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APPROXIMATION BY DE LA VALLÉE POUSSIN TYPE MARCINKIEWICZ MATRIX TRANSFORM MEANS OF WALSH–FOURIER SERIES

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Abstract. In the present paper, we discuss the rate of the approximation by the de la Vallée Poussin type Marcinkiewicz matrix transform of Walsh-Fourier series in $L^p(G^2)$ spaces ($1 \leqslant p < \infty$) and in $C(G^2)$. Namely, we prove

$$\left\|\sigma^T_{m,n}(f) - f\right\|_p \leqslant c \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-|m|}\right)$$

in some special cases. Moreover, we give an application for functions in Lipschitz classes $\mathrm{Lip}(\alpha, p, G^2)$ ($\alpha > 0$, $1 \leqslant p < \infty$) and $\mathrm{Lip}(\alpha, C(G^2))$ ($\alpha > 0$).

1. Definitions and notations

Let $\mathbb{P}$ be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let denote the discrete cyclic group of order 2 by $\mathbb{Z}_2$. The group operation is the modulo 2 addition. Let every subset be open. The normalized Haar measure μ on $\mathbb{Z}_2$ is given in the way that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. That is, the measure of a singleton is $1/2$. $G := \mathop{\times}\limits_{k=0}^{\infty} \mathbb{Z}_2$, which is called the Walsh group. The elements of Walsh group G are the $0,1$ sequences. That is, $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0,1\}$ where $k \in \mathbb{N}$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure μ is the product measure and the topology is the product topology. For an other topology on the Walsh group see e.g. [10].

Dyadic intervals are defined in the usual way

$$I_0(x) := G, \;\; I_n(x) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for $x \in G$, $n \in \mathbb{P}$. They form a base for the neighbourhoods of G. Let $0 := (0 : i \in \mathbb{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbb{N}$.

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Now, we introduce some concepts of Walsh-Fourier analysis. The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k},$$

where $x \in G$ and $k \in \mathbb{N}$. The sequence of the Walsh-Paley functions is the product system of the Rademacher functions. Namely, every natural number n can uniquely be expressed in the number system based 2, in the form

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0,1\},$$

where only a finite number of n_i's different from zero. Let the order of $n \in \mathbb{P}$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. It means $2^{|n|} \leqslant n < 2^{|n|+1}$. The Walsh-Paley functions are $w_0 := 1$ and for $n \in \mathbb{P}$

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

Two dimensional Fourier-coefficients, partial sum of the Fourier series and the nth Dirichlet kernel is defined as

$$\hat{f}(n^1,n^2) := \int_{G^2} f(x^1,x^2) w_{n^1}(x^1) w_{n^2}(x^2) d\mu(x^1,x^2),$$

$$S_{n^1,n^2}(f;x^1,x^2) := \sum_{j^1=0}^{n^1-1} \sum_{j^2=0}^{n^2-1} \hat{f}(j^1,j^2) w_{j^1}(x^1) w_{j^2}(x^2),$$

$$D_n := \sum_{k=0}^{n-1} w_k, \quad D_0 := 0.$$

Fejér kernels are defined as the arithmetical means of Dirichlet kernels, that is,

$$K_n := \frac{1}{n} \sum_{k=1}^{n} D_k.$$

In dimension 2, the Marcinkiewicz kernels are defined as follows

$$\mathscr{K}_n(x^1,x^2) := \frac{1}{n} \sum_{k=1}^{n} D_k(x^1) D_k(x^2).$$

Let $T := \left(t_{k,n}\right)_{k,n=1}^{\infty}$ be a doubly infinite matrix of non-negative numbers. It is always supposed that matrix T is triangular, so $t_{k,n} := 0$ if $k > n$. We also introduce the notation $\Delta t_{k,n} := t_{k,n} - t_{k+1,n}$, where $k \in \{1,\dots,n\}$.

Let us define the de la Vallée Poussin type Marcinkiewicz matrix transform means and kernels as follows:

$$\sigma_{m,n}^T(f;x^1,x^2) := \frac{1}{T_{m,n}} \sum_{k=m}^{n} t_{k,n} S_{k,k}(f;x^1,x^2),$$

$$K_{m,n}^T(x^1,x^2) := \frac{1}{T_{m,n}} \sum_{k=m}^{n} t_{k,n} D_k(x^1) D_k(x^2),$$

where $T_{m,n} := \sum_{k=m}^{n} t_{k,n}$.

It is very easy to verify that

$$\sigma_{m,n}^{T}(f;x) = \int_{G^2} f(u) K_{m,n}^{T}(u+x) d\mu(u),$$

where $x := (x^1, x^2)$ and $u := (u^1, u^2)$.

For two-dimensional variable $(x^1, x^2) \in G^2$ we use the notations

$$r_n^1(x^1,x^2) := r_n(x^1),\ \ D_n^1(x^1,x^2) := D_n(x^1),\ \ K_n^1(x^1,x^2) := K_n(x^1),$$
$$r_n^2(x^1,x^2) := r_n(x^2),\ \ D_n^2(x^1,x^2) := D_n(x^2),\ \ K_n^2(x^1,x^2) := K_n(x^2),$$

for any $n \in \mathbb{N}$.

Let us denote the set of Walsh polynomials P_n with order less than n by $\mathscr{P}_n$, where

$$P_n := \sum_{k=0}^{n-1} c_k w_k$$

and $c_k,\ k \in \{0,\dots,n-1\}$ complex numbers.

The two-dimensional Walsh polynomials are defined analogically. That is,

$$P_{n^1,n^2}(x^1,x^2) := \sum_{j^1=0}^{n^1-1} \sum_{j^2=0}^{n^2-1} c_{j^1,j^2} w_{j^1}(x^1) w_{j^2}(x^2)$$

and $c_{j^1,j^2},\ j^1 \in \{0,\dots,n^1-1\},\ j^2 \in \{0,\dots,n^2-1\}$ complex numbers. Let us denote the set of two-dimensional Walsh polynomials with order less than (n^1,n^2) by $\mathscr{P}_{n^1,n^2}$.

Let $L^p(G^2)$ denote the usual Lebesgue spaces on G^2 (with the corresponding norm $\|.\|_p$).

For the sake of brevity in notation, we agree to write $L^\infty(G^2)$ instead of $C(G^2)$ and set $\|f\|_\infty := \sup\{|f(x)| : x \in G^2\}$. Of course, it is clear that the space $L^\infty(G^2)$ is not the same as the space of continuous functions on G^2, i.e. it is a proper subspace of it. But since in the case of continuous functions the supremum norm and the $L^\infty(G^2)$ norm are the same, for convenience we hope the reader will be able to tolerate this simplification in notation.

