ for all $i,j=1,\cdots,m$, then

$$\left|\left|\left|\sum_{i=1}^{m}\left(A_i^{2}\sharp B_i^{2}\right)^{r}\right|\right|\right| \leqslant |||Z^r||| \leqslant \left|\left|\left|\left(\left(\sum_{i=1}^{m}A_i\right)^{\frac{rp}{2}}\left(\sum_{i=1}^{m}B_i\right)^{rp}\left(\sum_{i=1}^{m}A_i\right)^{\frac{rp}{2}}\right)^{\frac{1}{p}}\right|\right|\right|,$$

for all unitarily invariant norms, for all $p>0$ and $r\geqslant 1$ such that $rp\geqslant 1$. Our approach provides us with an alternative proof without using the method of majorization that was used in [7]. As a byproduct, we get a refinement to a result of Audenaert in 2015.

1. Introduction

Throughout this paper, the set of all matrices with complex entries and a size of n is referred to as $\mathbb{M}_n(\mathbb{C})$. The set of all Hermitian matrices in $\mathbb{M}_n(\mathbb{C})$ are represented by the symbol $\mathbb{H}_n$. We use the notation $A>0$ if $A\in\mathbb{H}_n$ is a positive definite matrix, that is, $\langle x,Ax\rangle>0$ for all $x\in\mathbb{C}^n-\{0\}$. The set of all positive definite matrices in $\mathbb{M}_n(\mathbb{C})$ is represented by the symbol $\mathbb{P}_n$. The symbol $|||.|||$ indicates any unitarily invariant norm on the space $\mathbb{M}_n(\mathbb{C})$. Let $A,B\in\mathbb{P}_n$. Then the matrix $A\sharp B$ is called the geometric mean of A and B and is given by

$$A\sharp B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}.$$

For all $t\in[0,1]$, the matrix $A\sharp_t B$ is called the t-geometric mean of A and B and is given by

$$A\sharp_t B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{t}A^{\frac{1}{2}}.$$

In his remarkable paper [5], Bourin asked the following question.

Mathematics subject classification (2020): Primary 15A60; Secondary 15B57, 47A30, 47B15.
Keywords and phrases: Bourin question, unitarily invariant norm, positive definite matrix, inequality.
* Corresponding author.

QUESTION 1. If $A,B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite matrices and $p,q > 0$, is it true that

$$|||A^{p+q} + B^{p+q}||| \leqslant |||(A^p + B^p)(A^q + B^q)|||?$$

In [8], Hayajneh and Kittaneh gave an affirmative answer for the trace norm and the Hilbert-Schmidt norm. In [2], Audenaert provided an affirmative answer to Question 1 by proving

$$\left|\left|\left|\sum_{i=1}^{m}(A_iB_i)\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{m}A_i^{\frac{1}{2}}B_i^{\frac{1}{2}}\right)^2\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{m}A_i\right)\left(\sum_{i=1}^{m}B_i\right)\right|\right|\right|, \tag{1}$$

for all $A_i, B_i \in \mathbb{P}_n$, $i = 1,\ldots,m$ such that $A_iB_i = B_iA_i$ and all unitarily invariant norms.

As a generalization of inequality (1), Dinh, Ahsani, and Tam established the following non-commutative inequality in [1, Theorem 3.1],

$$\left|\left|\left|\sum_{i=1}^{m}(A_i\sharp B_i)^2\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{m}A_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^{m}B_i\right)\left(\sum_{i=1}^{m}A_i\right)^{\frac{1}{2}}\right|\right|\right|, \tag{2}$$

for all $A_i, B_i \in \mathbb{P}_n$, $i = 1,\cdots,m$, and for all unitarily invariant norms. Also, the authors in the same paper [1, page 787] proposed the following conjecture.

CONJECTURE 1. If $A_i, B_i \in \mathbb{P}_n$ for all $i = 1,\cdots,m$, then

$$\left|\left|\left|\sum_{i=1}^{m}\left(A_i^2\sharp B_i^2\right)\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{m}A_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^{m}B_i\right)\left(\sum_{i=1}^{m}A_i\right)^{\frac{1}{2}}\right|\right|\right|, \tag{3}$$

for all unitarily invariant norms.

