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# AN INEQUALITY BETWEEN $L_{YJ}(\lambda,\mu,X)$ CONSTANT AND GENERALIZED JAMES CONSTANT

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*Abstract.* In this paper, a simple inequality

$$L_{YJ}(\lambda,\mu,X) \leqslant \frac{3\max\{\lambda^2,\mu^2\}+\min\{\lambda^2,\mu^2\}}{(\lambda+\mu)(\lambda^2+\mu^2)} J_{\lambda,\mu}(X)$$

was given for the constant $L_{YJ}(\lambda,\mu,X)$ and the generalized James constant $J_{\lambda,\mu}(X)$, which can be regarded as a further generalization of Alonso's inequality $C_{NJ}(X) \leqslant J(X)$.

## 1. Introduction

Let $S_X$ (resp. $B_X$) be the unit sphere (resp. the unit closed ball) of a real Banach space $X$. Throughout this paper, we always assume Banach space $X$ is nontrivial, i.e., $dim X \geqslant 2$.

Recall that the von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space $X$ was introduced by Clarkson [3] as the smallest constant C for which,

$$\frac{1}{C} \leqslant \frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \leqslant C,$$

holds for all $x,y \in X$ with $\|x\|^2+\|y\|^2 \neq 0$.

An equivalent form of this constant is

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x,y \in X, (x,y) \neq (0,0)\right\}.$$

The smaller constant than $C_{NJ}(X)$ is the following constant

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{4} : x,y \in S_X\right\},$$

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and we know that a relationship between them is as follows [1]

$$C_{NJ}(X) \leqslant 2\left[1 + C'_{NJ}(X) - \sqrt{2C'_{NJ}(X)}\right].$$

Another geometric constant is James constant which is defined by

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in B_X\}.$$

An important relationship between the von Neumann-Jordan constant and the James constant is Alonso's inequality [11, 13]

$$C_{NJ}(X) \leqslant J(X). \tag{1.1}$$

The properties of $C_{NJ}(X)$ and $J(X)$ have been investigated in many papers, and these classical constants has been generalized in many directions (for example, see [4, 6, 10, 12, 14]).

It is well known that a necessary and sufficient condition for a Banach space to be an inner product space is the parallelogram law. Instead of the parallelogram law, Moslehian and Rassias [9] have recently proved a new equivalent characterization of inner space which is called the Euler-Lagrange type identity. Motivated by the new characterization of inner space by Moslehian and Rassias, the authors in [7] introduced a new geometric constant $L_{YJ}(\lambda, \mu, X)$ for $\lambda, \mu > 0$ as follows

$$L_{YJ}(\lambda, \mu, X) = \sup\left\{\frac{\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)}, x, y \in X, (x, y) \neq (0, 0)\right\} \tag{1.2}$$

A similar constant

$$L'_{YJ}(\lambda, \mu, X) = \sup\left\{\frac{\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2}{2(\lambda^2 + \mu^2)}, x, y \in S_X\right\} \tag{1.3}$$

was also introduced by Liu et al. in [8].

Now let us collect some properties of these constants (see [7, 8]). For a Banach space $X$ , then

(i) $1 \leqslant L_{YJ}(\lambda, \mu, X) \leqslant 2$ and $1 \leqslant L'_{YJ}(\lambda, \mu, X) \leqslant 1 + \frac{2\lambda\mu}{\lambda^2 + \mu^2}$;

(ii) $X$ is Hilbert space if and only if $L_{YJ}(\lambda, \mu, X) = 1$ or $L'_{YJ}(\lambda, \mu, X) = 1$;

(iii)

$$L_{YJ}(\lambda, \mu, X) \leqslant \frac{2\lambda^2}{\lambda^2 + \mu^2} C_{NJ}(X) + \frac{2\sqrt{2}\lambda|\lambda - \mu|}{\lambda^2 + \mu^2} \sqrt{C_{NJ}(X)} + \frac{|\lambda - \mu|^2}{\lambda^2 + \mu^2};$$

(iv) $X$ is uniformly non-square if and only if $L'_{YJ}(\lambda, \mu, X) < 1 + \frac{2\lambda\mu}{\lambda^2 + \mu^2}$;