The partial modulus of continuity for $f \in L^p(G^2)$ are defined by

$$\omega_p^1(f,\delta) := \sup_{|t|<\delta} \|f(x^1+t,x^2) - f(x^1,x^2)\|_p,$$

$$\omega_p^2(f,\delta) := \sup_{|t|<\delta} \|f(x^1,x^2+t) - f(x^1,x^2)\|_p,$$

with the notation

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \quad \text{for all } x \in G,$$

and $\delta > 0$. In the case $f \in C(G^2)$ we change p by ∞.

We define the mixed modulus of continuity as follows

$$\omega_p^{1,2}(f,\delta_1,\delta_2)$$
$$:= \sup\{\|f(.+x^1,.+x^2)-f(.+x^1,.)-f(.,.+x^2)+f(.,.)\|_p : |x^1|\leqslant\delta_1, |x^2|\leqslant\delta_2\},$$

where $\delta_1,\delta_2>0$.

The i-th Lipschitz classes in $L^p(G^2)$ (for each $\alpha>0$) are defined as

$$\mathrm{Lip}(\alpha,p,i,G^2) := \{f\in L^p(G^2) : \omega_p^i(f,\delta) = O(\delta^\alpha) \text{ as } \delta\to 0\},$$

where $i\in\{1,2\}$.

2. Historical overview

In classical book of F. Schipp, W. R. Wade, P. Simon, and J. Pál [28], on p. 191. we can read inequality

$$\|\sigma_{2^s}(f)-f\|_X \leqslant \omega^{(X)}\left(f,2^{-s}\right)+\sum_{k=0}^{s-1}2^{k-s}\omega^{(X)}\left(f,2^{-k}\right), \tag{1}$$

where σ is the Fejér mean operator, X is a homogeneous Banach space (for example any L^p space, where $1\leqslant p<\infty$ and the space of continuous functions C) and $\omega^{(X)}$ is the modulus of continuity for functions in X. Our paper is motivated by the work of Móricz and Siddiqi [23] on the Walsh-Nörlund summation method and the result of Móricz and Rhoades [22] on the Walsh weighted mean method. Móricz and Siddiqi [23], and later Móricz and Rhoades [22] in these papers proved their generalized results in an analogous form to inequality (1). That was our aim in this paper, too, as you can see in Lemma 13.

As special cases, Móricz and Siddiqi obtained the earlier results given by Yano [35], Jastrebova [18] and Skvortsov [29] on the rate of the approximation by Cesàro means. The approximation properties of the Walsh-Cesàro means of negative order were studied by Goginava [16], the Vilenkin case was investigated by Shavardenidze [27] and Tepnadze [31]. A common generalization of these two results of Móricz and Siddiqi [23] and Móricz and Rhoades [22] was given by Nagy and the author [6].

In 2008, Fridli, Manchanda and Siddiqi generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces [15]. Recently, the author, Baramidze, Gát, Memić, Persson, Tephnadze and Wall presented some results with respect to this topic [2, 9, 21]. See [14, 33], as well. For the two-dimensional results see [8, 24, 25].

Matrix transforms means are common generalizations of several well-known summation methods. It follows by simple consideration that the Nörlund means, the Fejér (or the $(C,1)$) and the (C,α) means are special cases of the matrix transform summation method introduced above.

For matrix transforms means with respect to trigonometric system see e.g. results of Chandra [12] and Leindler [20], to Walsh system see paper of Blyumin [11].

We mention, that Iofina and Volosivets obtained similar – but one dimensional – results on Vilenkin systems with similar assumptions using different methods (independently form technics of Fridli, Móricz, Rhoades, Siddiqi and others) with respect to matrix transform means in [17].

For Marcinkiewicz means and other two-dimensional results on Walsh-Paley system see e.g. [5, 8, 25, 26].

De la Vallée Poussin means for Walsh-Paley system was introduced in [3], in a special case. In general see [4]. For de la Vallée Poussin's means in case of the trigonometric system see articles [19] and [30].

It is important to note that in the paper of Chripkó [13] some methods and results with respect to Jacobi-Fourier series gave us some ideas and used in this paper.

3. Auxiliary results

LEMMA 1. (Paley's lemma [28], p. 7) *For* $n \in \mathbb{N}$

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } \ x \in I_n, \\ 0, & \text{if } \ x \notin I_n. \end{cases}$$

LEMMA 2. ([28], p. 34.) *For* $j,n \in \mathbb{N},\ j < 2^n$ *we have*

$$D_{2^n+j} = D_{2^n} + r_n D_j.$$

The next lemma is also a simple one. It can be found in several articles in the literature. See for example article [1].

LEMMA 3. *For* $j,n \in \mathbb{N},\ j < 2^n$ *we have*

$$D_{2^n-j} = D_{2^n} - w_{2^n-1} D_j.$$

In 2018, Toledo improved Yano's [34] classical inequality.

LEMMA 4. (Toledo [32]) *For all* $n \in \mathbb{N}$

$$\|K_n\|_1 \leqslant \frac{17}{15}$$

holds.

In this paper we will use only the boundedness of $\|K_n\|_1$.

The next lemma is a well-known result, for proof see for example [8].

LEMMA 5. (Blahota, Nagy, Tephnadze [8]) *There exists a positive constant* c *such that*

$$\|\mathcal{K}_n\|_1 \leqslant c \quad \text{for all } n \in \mathbb{N}.$$

In the next lemma, we give a decomposition of the kernels $K_{m,n}^{T}(x^1,x^2)$, which is performed according to the dyadic blocks within the index range.