The authors in the same paper [1, Corollary 3.3] proved Conjecture 1 for the case of the trace norm $\|.\|_1$. In [7], Freewan and Hayajneh gave a proof of Conjecture 1 in its full generality. Actually, the authors found it more convenient to treat the following more general conjecture.

CONJECTURE 2. If $A_i, B_i \in \mathbb{P}_n$ for all $i = 1,\cdots,m$, and $t \in [0,1]$, then

$$\left|\left|\left|\sum_{i=1}^{m}(A_i^2\sharp_t B_i^2)^r\right|\right|\right| \leqslant \left|\left|\left|\left(\left(\sum_{i=1}^{m}A_i\right)^{(1-t)pr}\left(\sum_{i=1}^{m}B_i\right)^{2tpr}\left(\sum_{i=1}^{m}A_i\right)^{(1-t)pr}\right)^{\frac{1}{p}}\right|\right|\right|,$$

for all $2r \geqslant 1$ and for all $p > 0$ and for all unitarily invariant norms.

They proved this conjecture for the cases

$$t = \frac{1}{2}, \ \ p > 0, \ \ r \geqslant 1 \quad \text{such that} \quad pr \geqslant 1. \tag{4}$$

Note that the case $p=1$, $r=1$ is included in (4) and is nothing but the case that corresponds to Conjecture 1. Thus Conjecture 1 is completely settled in [7].

In this paper, we provide another proof to Conjecture 2 for the cases in (4) and for all unitarily invariant norms without using the method of majorisation. In fact, we give a refinement to the above result, namely if A_i and B_i are positive definite matrices and $Z=[Z_{ij}]$ is the block matrix such that $Z_{ij}=B_i^{\frac{1}{2}}\left(\sum_{k=1}^{m}A_k\right)B_j^{\frac{1}{2}}$ for all $i,j=1,\cdots,m$, then

$$\left|\left|\left|\sum_{i=1}^{m}\left(A_i^2\sharp B_i^2\right)^r\right|\right|\right| \leqslant |||Z^r||| \leqslant \left|\left|\left|\left(\left(\sum_{i=1}^{m}A_i\right)^{\frac{rp}{2}}\left(\sum_{i=1}^{m}B_i\right)^{rp}\left(\sum_{i=1}^{m}A_i\right)^{\frac{rp}{2}}\right)^{\frac{1}{p}}\right|\right|\right|,$$

for all unitarily invariant norms, for all $p>0$ and $r\geqslant 1$ such that $rp\geqslant 1$. The main ingredient of our proof is the remarkable matrix version of Cauchy-Schwarz inequality, namely Lemma 4. Also, in this paper we propose the following conjecture, which is a generalization of Conjecture 2.

CONJECTURE 3. Let $A_i,B_i\in\mathbb{P}_n$ for all $i=1,\cdots,m$ and let $t\in[0,1]$. Then

$$\left|\left|\left|\sum_{i=1}^{m}(A_i^s\sharp_t B_i^s)^r\right|\right|\right| \leqslant \left|\left|\left|\left(\left(\sum_{i=1}^{m}A_i\right)^{\frac{(1-t)srp}{2}}\left(\sum_{i=1}^{m}B_i\right)^{tsrp}\left(\sum_{i=1}^{m}A_i\right)^{\frac{(1-t)srp}{2}}\right)^{\frac{1}{p}}\right|\right|\right|,$$

for all $p>0$, for all $r>0$ and $s>0$ such that $sr\geqslant 1$ and for all unitarily invariant norms.

2. Preliminaries and definitions

We begin with a few fundamental well-known facts that will be used to demonstrate our main results.

LEMMA 1. *If $A,B\in\mathbb{P}_n$, then there exists a unitary $U\in\mathbb{M}_n(\mathbb{C})$ such that* $A\sharp B = A^{\frac{1}{2}}UB^{\frac{1}{2}}$.

Proof. See [4, page 108]. □

LEMMA 2. *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ such that the product AB is normal. Then for every unitarily invariant norms, we have $|||AB|||\leqslant|||BA|||$.*

Proof. See [3, page 253]. □

LEMMA 3. *If $Z = [Z_{i,j}] \in \mathbb{M}_{mn}(\mathbb{C})$ is a block matrix such that $Z_{i,j} \in \mathbb{M}_n(\mathbb{C})$ is a normal matrix for all $i, j \in \{1, 2, \cdots, m\}$ or Z is Hermitian, then*

$$|||Z||| \leqslant \left|\left|\left| \sum_{i,j=1}^{m} |Z_{i,j}| \right|\right|\right|,$$

for every unitarily invariant norms.