(v)

$$\frac{2\min\{\lambda^2,\mu^2\}C'_{NJ}(X)}{\lambda^2+\mu^2}\leqslant L'_{YJ}(\lambda,\mu,X)$$
$$\leqslant \frac{2\lambda^2}{\lambda^2+\mu^2}C'_{NJ}(X)+\frac{2\sqrt{2}\lambda|\lambda-\mu|}{\lambda^2+\mu^2}\sqrt{C'_{NJ}(X)}+\frac{|\lambda-\mu|^2}{\lambda^2+\mu^2}.$$

To further estimate the upper bound of the constant $L_{YJ}(\lambda,\mu,X)$, in this paper, we will give a relationship among the constant $L_{YJ}(\lambda,\mu,X)$ and generalized James constant $J_{\lambda,\mu}(X)$, which is defined as

$$J_{\lambda,\mu}(X)=\sup\Big\{\min\{\|\lambda x+\mu y\|,\|\mu x-\lambda y\|\}: x,y\in B_X\Big\}.$$

And we point out that this relationship inequality covers Alonso's inequality (1.1).

## 2. Main results

LEMMA 2.1. *Let $\lambda,\mu>0$ and $X$ be a Banach space. If there exist two positive numbers $E$ and $F$ such that*

$$\sup\big\{\|\lambda x+\mu y\|^2+\|\mu x-\lambda y\|^2: \quad x,y\in S_X\big\}\leqslant E$$

*and*

$$\sup\{\lambda\|\lambda x+\mu y\|+\mu\|\mu x-\lambda y\|: x,y\in S_X\}\leqslant F,$$

*then*

$$L_{YJ}(\lambda,\mu,X)\leqslant 1+\frac{E-2F+\sqrt{(E-2F)^2+4(F-\lambda^2-\mu^2)^2}}{2(\lambda^2+\mu^2)}. \tag{2.1}$$

*Proof.* Let $x,y\in S_X$, then clearly

$$\|\lambda x+\mu t y\|\leqslant t\|\lambda x+\mu y\|+(1-t)\lambda$$

and

$$\|\mu x-t\lambda y\|\leqslant t\|\mu x-\lambda y\|+(1-t)\mu$$

for every $t\in[0,1]$, which implies

$$\begin{aligned}&\|\lambda x+\mu t y\|^2+\|\mu x-t\lambda y\|^2\\ &\leqslant [t\|\lambda x+\mu y\|+(1-t)\lambda]^2+[t\|\mu x-\lambda y\|+(1-t)\mu]^2\\ &\leqslant Et^2+2t(1-t)F+(1-t)^2\left(\lambda^2+\mu^2\right).\end{aligned}$$

Thus,

$$\begin{aligned} L_{YJ}(\lambda,\mu,X) &= \sup\left\{ \frac{\|\lambda x+\mu t y\|^2+\|\mu x-t\lambda y\|^2}{(\lambda^2+\mu^2)(1+t^2)} \mid x,y\in S_X, 0\leqslant t\leqslant 1\right\} \\ &\leqslant \sup\left\{ \frac{Et^2+2t(1-t)F+(1-t)^2(\lambda^2+\mu^2)}{(\lambda^2+\mu^2)(1+t^2)} \mid 0\leqslant t\leqslant 1\right\} \\ &= 1+\sup\left\{ \frac{(E-2F)t^2+2\left(F-\lambda^2-\mu^2\right)t}{(\lambda^2+\mu^2)(1+t^2)} \mid 0\leqslant t\leqslant 1\right\} \\ &\leqslant 1+\frac{E-2F+\sqrt{(E-2F)^2+4\left(F-\lambda^2-\mu^2\right)^2}}{2\left(\lambda^2+\mu^2\right)}, \end{aligned}$$

where the last inequality is obtained by

$$\begin{aligned} &\left(E-2F-\sqrt{(E-2F)^2+4\left(F-\lambda^2-\mu^2\right)^2}\right)t^2 \\ &\quad +4\left(F-\lambda^2-\mu^2\right)t-\left(E-2F+\sqrt{(E-2F)^2+4\left(F-\lambda^2-\mu^2\right)^2}\right) \\ &\leqslant 0 \end{aligned}$$

for any $t\in[0,1]$.