LEMMA 6. *Let $m,n\in\mathbb{P}$ and suppose that $|m|<|n|$. Then we have*

$$
\begin{aligned}
T_{m,n}K_{m,n}^{T} = {} & D^1_{2^{|m|+1}}D^2_{2^{|m|+1}}\sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n}\\
&+w^1_{2^{|m|+1}-1}w^2_{2^{|m|+1}-1}\sum_{k=1}^{2^{|m|+1}-m-1}\Delta t_{2^{|m|+1}-k-1,n}k\mathscr{K}_k\\
&+w^1_{2^{|m|+1}-1}w^2_{2^{|m|+1}-1}t_{m,n}(2^{|m|+1}-m)\mathscr{K}_{2^{|m|+1}-m}\\
&-\sum_{i=1}^{2}D^{3-i}_{2^{|m|+1}}w^i_{2^{|m|+1}-1}\sum_{k=1}^{2^{|m|+1}-m-1}\Delta t_{2^{|m|+1}-k-1,n}kK^i_k\\
&-\sum_{i=1}^{2}D^{3-i}_{2^{|m|+1}}w^i_{2^{|m|+1}-1}t_{m,n}(2^{|m|+1}-m)K^i_{2^{|m|+1}-m}\\
&+\sum_{j=0}^{|n|-|m|-2}D^1_{2^{|m|+j+1}}D^2_{2^{|m|+j+1}}\sum_{k=0}^{2^{|m|+j+1}-1}t_{2^{|m|+j+1}+k,n}\\
&+\sum_{j=0}^{|n|-|m|-2}r^1_{|m|+j+1}r^2_{|m|+j+1}\sum_{k=1}^{2^{|m|+j+1}-1}\Delta t_{2^{|m|+j+1}+k,n}k\mathscr{K}_k\\
&+\sum_{j=0}^{|n|-|m|-2}r^1_{|m|+j+1}r^2_{|m|+j+1}t_{2^{|m|+j+2}-1,n}(2^{|m|+j+1}-1)\mathscr{K}_{2^{|m|+j+1}-1}\\
&+\sum_{i=1}^{2}\sum_{j=0}^{|n|-|m|-2}D^{3-i}_{2^{|m|+j+1}}r^i_{|m|+j+1}\sum_{k=1}^{2^{|m|+j+1}-2}\Delta t_{2^{|m|+j+1}+k,n}kK^i_k\\
&+\sum_{i=1}^{2}\sum_{j=0}^{|n|-|m|-2}D^{3-i}_{2^{|m|+j+1}}r^i_{|m|+j+1}t_{2^{|m|+j+2}-1,n}(2^{|m|+j+1}-1)K^i_{2^{|m|+j+1}-1}\\
&+D^1_{2^{|n|}}D^2_{2^{|n|}}\sum_{k=0}^{n-2^{|n|}}t_{2^{|n|}+k,n}+r^1_{|n|}r^2_{|n|}\sum_{k=1}^{n-2^{|n|}}\Delta t_{2^{|n|}+k,n}k\mathscr{K}_k\\
&+\sum_{i=1}^{2}D^{3-i}_{2^{|n|}}r^i_{|n|}\sum_{k=1}^{n-2^{|n|}}\Delta t_{2^{|n|}+k,n}kK^i_k\\
=: {} & \sum_{i=1}^{13}K_{i,m,n}.
\end{aligned}
$$

Proof. Divide the kernel function $K_{m,n}^{T}$ into parts:

$$
\begin{aligned}
T_{m,n}K_{m,n}^{T} &= \sum_{l=m}^{2^{|m|+1}-1}t_{l,n}D^1_lD^2_l+\sum_{j=|m|+1}^{|n|-1}\sum_{l=2^j}^{2^{j+1}-1}t_{l,n}D^1_lD^2_l+\sum_{l=2^{|n|}}^{n}t_{l,n}D^1_lD^2_l\\
&=: K^A_{m,n}+K^B_{m,n}+K^C_n.
\end{aligned}
$$

Now, we apply Lemma 3 for the expression $K_{m,n}^A$ and Lemma 2 for $K_{m,n}^B$ and K_n^C. We get

$$\begin{aligned}
K_{m,n}^A &= \sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n} D^1_{2^{|m|+1}-k} D^2_{2^{|m|+1}-k} \\
&= \sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n} \left(D^1_{2^{|m|+1}} - w^1_{2^{|m|+1}-1} D^1_k\right)\left(D^2_{2^{|m|+1}} - w^2_{2^{|m|+1}-1} D^2_k\right) \\
&= D^1_{2^{|m|+1}} D^2_{2^{|m|+1}} \sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n} + w^1_{2^{|m|+1}-1} w^2_{2^{|m|+1}-1} \sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n} D^1_k D^2_k \\
&\quad - \sum_{i=1}^{2} D^{3-1}_{2^{|m|+1}} w^i_{2^{|m|+1}-1} \sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n} D^i_k.
\end{aligned}$$

Using the Abel-transform we obtain for $D_k^i,\ i \in \{1,2\}$ and $D_k^1 D_k^2$

$$\sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n} D^i_k = \sum_{k=1}^{2^{|m|+1}-m-1} \Delta t_{2^{|m|+1}-k-1,n} k K^i_k + t_{m,n}(2^{|m|+1}-m) K^i_{2^{|m|+1}-m}.$$

$$\sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n} D^1_k D^2_k = \sum_{k=1}^{2^{|m|+1}-m-1} \Delta t_{2^{|m|+1}-k-1,n} k \mathscr{K}_k + t_{m,n}(2^{|m|+1}-m) \mathscr{K}_{2^{|m|+1}-m}.$$

Let us see $K_{m,n}^B$.

$$\begin{aligned}
K_{m,n}^B &= \sum_{j=|m|+1}^{|n|-1} \sum_{k=0}^{2^j-1} t_{2^j+k,n} D^1_{2^j+k} D^2_{2^j+k} \\
&= \sum_{j=|m|+1}^{|n|-1} \sum_{k=0}^{2^j-1} t_{2^j+k,n} \left(D^1_{2^j} + r^1_j D^1_k\right)\left(D^2_{2^j} + r^2_j D^2_k\right) \\
&= \sum_{j=|m|+1}^{|n|-1} D^1_{2^j} D^2_{2^j} \sum_{k=0}^{2^j-1} t_{2^j+k,n} + \sum_{j=|m|+1}^{|n|-1} r^1_j r^2_j \sum_{k=1}^{2^j-1} t_{2^j+k,n} D^1_k D^2_k \\
&\quad + \sum_{i=1}^{2} \sum_{j=|m|+1}^{|n|-1} r^i_j D^{3-i}_{2^j} \sum_{k=1}^{2^j-1} t_{2^j+k,n} D^i_k \\
&= \sum_{j=0}^{|n|-|m|-2} D^1_{2^{|m|+j+1}} D^2_{2^{|m|+j+1}} \sum_{k=0}^{2^{|m|+j+1}-1} t_{2^{|m|+j+1}+k,n} \\
&\quad + \sum_{j=0}^{|n|-|m|-2} r^1_{|m|+j+1} r^2_{|m|+j+1} \sum_{k=1}^{2^{|m|+j+1}-1} t_{2^{|m|+j+1}+k,n} D^1_k D^2_k \\
&\quad + \sum_{i=1}^{2} \sum_{j=0}^{|n|-|m|-2} r^i_{|m|+j+1} D^{3-i}_{2^{|m|+j+1}} \sum_{k=1}^{2^{|m|+j+1}-1} t_{2^{|m|+j+1}+k,n} D^i_k.
\end{aligned}$$