Proof. See [5, page 7]. □

The following lemma is the celebrated matrix version of the well known inequality, namely, Cauchy-Schwarz inequality. This inequality is the main ingredient in the proof of our main theorem.

LEMMA 4. *Let $X, Y \in \mathbb{M}_n(\mathbb{C})$. Then for all unitarily invariant norms*

$$|||X^*Y||| \leqslant |||XX^*|||^{\frac{1}{2}} \, |||YY^*|||^{\frac{1}{2}}.$$

Proof. See [3, page 266]. □

To prove our main result in Theorem 2, we need to use Lemma 5, Lemma 6, and Lemma 7.

LEMMA 5. *Let $A_i \geqslant 0$ for all $i = 1, \cdots, m$. Let $f : [0, \infty) \longrightarrow [0, \infty)$ be a convex function with $f(0) = 0$. Then for all unitarily invariant norms, we have*

$$\left|\left|\left| \sum_{i=1}^{m} f(A_i) \right|\right|\right| \leqslant \left|\left|\left| f\left(\sum_{i=1}^{m} A_i \right) \right|\right|\right|.$$

Proof. See [6, Theorem 1.2]. □

LEMMA 6. *Let $A, B \in \mathbb{P}_n$ and $r \geqslant 1$. Then for all unitarily invariant norms*

$$|||A||| \leqslant |||B||| \Longrightarrow \left|\left|\left| A^r \right|\right|\right| \leqslant \left|\left|\left| B^r \right|\right|\right|.$$

Proof. See [7, Lemma 2.9]. □

LEMMA 7. *Let $A, B \in \mathbb{P}_n$ and let $r \geqslant 1$ and $p > 0$. Then for all unitarily invariant norms*

$$\left|\left|\left| (BAB)^{pr} \right|\right|\right| \leqslant \left|\left|\left| (B^r A^r B^r)^p \right|\right|\right|.$$

Proof. See [3, page 258] and [7, Lemma 2.13]. □

3. Main results

We begin with the following theorem before proving our main theorem.

THEOREM 1. *Let $A_i, B_i \in \mathbb{P}_n$ for all $i = 1,\cdots,m$ and let $Z = [Z_{ij}]$ be the block matrix such that $Z_{ij} = B_i^{\frac{1}{2}} \left(\sum_{k=1}^{m} A_k\right) B_j^{\frac{1}{2}}$ for all $i, j = 1,\cdots,m$. Then for all unitarily invariant norms*

$$\left|\left|\left|\sum_{i=1}^{m} A_i^2 \sharp B_i^2\right|\right|\right| \leqslant |||Z||| = \left|\left|\left|\left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} B_i\right) \left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}}\right|\right|\right|.$$

Proof. Using Lemma 1, we have $A_i^2 \sharp B_i^2 = A_i U_i B_i$ for some unitary $U_i \in \mathbb{M}_n(\mathbb{C})$ and for all $i = 1,\cdots,m$. Let $X, Y \in \mathbb{M}_{mn}(\mathbb{C})$ be given by

$$X = \begin{bmatrix} B_1^{\frac{1}{2}} U_1^* A_1^{\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 & & B_m^{\frac{1}{2}} U_m^* A_m^{\frac{1}{2}} \end{bmatrix} \text{ and } Y = \begin{bmatrix} B_1^{\frac{1}{2}} & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ B_m^{\frac{1}{2}} & 0 \cdots 0 \end{bmatrix} \begin{bmatrix} A_1^{\frac{1}{2}} & \cdots & A_m^{\frac{1}{2}} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