In fact, when $E-2F-\sqrt{(E-2F)^2+4\left(F-\lambda^2-\mu^2\right)^2}=0$, it is obviously valid. When $E-2F-\sqrt{(E-2F)^2+4\left(F-\lambda^2-\mu^2\right)^2}\neq 0$, we have

$$\begin{aligned} &\left(E-2F-\sqrt{(E-2F)^2+4\left(F-\lambda^2-\mu^2\right)^2}\right)t^2 \\ &\quad +4\left(F-\lambda^2-\mu^2\right)t-\left(E-2F+\sqrt{(E-2F)^2+4\left(F-\lambda^2-\mu^2\right)^2}\right) \\ &=\left(E-2F-\sqrt{(E-2F)^2+4\left(F-\lambda^2-\mu^2\right)^2}\right) \\ &\quad \times\left(t+\frac{2(F-\lambda^2-\mu^2)}{E-2F-\sqrt{(E-2F)^2+4\left(F-\lambda^2-\mu^2\right)^2}}\right)^2 \\ &\leqslant 0 \end{aligned}$$

for any real $t$. □

LEMMA 2.2. *If* $0<\mu<1$ *and* $x,y\in S_X$, *and* $F_{x,y}=\|x+\mu y\|+\mu\|\mu x-y\|$, *then we have*

$$F_{x,y}\leqslant(1+\mu)^2-\frac{1+\mu^2}{J_{1,\mu}(X)}\mu(1+\mu)+\mu\left(1+\mu^2\right),$$

*where* $J_{1,\mu}(X)=\sup\{\min\{\|x+\mu y\|,\|\mu x-y\|\}\mid x,y\in B_X\}$.

*Proof.* a) If $\max\{\|x+\mu y\|,\|\mu x-y\|\}\leqslant J_{1,\mu}(X)$, then

$$F_{x,y}\leqslant(1+\mu)J_{1,\mu}(X)\leqslant(1+\mu)^2+\mu\left(1+\mu^2\right)\left(1-\frac{1+\mu}{J_{1,\mu}(X)}\right),$$

which follows from

$$J_{1,\mu}(X)\geqslant\mu\geqslant\frac{\mu\left(1+\mu^2\right)}{1+\mu}. \tag{2.2}$$

b) If $\varepsilon:=\|\mu x-y\|>J_{1,\mu}(X)$, then for every sufficient small positive number $\sigma$, we have $\varepsilon>J_{1,\mu}(X)+\sigma$.

Letting

$$x'=\frac{J_{1,\mu}(X)+\sigma}{\varepsilon}x+\left(1-\frac{J_{1,\mu}(X)+\sigma}{\varepsilon}\right)\frac{x+\mu y}{\|x+\mu y\|},$$

$$y'=\frac{J_{1,\mu}(X)+\sigma}{\varepsilon}y+\mu\left(1-\frac{J_{1,\mu}(X)+\sigma}{\varepsilon}\right)\frac{x+\mu y}{\|x+\mu y\|},$$

we have $\|x'\|\leqslant 1,\|y'\|\leqslant 1$ and

$$\left\|\mu x'-y'\right\|=J_{1,\mu}+\sigma>J_{1,\mu}(X),$$

which implies

$$\left\|x'+\mu y'\right\|\leqslant J_{1,\mu}(X),$$

that is,

$$\frac{J_{1,\mu}(X)+\sigma}{\varepsilon}\|x+\mu y\|+\left(1-\frac{J_{1,\mu}(X)+\sigma}{\varepsilon}\right)\left(1+\mu^2\right)\leqslant J_{1,\mu}(X).$$

Hence, we can get

$$\|x+\mu y\|\leqslant\varepsilon+\left(1-\frac{\varepsilon}{J_{1,\mu}(X)}\right)\left(1+\mu^2\right) \tag{2.3}$$

by $\sigma\to 0^+$.