Using the Abel-transform for $D_k^i,\ i\in\{1,2\}$ and $D_k^1D_k^2$

$$\sum_{k=1}^{2^{|m|+j+1}-1} t_{2^{|m|+j+1}+k,n}D_k^i$$
$$=\sum_{k=1}^{2^{|m|+j+1}-2} \Delta t_{2^{|m|+j+1}+k,n}kK_k^i+t_{2^{|m|+j+2}-1,n}(2^{|m|+j+1}-1)K^i_{2^{|m|+j+1}-1},$$

$$\sum_{k=1}^{2^{|m|+j+1}-1} t_{2^{|m|+j+1}+k,n}D_k^1D_k^2$$
$$=\sum_{k=1}^{2^{|m|+j+1}-2} \Delta t_{2^{|m|+j+1}+k,n}k\mathscr{K}_k+t_{2^{|m|+j+2}-1,n}(2^{|m|+j+1}-1)\mathscr{K}_{2^{|m|+j+1}-1}.$$

Similarly,

$$\begin{aligned}
K_n^C &= \sum_{k=0}^{n-2^{|n|}} t_{2^{|n|}+k,n}D^1_{2^{|n|}+k}D^2_{2^{|n|}+k}\\
&= \sum_{k=0}^{n-2^{|n|}} t_{2^{|n|}+k,n}\left(D^1_{2^{|n|}}+r^1_{|n|}D^1_k\right)\left(D^2_{2^{|n|}}+r^2_{|n|}D^2_k\right)\\
&=D^1_{2^{|n|}}D^2_{2^{|n|}}\sum_{k=0}^{n-2^{|n|}} t_{2^{|n|}+k,n}+r^1_{|n|}r^2_{|n|}\sum_{k=0}^{n-2^{|n|}} t_{2^{|n|}+k,n}D_k^1D_k^2\\
&\quad+\sum_{i=1}^{2} D^{3-1}_{2^{|n|}}r^i_{|n|}\sum_{k=0}^{n-2^{|n|}} t_{2^{|n|}+k,n}D_k^i
\end{aligned}$$

Using $t_{n+1,n}=0$, for $D_k^i,\ i\in\{1,2\}$ and $D_k^1D_k^2$ the Abel-transform again, we obtain

$$\begin{aligned}
\sum_{k=1}^{n-2^{|n|}} t_{2^{|n|}+k,n}D_k^i &= \sum_{k=1}^{n-2^{|n|}-1} \Delta t_{2^{|n|}+k,n}kK_k^i+t_{n,n}(n-2^{|n|})K^i_{n-2^{|n|}}\\
&= \sum_{k=1}^{n-2^{|n|}} \Delta t_{2^{|n|}+k,n}kK_k^i,
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{n-2^{|n|}} t_{2^{|n|}+k,n}D_k^1D_k^2 &= \sum_{k=1}^{n-2^{|n|}-1} \Delta t_{2^{|n|}+k,n}k\mathscr{K}_k+t_{n,n}(n-2^{|n|})\mathscr{K}_{n-2^{|n|}}\\
&= \sum_{k=1}^{n-2^{|n|}} \Delta t_{2^{|n|}+k,n}k\mathscr{K}_k.
\end{aligned}$$

It completes the proof of Lemma 6. □

LEMMA 7. *Let $m,n \in \mathbb{P}$ and suppose that $m < n$, but $|m| = |n|$. Then we have*

$$
\begin{aligned}
T_{m,n}K^T_{m,n} = D^1_{2^{|m|+1}} D^2_{2^{|m|+1}} &\sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n} \\
&+ w^1_{2^{|m|+1}-1} w^2_{2^{|m|+1}-1} \sum_{k=1}^{2^{|m|+1}-m-1} \Delta t_{2^{|m|+1}-k-1,n} k\mathscr{K}_k \\
&+ w^1_{2^{|m|+1}-1} w^2_{2^{|m|+1}-1} t_{m,n}(2^{|m|+1}-m)\mathscr{K}_{2^{|m|+1}-m} \\
&- \sum_{i=1}^{2} D^i_{2^{|m|+1}} w^{3-i}_{2^{|m|+1}-1} \sum_{k=1}^{2^{|m|+1}-m-1} \Delta t_{2^{|m|+1}-k-1,n} kK^{3-i}_k \\
&- \sum_{i=1}^{2} D^i_{2^{|m|+1}} w^{3-i}_{2^{|m|+1}-1} t_{m,n}(2^{|m|+1}-m) K^{3-i}_{2^{|m|+1}-m}
\end{aligned}
$$

Proof. Since $t_{k,n} = 0$ if $k > n$, we get from the proof of Lemma 6, that in this case

$$
T_{m,n}K^T_{m,n} = \sum_{l=m}^{2^{|m|+1}-1} t_{l,n} D^1_l D^2_l = K^A_{m,n}. \quad \square
$$

LEMMA 8. (Nagy [24]) *Let $P \in \mathscr{P}_{2^A,2^B}, f \in L^p(G^2)$, where $A,B \in \mathbb{P}$ and $1 \leqslant p \leqslant \infty$. Then there exists a positive constant c such that*

$$
\left\| \int_{G^2} (f(.+x) - f(.)) r_A(x^1) r_B(x^2) P(x) d\mu(x) \right\|_p \leqslant c\|P\|_1 \omega^{1,2}_p(f, 2^{-A}, 2^{-B}),
$$

with the notation $x = (x^1, x^2) \in G^2$.

LEMMA 9. (Blahota and Nagy [7]) *Let $P \in \mathscr{P}_{2^A}$, $f \in L^p(G^2)$ $(1 \leqslant p \leqslant \infty)$ and $A \in \mathbb{P}$. Then*

$$
\left\| \int_{G^2} (f(.+x) - f(.)) D_{2^A}(x^{3-i}) r_A(x^i) P(x^i) d\mu(x) \right\|_p \leqslant 2\|P\|_1 \omega^i_p(f, 2^{-A}),
$$

where $i \in \{1,2\}$.

Next two lemmas can be proved similarly to the cases Lemma 8 and Lemma 9, but using the following method based on Watari's estimation is simpler.

LEMMA 10. *Let $P \in \mathscr{P}_{2^A,2^B}, f \in L^p(G^2)$, where $A,B \in \mathbb{P}$ and $1 \leqslant p \leqslant \infty$. Then*

$$
\left\| \int_{G^2} (f(.+x) - f(.)) w_{2^{A+1}-1}(x^1) w_{2^{B+1}-1}(x^2) P(x) d\mu(x) \right\|_p \leqslant \|P\|_1 \omega^{1,2}_p(f, 2^{-A}, 2^{-B}),
$$

with the notation $x = (x^1, x^2) \in G^2$.

Proof. We carry out proofs of this lemma and the next one in spaces $L^p(G^2)$ $(1 \leqslant p < \infty)$. In the space $C(G^2)$ the proof is similar, it is even simpler.