Then

$$\begin{aligned}
&\left|\left|\left|\sum_{i=1}^{m} \left(A_i^2 \sharp B_i^2\right)\right|\right|\right| \\
&= \left|\left|\left|\sum_{i=1}^{m} (A_i U_i B_i)\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} A_1^{\frac{1}{2}} & \cdots & A_m^{\frac{1}{2}} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} A_1^{\frac{1}{2}} U_1 B_1^{\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 & & A_m^{\frac{1}{2}} U_m B_m^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} B_1^{\frac{1}{2}} & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ B_m^{\frac{1}{2}} & 0 \cdots 0 \end{bmatrix}\right|\right|\right| \\
&\leqslant |||X^*Y||| \quad \text{(by Lemma 2)} \\
&\leqslant |||XX^*|||^{\frac{1}{2}} \, |||YY^*|||^{\frac{1}{2}} \quad \text{(by Lemma 4)} \\
&\leqslant \left|\left|\left|\begin{bmatrix} U_1^* A_1 U_1 B_1^{\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 & & U_m^* A_m U_m B_m^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} B_1^{\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 & & B_m^{\frac{1}{2}} \end{bmatrix}\right|\right|\right|^{\frac{1}{2}} |||YY^*|||^{\frac{1}{2}} \\
&\qquad \text{(by Lemma 2)}
\end{aligned}$$

$$= \left|\left|\left| \begin{bmatrix} A_1^2 \sharp B_1^2 & & 0 \\ & \ddots & \\ 0 & & A_m^2 \sharp B_m^2 \end{bmatrix} \right|\right|\right|^{\frac{1}{2}} |||YY^*|||^{\frac{1}{2}}$$

$$\leqslant \left|\left|\left| \sum_{i=1}^{m} \left(A_i^2 \sharp B_i^2 \right) \right|\right|\right|^{\frac{1}{2}} |||YY^*|||^{\frac{1}{2}}. \quad \text{(by Lemma 3)}$$

Hence

$$\left|\left|\left| \sum_{i=1}^{m} \left(A_i^2 \sharp B_i^2 \right) \right|\right|\right| \leqslant |||YY^*|||. \tag{5}$$

But

$$YY^* = \begin{bmatrix} B_1^{\frac{1}{2}} & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ B_m^{\frac{1}{2}} & 0 \cdots 0 \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{m} A_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1^{\frac{1}{2}} & \cdots & B_m^{\frac{1}{2}} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = Z. \tag{6}$$

From (5) and (6), we get

$$\left|\left|\left| \sum_{i=1}^{m} \left(A_i^2 \sharp B_i^2 \right) \right|\right|\right| \leqslant |||Z|||.$$

Now, using (6) and Lemma 2, we have

$$|||Z|||$$

$$= \left|\left|\left| \begin{bmatrix} B_1^{\frac{1}{2}} & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ B_m^{\frac{1}{2}} & 0 \cdots 0 \end{bmatrix} \begin{bmatrix} \left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1^{\frac{1}{2}} & \cdots & B_m^{\frac{1}{2}} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \right|\right|\right|$$

$$= \left|\left|\left| \begin{bmatrix} \left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1^{\frac{1}{2}} & \cdots & B_m^{\frac{1}{2}} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} B_1^{\frac{1}{2}} & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ B_m^{\frac{1}{2}} & 0 \cdots 0 \end{bmatrix} \begin{bmatrix} \left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \right|\right|\right|$$

$$= \left|\left|\left| \left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} B_i\right) \left(\sum_{i=1}^{m} A_i\right)^{\frac{1}{2}} \right|\right|\right|.$$

This completes the proof. □

Now, let us prove our main theorem.

THEOREM 2. *Let $A_i, B_i \in \mathbb{P}_n$ for all $i = 1, \cdots, m$ and let $p > 0$ and $r \geqslant 1$ such that $rp \geqslant 1$. Let $Z = [Z_{ij}]$ be the block matrix such that $Z_{ij} = B_i^{\frac{1}{2}} \left(\sum_{k=1}^{m} A_k \right) B_j^{\frac{1}{2}}$ for all $i, j = 1, \cdots, m$. Then for every unitarily invariant norms*

$$\left|\left|\left| \sum_{i=1}^{m} \left(A_i^2 \sharp B_i^2 \right)^r \right|\right|\right| \leqslant |||Z^r||| \leqslant \left|\left|\left| \left(\left(\sum_{i=1}^{m} A_i \right)^{\frac{rp}{2}} \left(\sum_{i=1}^{m} B_i \right)^{rp} \left(\sum_{i=1}^{m} A_i \right)^{\frac{rp}{2}} \right)^{\frac{1}{p}} \right|\right|\right|.$$