Therefore, by $\varepsilon>J_{1,\mu}(X)$, we obtain

$$\begin{aligned}F_{x,y}&\leqslant(1+\mu)\varepsilon+\left(1-\frac{\varepsilon}{J_{1,\mu}(X)}\right)\left(1+\mu^2\right)\\&\leqslant(1+\mu)\varepsilon+\left(1-\frac{\varepsilon}{J_{1,\mu}(X)}\right)\mu\left(1+\mu^2\right).\end{aligned}$$

c) If $\varepsilon := \|x+\mu y\| > J_{1,\mu}(X)$.
Similar to (b), it can concluded that

$$\|\mu x - y\| \leqslant \varepsilon + \left(1 - \frac{\varepsilon}{J_{1,\mu}(X)}\right)\left(1+\mu^2\right), \tag{2.4}$$

which implies

$$F_{x,y} \leqslant (1+\mu)\varepsilon + \left(1 - \frac{\varepsilon}{J_{1,\mu}(X)}\right)\mu\left(1+\mu^2\right).$$

Therefore, from b) and $c)$, we have

$$\begin{aligned} F_{x,y} &\leqslant \max\left\{(1+\mu)\varepsilon + \left(1 - \frac{\varepsilon}{J_{1,\mu}(X)}\right)\mu\left(1+\mu^2\right) : \varepsilon \in [J_{1,\mu}(X), 1+\mu]\right\} \\ &= \left(1+\mu - \frac{\mu\left(1+\mu^2\right)}{J_{1,\mu}(X)}\right)(1+\mu) + \mu\left(1+\mu^2\right) \end{aligned}$$

by $(2.2)$. □

LEMMA 2.3. *Let* $0 < \mu \leqslant \lambda$. *If* $J_{\lambda,\mu}(X) = \sup\left\{\min\{\|\lambda x + \mu y\|, \|\mu x - \lambda y\|\} : x, y \in B_X\right\}$, *then*

$$J_{\lambda,\mu}(X) \geqslant \frac{\lambda^2+\mu^2}{\sqrt{2}\lambda}.$$

*Proof.* In [5], the author proved that there exist $u,v \in S_X$ such that $\|u+v\| = \|u-v\| \geqslant \sqrt{2}$. Now by taking $x = \frac{1}{\lambda}\left[\frac{\lambda+\mu}{2}u + \frac{\lambda-\mu}{2}v\right]$ and $y = \frac{1}{\lambda}\left[\frac{\mu-\lambda}{2}u + \frac{\lambda+\mu}{2}v\right]$, we can get

$$\|x\| \leqslant \frac{1}{\lambda}\left[\frac{\lambda+\mu}{2} + \frac{\lambda-\mu}{2}\right] = 1, \quad \|y\| \leqslant \frac{1}{\lambda}\left[\frac{\lambda-\mu}{2} + \frac{\lambda+\mu}{2}\right] = 1,$$

$$\begin{aligned} \|\lambda x + \mu y\| &= \left\|\lambda\left(\frac{\lambda+\mu}{2\lambda}u + \frac{\lambda-\mu}{2\lambda}v\right) + \mu\left(\frac{\mu-\lambda}{2\lambda}u + \frac{\lambda+\mu}{2\lambda}v\right)\right\| \\ &= \left\|\frac{\left(\lambda^2+\mu^2\right)(u+v)}{2\lambda}\right\| \geqslant \frac{\lambda^2+\mu^2}{\sqrt{2}\lambda}, \end{aligned}$$

and

$$\begin{aligned} \|\mu x - \lambda y\| &= \left\|\mu\left(\frac{\lambda+\mu}{2\lambda}u + \frac{\lambda-\mu}{2\lambda}v\right) - \lambda\left(\frac{\mu-\lambda}{2\lambda}u + \frac{\lambda+\mu}{2\lambda}v\right)\right\| \\ &= \left\|\frac{\left(\mu^2+\lambda^2\right)(u-v)}{2\lambda}\right\| \geqslant \frac{\lambda^2+\mu^2}{\sqrt{2}\lambda}. \end{aligned}$$

Therefore, $J_{\lambda,\mu}(X) \geqslant \frac{\lambda^2+\mu^2}{\sqrt{2}\lambda}$. □

LEMMA 2.4. *Let* $E_{x,y}=\|x+\mu y\|^2+\|\mu x-y\|^2$ *for* $x,y\in S_X$, *if* $0<\mu<1$, *then*

$$E_{x,y}\leqslant(1+\mu)^2+\left[\left(1-\frac{1+\mu^2}{J_{1,\mu}(X)}\right)(1+\mu)+1+\mu^2\right]^2$$

*for any* $x,y\in S_X$.