Because of the orthonormality of the Walsh-Paley system

$$\int_{G^2} w_{2^{A+1}-1}(x^1) w_{2^{B+1}-1}(x^2) P(x) d\mu(x) = 0 \tag{2}$$

holds. Since $S_{2^A,2^B}(f) \in \mathscr{P}_{2^A,2^B}$, using equality (2) twice we obtain

$$\begin{aligned}
&\int_{G^2} (f(t+x) - f(t)) w_{2^{A+1}-1}(x^1) w_{2^{B+1}-1}(x^2) P(x) d\mu(x) \\
&\quad = \int_{G^2} f(t+x) w_{2^{A+1}-1}(x^1) w_{2^{B+1}-1}(x^2) P(x) d\mu(x) \\
&\quad = \int_{G^2} (f(t+x) - S_{2^A,2^B}(f;t+x)) w_{2^{A+1}-1}(x^1) w_{2^{B+1}-1}(x^2) P(x) d\mu(x).
\end{aligned}$$

So, by the generalized Minkowski inequality it follows

$$\begin{aligned}
&\left\| \int_{G^2} (f(.+x) - f(.)) w_{2^{A+1}-1}(x^1) w_{2^{B+1}-1}(x^2) P(x) d\mu(x) \right\|_p \\
&\quad = \left(\int_{G^2} \left| \int_{G^2} (f(t+x) - S_{2^A,2^B}(f;t+x)) \right.\right. \\
&\qquad \left.\left. \times\ w_{2^{A+1}-1}(x^1) w_{2^{B+1}-1}(x^2) P(x) d\mu(x) \right|^p d\mu(t) \right)^{1/p} \\
&\quad \leqslant \int_{G^2} \left(\int_{G^2} \left| f(t+x) - S_{2^A,2^B}(f;t+x) \right|^p d\mu(t) \right)^{1/p} |P(x)| d\mu(x)
\end{aligned}$$

The two-dimensional version of the Watari-Efimov inequality says

$$\left\| f - S_{2^A,2^B}(f) \right\|_p \leqslant \omega_p^{1,2}(f, 2^{-A}, 2^{-B}),$$

so this completes the proof of Lemma 10. □

LEMMA 11. *Let $P \in \mathscr{P}_{2^B}$, $f \in L^p(G^2)$ $(1 \leqslant p \leqslant \infty)$ and $A, B \in \mathbb{P}$. Then*

$$\left\| \int_{G^2} (f(.+x) - f(.)) D_{2^{A-1}}(x^{3-i}) w_{2^{B+1}-1}(x^i) P(x^i) d\mu(x) \right\|_p \leqslant \|P\|_1 \omega_p^i(f, 2^{-B}),$$

where $i \in \{1,2\}$.

Proof. Using Paley's lemma (Lemma 1) and the Watari-Efimov inequality this proof is similar to the proof of Lemma 10. □

The next lemma is a special case of the main result of paper [7], there was named as Theorem 1. (See also [8]) On the other hand, we use only a part of that statement, only for non-decreasing $t_{k,n}$ sequences. On that paper authors used another definitions and notations. We cite it with our forms.

LEMMA 12. (Blahota and Nagy [7]) *Let* $f \in L^p(G^2)$, *where* $1 \leqslant p \leqslant \infty$ *and* $n \in \mathbb{P}$. *Let the finite sequences* $\{t_{k,n} : 1 \leqslant k \leqslant n\}$ *of non-negative numbers be non-decreasing for all* n. *We suppose that*

$$\sum_{k=1}^{n} t_{k,n} = 1$$

and

$$t_{n,n} = O(n^{-1}).$$

Then

$$\|\sigma_{1,n}^T(f) - f\|_p \leqslant c \sum_{i=1}^{2} \sum_{j=0}^{|n|-1} 2^j t_{2^{j+1}-1,n} \omega_p^i \left(f, 2^{-j}\right) + c \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-|n|}\right).$$

The statement of next lemma has mostly analogous form as inequality (1), Lemma 12, results of Móricz, Siddiqi [23] and others.

LEMMA 13. *Let* $f \in L^p(G^2)$, *where* $1 \leqslant p \leqslant \infty$ *and* $m, n \in \mathbb{P}$, *where* $|m| < |n|$. *Let the finite sequences of non-negative numbers* $\{t_{k,n} : m \leqslant k \leqslant n\}$ *be non-increasing for all* n. *We suppose that*

$$\frac{t_{m,n}}{T_{m,n}} = O(m^{-1}) \tag{3}$$

hold. Then

$$\left\|\sigma_{m,n}^T(f) - f\right\|_p \leqslant \frac{c}{T_{m,n}} \sum_{i=1}^{2} \sum_{j=0}^{|n|-|m|-2} 2^{|m|+j+1} t_{2^{|m|+j+1},n} \omega_p^i \left(f, 2^{-|m|-j-1}\right)$$
$$+ c \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-|m|}\right).$$

Proof. We prove the lemma for $L^p(G^2)$ spaces $1 \leqslant p < \infty$. For $C(G^2)$ the proof is similar. Let us set $f \in L^p(G^2)$. It follows

$$\begin{aligned}
\left\|\sigma_{m,n}^T(f) - f\right\|_p &= \left(\int_{G^2} \left|\sigma_{m,n}^T(f;x) - f(x)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&= \left(\int_{G^2} \left|\int_{G^2} K_{m,n}^T(u)(f(x+u) - f(x)) d\mu(u)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&\leqslant \sum_{i=1}^{13} \left\|\int_{G^2} \frac{1}{T_{m,n}} K_{i,m,n}(u)(f(.+u) - f(.)) d\mu(u)\right\|_p \\
&=: \frac{1}{T_{m,n}} \sum_{i=1}^{13} I_{i,m,n}.
\end{aligned}$$

Using generalized Minkowski's inequality [36, vol. 1, p. 19] and inequality

$$|f(x+u) - f(x)| \leqslant |f(x^1+u^1, x^2+u^2) - f(x^1+u^1, x^2)| + |f(x^1+u^1, x^2) - f(x^1, x^2)|$$

we write that for any $j \in \mathbb{N}$

$$
\begin{aligned}
&\left\| \int_{G^2} D_{2^j}(u^1) D_{2^j}(u^2) (f(.+u) - f(.)) d\mu(u) \right\|_p \\
&\leqslant \int_{G^2} D_{2^j}(u^1) D_{2^j}(u^2) \left(\int_{G^2} |f(x+u) - f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} d\mu(u) \\
&\leqslant \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-j}\right). \qquad (4)
\end{aligned}
$$