Proof. Using Theorem 1 and Lemma 6, we have

$$\left|\left|\left| \left(\sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right)^r \right|\right|\right| \leqslant |||Z^r||| = \left|\left|\left| \left(\left(\sum_{i=1}^{m} A_i \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} B_i \right) \left(\sum_{i=1}^{m} A_i \right)^{\frac{1}{2}} \right)^r \right|\right|\right|. \tag{7}$$

Now, note that

$$\begin{aligned}
\left|\left|\left| \sum_{i=1}^{m} \left(A_i^2 \sharp B_i^2 \right)^r \right|\right|\right| &\leqslant \left|\left|\left| \left(\sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right)^r \right|\right|\right| \\
&\qquad \text{(because } f(x) = x^r \text{ is convex and by Lemma 5)} \\
&\leqslant |||Z^r||| \quad \text{(by (7))} \\
&= \left|\left|\left| \left(\left(\sum_{i=1}^{m} A_i \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} B_i \right) \left(\sum_{i=1}^{m} A_i \right)^{\frac{1}{2}} \right)^r \right|\right|\right| \quad \text{(by (7))} \\
&\leqslant \left|\left|\left| \left(\left(\sum_{i=1}^{m} A_i \right)^{\frac{rp}{2}} \left(\sum_{i=1}^{m} B_i \right)^{rp} \left(\sum_{i=1}^{m} A_i \right)^{\frac{rp}{2}} \right)^{\frac{1}{p}} \right|\right|\right|. \\
&\qquad \text{(by Lemma 7 and because } pr \geqslant 1)
\end{aligned}$$

This completes the proof. □

THEOREM 3. *Let $A_i, B_i \in \mathbb{P}_n$ for all $i = 1, \cdots, m$. Then for all unitarily invariant norms*

$$\left|\left|\left| \sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right|\right|\right| \leqslant \left|\left|\left| \sum_{1 \leqslant i,j \leqslant m} \left| B_i^{\frac{1}{2}} \left(\sum_{k=1}^{m} A_k \right) B_j^{\frac{1}{2}} \right| \right|\right|\right|.$$

Proof. Let $Z = [Z_{ij}]$ be the block matrix such that $Z_{ij} = B_i^{\frac{1}{2}} \left(\sum_{k=1}^{m} A_k \right) B_j^{\frac{1}{2}}$ for all $i, j = 1, \cdots, m$. We conclude that $Z \in \mathbb{H}_{mn}$ from (6). From Theorem 1 and Lemma 3, we get

$$\left|\left|\left| \sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right|\right|\right| \leqslant |||Z||| \leqslant \left|\left|\left| \sum_{1 \leqslant i,j \leqslant m} \left| B_i^{\frac{1}{2}} \left(\sum_{k=1}^{m} A_k \right) B_j^{\frac{1}{2}} \right| \right|\right|\right|,$$

for all unitarily invariant norms. This completes the proof. □

REFERENCES

[1] T. H. DINH, S. AHSANI, AND T.Y. TAM, *Geometry and inequalities of geometric mean*, Czechoslovak Math. J., **66**, 3 (2016), 777–792.

[2] K. AUDENAERT, *A norm inequality for pairs of commuting positive semidefinite matrices*, Electronic Journal of Linear Algebra, **30**, (2015), 80–84.

[3] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[4] R. BHATIA, *Positive Definite Matrices*, Princeton University Press, Princeton, 2007.

[5] J. C. BOURIN, *Matrix subadditivity inequalities and block-matrices*, Internat. J. Math., **20**, 6 (2009), 679–691.

[6] J. C. BOURIN AND M. UCHIYAMA, *A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$*, Linear Algebra Appl., **423**, 2–3 (2007), 512–518.

[7] S. FREEWAN AND M. HAYAJNEH, *On norm inequalities related to the geometric mean*, Linear Algebra Appl., **670**, (2023), 104–120.

[8] S. HAYAJNEH AND F. KITTANEH, *Trace inequalities and a question of Bourin*, Bull. Aust. Math., **88**, 3 (2013), 384–389.

(Received July 3, 2023)

Shaima'a Freewan
Department of Mathematics
Yarmouk University
Irbid, Jordan
e-mail: shyf725@gmail.com

Mostafa Hayajneh
Department of Mathematics
Yarmouk University
Irbid, Jordan
e-mail: hayaj86@yahoo.com