*Proof.* Assume $H(\varepsilon)=\varepsilon^2+\left[\left(1-\frac{1+\mu^2}{J_{1,\mu}(X)}\right)\varepsilon+1+\mu^2\right]^2$, then we can prove that

$$E_{x,y}\leqslant\max\left\{H(\varepsilon)\mid J_{1,\mu}(X)\leqslant\varepsilon\leqslant1+\mu\right\}.$$

In fact, if $\max\{\|x+\mu y\|,\|\mu x-y\|\}\leqslant J_{1,\mu}(X)$, then $E_{x,y}\leqslant 2J_{1,\mu}(X)^2=H\left(J_{1,\mu}(X)\right)$. On the other hand, if $\varepsilon=:\|\mu x-y\|$ or $\varepsilon=:\|x+\mu y\|>J_{1,\mu}(X)$, then $\varepsilon\leqslant1+\mu$ and by (2.3) or (2.4), we have $E_{x,y}\leqslant\varepsilon^2+\left(\varepsilon+\left(1-\frac{\varepsilon}{J_{1,\mu}(X)}\right)(1+\mu^2)\right)^2=H(\varepsilon)$.

Because $H(\varepsilon)$ is convex, we have

$$E_{x,y}\leqslant\max\left\{H\left(J_{1,\mu}(X)\right),H(1+\mu)\right\}.$$

Finally, by applying Lemma2.3, we obtain

$$\begin{aligned}
&H(1+\mu)-H\left(J_{1,\mu}(X)\right)\\
&=\left(1+\mu-J_{1,\mu}(X)\right)\\
&\quad\times\left[2\left(1+\mu+J_{1,\mu}(X)\right)+\frac{1+\mu-J_{1,\mu}(X)}{J_{1,\mu}(X)^2}\left(1+\mu^2\right)^2-\frac{2(1+\mu)\left(1+\mu^2\right)}{J_{1,\mu}(X)}\right]\\
&=\frac{\left(1+\mu-J_{1,\mu}(X)\right)}{J_{1,\mu}(X)}\\
&\quad\times\left[2(1+\mu)J_{1,\mu}(X)+\frac{(1+\mu)}{J_{1,\mu}(X)}\left(1+\mu^2\right)^2+2J_{1,\mu}(X)^2-\left(1+\mu^2\right)^2-2(1+\mu)\left(1+\mu^2\right)\right]\\
&\geqslant\frac{1+\mu-J_{1,\mu}(X)}{J_{1,\mu}(X)}\\
&\quad\times\left[2\sqrt{2(1+\mu)^2\left(1+\mu^2\right)^2}+2\cdot\left(\frac{1+\mu^2}{\sqrt{2}}\right)^2-\left(1+\mu^2\right)^2-2(1+\mu)\left(1+\mu^2\right)\right]\\
&=\frac{1+\mu-J_{1,\mu}(X)}{J_{1,\mu}(X)}\left[2\sqrt{2}(1+\mu)\left(1+\mu^2\right)-2(1+\mu)\left(1+\mu^2\right)\right]\\
&\geqslant0,
\end{aligned}$$

which complete the proof. □

LEMMA 2.5. *If* $0<\mu\leqslant1$, *then*

$$L_{YJ}(1,\mu,X)\leqslant\frac{3+\mu^2}{(1+\mu)(1+\mu^2)}J_{1,\mu}(X). \tag{2.5}$$