Applying inequality (4), Lemma 1 for the expressions $I_{1,m,n}$, $I_{6,m,n}$ and $I_{11,m,n}$, we obtain that

$$
\begin{aligned}
I_{1,m,n} &\leqslant \sum_{k=1}^{2^{|m|+1}-m} t_{2^{|m|+1}-k,n} \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-|m|-1}\right) \\
&\leqslant \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-|m|-1}\right) \sum_{k=m}^{2^{|m|+1}-1} t_{k,n} \\
&\leqslant T_{m,n} \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-|m|}\right),
\end{aligned}
$$

$$
\begin{aligned}
I_{6,m,n} &\leqslant \sum_{i=1}^{2} \sum_{j=0}^{|n|-|m|-2} \omega_p^i \left(f, 2^{-|m|-j-1}\right) \sum_{k=0}^{2^{|m|+j+1}-1} t_{2^{|m|+j+1}+k,n} \\
&\leqslant \sum_{i=1}^{2} \sum_{j=0}^{|n|-|m|-2} \omega_p^i \left(f, 2^{-|m|-j-1}\right) 2^{|m|+j+1} t_{2^{|m|+j+1},n}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{11,m,n} &\leqslant \sum_{k=0}^{n-2^{|n|}} t_{2^{|n|}+k,n} \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-|n|}\right) \\
&\leqslant \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-|n|}\right) \sum_{k=2^{|n|}}^{n} t_{k,n} \\
&\leqslant T_{m,n} \sum_{i=1}^{2} \omega_p^i \left(f, 2^{-|m|}\right).
\end{aligned}
$$

We discuss expressions $I_{9,m,n}, I_{10,m,n}$ and $I_{13,m,n}$ based on Lemma 4 and Lemma 9.

We get

$$
\begin{aligned}
I_{9,m,n} &\leqslant c \sum_{i=1}^{2} \sum_{j=0}^{|n|-|m|-2} \sum_{k=1}^{2^{|m|+j+1}-2} \left| \Delta t_{2^{|m|+j+1}+k,n} \right| k \omega_p^i \left(f, 2^{-|m|-j-1}\right) \left\| K_k^i \right\|_1 \\
&\leqslant c \sum_{i=1}^{2} \sum_{j=0}^{|n|-|m|-2} \sum_{k=1}^{2^{|m|+j+1}-2} \left| \Delta t_{2^{|m|+j+1}+k,n} \right| k \omega_p^i \left(f, 2^{-|m|-j-1}\right).
\end{aligned}
$$

We write

$$\sum_{k=1}^{2^{|m|+j+1}-2}\left|\Delta t_{2^{|m|+j+1}+k,n}\right|k=\sum_{k=1}^{2^{|m|+j+1}-2}t_{2^{|m|+j+1}+k,n}-(2^{|m|+j+1}-2)t_{2^{|m|+j+2}-1,n}$$
$$\leqslant\sum_{k=1}^{2^{|m|+j+1}-2}t_{2^{|m|+j+1}+k,n}\leqslant 2^{|m|+j+1}t_{2^{|m|+j+1},n} \tag{5}$$

and

$$I_{9,m,n}\leqslant c\sum_{i=1}^{2}\sum_{j=0}^{|n|-|m|-2}2^{|m|+j+1}t_{2^{|m|+j+1},n}\omega_p^i\left(f,2^{-|m|-j-1}\right).$$

Since $t_{2^{|m|+j+2}-1,n}\leqslant t_{2^{|m|+j+1},n}$, easy to see that

$$I_{10,m,n}\leqslant c\sum_{i=1}^{2}\sum_{j=0}^{|n|-|m|-2}t_{2^{|m|+j+2}-1,n}(2^{|m|+j+1}-1)\omega_p^i\left(f,2^{-|m|-j-1}\right)\left\|K^i_{2^{|m|+j+1}-1}\right\|_1$$
$$\leqslant c\sum_{i=1}^{2}\sum_{j=0}^{|n|-|m|-2}2^{|m|+j+1}t_{2^{|m|+j+1},n}\omega_p^i\left(f,2^{-|m|-j-1}\right).$$

Now, we estimate the expression $I_{13,m,n}$.

$$I_{13,m,n}\leqslant c\sum_{i=1}^{2}\sum_{k=1}^{n-2^{|n|}}\left|\Delta t_{2^{|n|}+k,n}\right|k\omega_p^i\left(f,2^{-|n|}\right)\left\|K_k^i\right\|_1$$
$$\leqslant c\sum_{i=1}^{2}\omega_p^i\left(f,2^{-|n|}\right)\sum_{k=1}^{n-2^{|n|}}\left|\Delta t_{2^{|n|}+k,n}\right|k.$$

We obtain

$$\sum_{k=1}^{n-2^{|n|}}\left|\Delta t_{2^{|n|}+k,n}\right|k=\sum_{k=1}^{n-2^{|n|}}t_{2^{|n|}+k,n}$$
$$\leqslant\sum_{k=m}^{n}t_{k,n}=T_{m,n}$$

and

$$I_{13,m,n}\leqslant cT_{m,n}\sum_{i=1}^{2}\omega_p^i\left(f,2^{-|n|}\right)$$
$$\leqslant cT_{m,n}\sum_{i=1}^{2}\omega_p^i\left(f,2^{-|m|}\right).$$

For estimating expressions $I_{7,m,n},I_{8,m,n}$ and $I_{12,m,n}$ we will use Lemma 12 and

Lemma 8. At first

$$I_{7,m,n} \leqslant \sum_{j=0}^{|n|-|m|-2} \sum_{k=1}^{2^{|m|+j+1}-2} \left|\Delta t_{2^{|m|+j+1}+k,n}\right| k\omega_p^{1,2}\left(f,2^{-|m|-j-1},2^{-|m|-j-1}\right)\|\mathscr{K}_k\|_1$$

$$\leqslant c\sum_{j=0}^{|n|-|m|-2} \omega_p^{1,2}\left(f,2^{-|m|-j-1},2^{-|m|-j-1}\right)\sum_{k=1}^{2^{|m|+j+1}-2}\left|\Delta t_{2^{|m|+j+1}+k,n}\right| k,$$

and using inequality (5) follows

$$I_{7,m,n} \leqslant c\sum_{j=0}^{|n|-|m|-2} 2^{|m|+j+1} t_{2^{|m|+j+1},n}\,\omega_p^{1,2}\left(f,2^{-|m|-j-1},2^{-|m|-j-1}\right).$$

Let us see $I_{8,m,n}$. We obtain

$$I_{8,m,n} \leqslant c\sum_{j=0}^{|n|-|m|-2} t_{2^{|m|+j+2}-1,n}\left(2^{|m|+j+1}-1\right)$$

$$\times\,\omega_p^{1,2}\left(f,2^{-|m|-j-1},2^{-|m|-j-1}\right)\left\|\mathscr{K}_{2^{|m|+j+1}-1}\right\|_1$$

$$\leqslant c\sum_{j=0}^{|n|-|m|-2} t_{2^{|m|+j+1},n} 2^{|m|+j+1}\omega_p^{1,2}\left(f,2^{-|m|-j-1},2^{-|m|-j-1}\right).$$

Let us investigate the expression $I_{12,m,n}$.

$$I_{12,m,n} \leqslant c\sum_{k=1}^{n-2^{|n|}}\left|\Delta t_{2^{|n|}+k,n}\right| k\omega_p^{1,2}\left(f,2^{-|n|},2^{-|n|}\right)\|\mathscr{K}_k\|_1$$

$$\leqslant c\omega_p^{1,2}\left(f,2^{-|n|},2^{-|n|}\right)\sum_{k=1}^{n-2^{|n|}}\left|\Delta t_{2^{|n|}+k,n}\right| k.$$

As in case $I_{13,m,n}$, we obtain

$$I_{12,m,n} \leqslant cT_{m,n}\omega_p^{1,2}\left(f,2^{-|m|},2^{-|m|}\right).$$

For proving estimates of $I_{2,m,n}$ and $I_{3,m,n}$ we will use Lemma 5 and Lemma 10.

$$I_{2,m,n} \leqslant c\sum_{k=1}^{2^{|m|+1}-m-1}\left|\Delta t_{2^{|m|+1}-k-1,n}\right| k\omega_p^{1,2}\left(f,2^{-|m|},2^{-|m|}\right)\|\mathscr{K}_k\|_1$$

$$\leqslant c\omega_p^{1,2}\left(f,2^{-|m|},2^{-|m|}\right)\sum_{k=1}^{2^{|m|+1}-m-1}\left|\Delta t_{2^{|m|+1}-k-1,n}\right| k.$$

We have

$$\sum_{k=0}^{2^{|m|+1}-m-1}\left|\Delta t_{2^{|m|+1}-k-1,n}\right| k = -\sum_{k=0}^{2^{|m|+1}-m-2} t_{2^{|m|+1}-k-1,n} + t_{m,n}(2^{|m|+1}-m-1)$$

$$\leqslant t_{m,n}\left(2^{|m|+1}-m-1\right),$$

from condition (3)

$$\begin{aligned} I_{2,m,n} &\leqslant ct_{m,n} m\omega_p^{1,2}\left(f,2^{-|m|},2^{-|m|}\right) \\ &\leqslant cT_{m,n}\omega_p^{1,2}\left(f,2^{-|m|},2^{-|m|}\right) \end{aligned}$$

and

$$\begin{aligned} I_{3,m,n} &\leqslant ct_{m,n}(2^{|m|+1}-m)\omega_p^{1,2}\left(f,2^{-|m|},2^{-|m|}\right)\left\|\mathscr{K}_{2^{|m|+1}-m}\right\|_1 \\ &\leqslant ct_{m,n} m\omega_p^{1,2}\left(f,2^{-|m|},2^{-|m|}\right) \\ &\leqslant cT_{m,n}\omega_p^{1,2}\left(f,2^{-|m|},2^{-|m|}\right). \end{aligned}$$

Last two parts are $I_{4,m,n}$ and $I_{5,m,n}$. Their estimations are based on Lemma 4 and Lemma 11. Indeed,

$$\begin{aligned} I_{4,m,n} &\leqslant c\sum_{i=1}^{2}\sum_{k=1}^{2^{|m|+1}-m-1}\left|\Delta t_{2^{|m|+1}-k-1,n}\right| k\omega_p^i\left(f,2^{-|m|}\right)\left\|K_k^i\right\|_1 \\ &\leqslant c\sum_{i=1}^{2}\omega_p^i\left(f,2^{-|m|}\right)\sum_{k=1}^{2^{|m|+1}-m-1}\left|\Delta t_{2^{|m|+1}-k-1,n}\right| k, \end{aligned}$$

which is a similar expression what we obtained in case $I_{2,m,n}$, so analogously

$$I_{4,m,n} \leqslant cT_{m,n}\sum_{i=1}^{2}\omega_p^i\left(f,2^{-|m|}\right).$$

Since

$$\begin{aligned} I_{5,m,n} &\leqslant c\sum_{i=1}^{2} t_{m,n}(2^{|m|+1}-m)\omega_p^i\left(f,2^{-|m|}\right)\left\|\mathscr{K}_{2^{|m|+1}-m}\right\|_1 \\ &\leqslant c\sum_{i=1}^{2} t_{m,n} m\omega_p^i\left(f,2^{-|m|}\right), \end{aligned}$$

from condition (3) we get

$$I_{5,m,n} \leqslant cT_{m,n}\sum_{i=1}^{2}\omega_p^i\left(f,2^{-|m|}\right)$$

immediately, as in case $I_{3,m,n}$.

At the end, inequality

$$\omega_p^{1,2}\left(f,2^{-A},2^{-A}\right) \leqslant \sum_{i=1}^{2}\omega_p^i\left(f,2^{-A}\right)$$

(it needs in cases $I_{2,m,n}, I_{3,m,n}, I_{7,m,n}, I_{8,m,n}$ and $I_{12,m,n}$) completes the proof of Lemma 13. □

4. The rate of the approximation by T-means

COROLLARY 1. *Let $f \in L^p(G^2)$, $1 \leqslant p \leqslant \infty$ and $m,n \in \mathbb{P}$, where $m < n$. Let the finite sequences $\{t_{k,n} : m \leqslant k \leqslant n\}$ of nonnegative numbers be non-decreasing for all n. We suppose that*

$$\frac{t_{n,n}}{T_{m,n}} = O(n^{-1}) \tag{6}$$

holds. Then

$$\left\|\sigma^T_{m,n}(f) - f\right\|_p \leqslant c\sum_{i=1}^{2} \omega_p^i\left(f, 2^{-|m|}\right).$$

Proof. The proof is a simple consequence of Lemma 12.

Namely, in the statement of Lemma 12 let us choose $t_{1,n} := \ldots := t_{m-1,n} = 0$. Then, using monotony and condition (6) we get

$$\begin{aligned}
\|\sigma^T_{m,n}(f) - f\|_p &= \|\sigma^T_{1,n}(f) - f\|_p \\
&\leqslant \frac{c}{T_{m,n}} \sum_{i=1}^{2} \sum_{j=0}^{|n|-1} 2^j t_{2^{j+1}-1,n} \omega_p^i\left(f, 2^{-j}\right) + c\sum_{i=1}^{2} \omega_p^i\left(f, 2^{-|n|}\right) \\
&= \frac{c}{T_{m,n}} \sum_{i=1}^{2} \sum_{j=|m|}^{|n|-1} 2^j t_{2^{j+1}-1,n} \omega_p^i\left(f, 2^{-j}\right) + c\sum_{i=1}^{2} \omega_p^i\left(f, 2^{-|n|}\right) \\
&\leqslant \frac{c}{T_{m,n}} n t_{n,n} \sum_{i=1}^{2} \omega_p^i\left(f, 2^{-|m|}\right) + c\sum_{i=1}^{2} \omega_p^i\left(f, 2^{-|n|}\right) \\
&\leqslant c\sum_{i=1}^{2} \omega_p^i\left(f, 2^{-|m|}\right) + c\sum_{i=1}^{2} \omega_p^i\left(f, 2^{-|n|}\right) \\
&\leqslant c\sum_{i=1}^{2} \omega_p^i\left(f, 2^{-|m|}\right). \qquad \square
\end{aligned}$$

LEMMA 14. *Let $f \in L^p(G^2)$, where $1 \leqslant p \leqslant \infty$ and $m,n \in \mathbb{P}$, where $m < n$, but $|m| = |n|$. Let the finite sequences of non-negative numbers $\{t_{k,n} : m \leqslant k \leqslant n\}$ be non-increasing for all n. We suppose that*

$$\frac{t_{m,n}}{T_{m,n}} = O(m^{-1})$$

holds. Then

$$\left\|\sigma^T_{m,n}(f) - f\right\|_p \leqslant c\sum_{i=1}^{2} \omega_p^i\left(f, 2^{-|m|}\right).$$

Proof. It is trivial, according to Lemma 7 and the proof of Lemma 13 in cases $I_{1,m,n}, \ldots, I_{5,m,n}$. $\square$

THEOREM 1. *Let $f \in L^p(G^2)$, where $1 \leqslant p \leqslant \infty$ and $m,n \in \mathbb{P}$, where $|m| < |n|$. Let the finite sequences $\{t_{k,n} : m \leqslant k \leqslant n\}$ of nonnegative numbers be non-increasing for all n. We suppose that*

$$\frac{t_{m,n}}{T_{m,n}} = O(m^{-1})$$

holds. Then

$$\left\| \sigma^T_{m,n}(f) - f \right\|_p \leqslant c \sum_{i=1}^{2} \omega^i_p \left(f, 2^{-|m|} \right).$$

Proof. The proof is a consequence of Lemma 13.
We using inequality

$$2^{|m|+j+1} t_{2^{|m|+j+1},n} \leqslant 2 \sum_{i=1}^{2^{|m|+j}} t_{2^{|m|+j}+i,n}$$

for $j \in \{1, \ldots, |n| - |m| - 2\}$ and

$$2^{|m|+1} t_{2^{|m|+1},n} \leqslant m t_{m,n} + \sum_{i=m-2^{|m|}+1}^{2^{|m|}} t_{2^{|m|}+i,n}.$$

So

$$\begin{aligned}
\sum_{j=0}^{|n|-|m|-2} 2^{|m|+j+1} t_{2^{|m|+j+1},n} &\leqslant m t_{m,n} + \sum_{i=m-2^{|m|}+1}^{2^{|m|}} t_{2^{|m|}+i,n} + 2 \sum_{j=1}^{|n|-|m|-2} \sum_{i=1}^{2^{|m|+j}} t_{2^{|m|+j}+i,n} \\
&\leqslant c T_{m,n} + 2 \sum_{k=m}^{2^{|n|-1}} t_{k,n} \\
&\leqslant c T_{m,n} + 2 T_{m,n} = c T_{m,n}.
\end{aligned}$$

It implies that

$$\frac{1}{T_{m,n}} \sum_{j=0}^{|n|-|m|-2} 2^{|m|+j+1} t_{2^{|m|+j+1},n} \sum_{i=1}^{2} \omega^i_p \left(f, 2^{-|m|-j-1} \right) \leqslant c \sum_{i=1}^{2} \omega^i_p \left(f, 2^{-|m|} \right). \quad \square$$

At the end of this section we summarize our results.

COROLLARY 2. *Let $f \in L^p(G^2)$, where $1 \leqslant p \leqslant \infty$ and $m,n \in \mathbb{P}$, where $m < n$. Let the members of the finite sequences $\{t_{k,n} : m \leqslant k \leqslant n\}$ be non-negative numbers. If the finite sequence $\{t_{k,n} : m \leqslant k \leqslant n\}$ is nondecreasing for all n, then we suppose*

$$\frac{t_{n,n}}{T_{m,n}} = O(n^{-1})$$

and if the finite sequence $\{t_{k,n} : m \leqslant k \leqslant n\}$ *is nonincreasing for all* n, *then we suppose*

$$\frac{t_{m,n}}{T_{m,n}} = O(m^{-1}).$$

Then

$$\left\|\sigma^T_{m,n}(f) - f\right\|_p \leqslant c \sum_{i=1}^{2} \omega^i_p \left(f, 2^{-|m|}\right)$$

holds, where c *depends only on* p.

5. Application

THEOREM 2. *Let* $f \in \mathrm{Lip}(\alpha, p, G^2)$, *where* $1 \leqslant p < \infty$ *for some* $\alpha > 0$ *or* $f \in$ *Lip*$(\alpha, C(G^2))$. *For the de la Vallée Poussin type Marcinkiewicz matrix transform* $\sigma^T_{m,n}$ *we suppose that the corresponding conditions in Corollary 2 are satisfied. Then*

$$\left\|\sigma^T_{m,n}(f) - f\right\|_p = O(m^{-\alpha}).$$

Proof. It is a simple consequence of Corollary 2 for Lipschitz functions. □

A very specific case can be formulated as a statement - or rather as an example - one may say, concerning de la Vallée Poussin means.

EXAMPLE 1. As a special case, let

$$t_{k,n} := \begin{cases} (qm)^{-1}, & \text{if } m \leqslant k \leqslant n, \\ 0, & \text{otherwise }, \end{cases}$$

where $q > 0$ and $n := \lfloor (q+1)m \rfloor - 1$. In this case we get the rate of the norm convergence of de la Vallée Poussin mean for Lipschitz functions, namely if $f \in \mathrm{Lip}(\alpha, p, G)$, then

$$\left\| \frac{1}{qm} \sum_{k=m}^{n} S_{k,k}(f) - f \right\|_p = O(m^{-\alpha}).$$

Proof. This statement is a simple corollary of Theorem 2. □

Although outside the scope of this manuscript, we believe that the methods of this paper could be used to investigate the almost everywhere convergence of the corresponding means. For the specific case of de la Vallée Poussin's means (for the trigonometric system) in question, see articles [19] and [30].

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