*Proof.* If $\mu=1$, then (2.5) is just (1.1). Now we can suppose that $0<\mu<1$. Let $E=(1+\mu)^2+\left[1+\mu+\left(1+\mu^2\right)\left(1-\frac{1+\mu}{J_{1,\mu}(X)}\right)\right]^2$ and $F=(1+\mu)^2-\frac{1+\mu^2}{J_{1,\mu}(X)}\mu(1+\mu)+\mu\left(1+\mu^2\right)$, we have

$$\begin{aligned} E-2F &= \left(1-\frac{1+\mu}{J_{1,\mu}(X)}\right)^2\left(1+\mu^2\right)^2+2(1+\mu)\left(1+\mu^2\right)\left(1-\frac{1+\mu}{J_{1,\mu}(X)}\right) \\ &\quad +\frac{2(1+\mu^2)}{J_{1,\mu}(X)}\mu(1+\mu)-2\mu\left(1+\mu^2\right) \\ &= \left(1-\frac{1+\mu}{J_{1,\mu}(X)}\right)\left(1+\mu^2\right)\left[\left(1+\mu^2\right)\left(1-\frac{1+\mu}{J_{1,\mu}(X)}\right)+2\right] \end{aligned}$$

and

$$F-\left(1+\mu^2\right)=\mu\left[\left(1+\mu^2\right)\left(1-\frac{1+\mu}{J_{1,\mu}(X)}\right)+2\right].$$

By Lemma 2.1, Lemma 2.2 and Lemma 2.4, we can get

$$\begin{aligned} L_{YJ}(1,\mu,X) &\leqslant 1+\frac{E-2F+\sqrt{(E-2F)^2+4\left(F-1-\mu^2\right)^2}}{2\left(1+\mu^2\right)} \\ &= 1+\frac{1-\frac{1+\mu}{J_{1,\mu}(X)}+\sqrt{\left(1-\frac{1+\mu}{J_{1,\mu}(X)}\right)^2+\frac{4\mu^2}{(1+\mu^2)^2}}}{2} \\ &\quad \times\left(3+\mu^2-\frac{(1+\mu)\left(1+\mu^2\right)}{J_{1,\mu}(X)}\right) \end{aligned} \tag{2.6}$$

So, in order to get (2.5) we just need prove

$$J_{1,\mu}(X)\geqslant\frac{(1+\mu)\left(1+\mu^2\right)}{3+\mu^2} \tag{2.7}$$

and

$$1-\frac{1+\mu}{J_{1,\mu}(X)}+\sqrt{\left(1-\frac{1+\mu}{J_{1,\mu}(X)}\right)^2+\frac{4\mu^2}{\left(1+\mu^2\right)^2}}\leqslant\frac{2J_{1,\mu}(X)}{(1+\mu)\left(1+\mu^2\right)}. \tag{2.8}$$

In fact, (2.7) is clear by $\sqrt{2}(1+\mu)\leqslant 2\sqrt{2}\leqslant 3+\mu^2$ and $J_{1,\mu}(X)\geqslant\frac{1+\mu^2}{\sqrt{2}}\geqslant\frac{(1+\mu)\left(1+\mu^2\right)}{3+\mu^2}$. Moreover, (2.8) is equivalent to

$$J_{1,\mu}(X)^2-(1+\mu)\left(1+\mu^2\right)J_{1,\mu}(X)+(1+\mu)^2\geqslant 0.$$

Because

$$\Delta:=(1+\mu)^2\left(1+\mu^2\right)^2-4(1+\mu)^2=(1+\mu)^2\left[\left(1+\mu^2\right)^2-4\right]\leqslant 0,$$

and hence we can see (2.8) is valid. $\square$

THEOREM 2.1. *If $\lambda,\mu>0$, then*

$$L_{YJ}(\lambda,\mu,X)\leqslant\frac{3\max\{\lambda^2,\mu^2\}+\min\{\lambda^2,\mu^2\}}{(\lambda+\mu)(\lambda^2+\mu^2)}J_{\lambda,\mu}(X).$$

*and*

$$\begin{aligned}L_{YJ}(\lambda,\mu,X)\leqslant 1+&\frac{1-\frac{\lambda+\mu}{J_{\lambda,\mu}(X)}+\sqrt{\left(1-\frac{\lambda+\mu}{J_{\lambda,\mu}(X)}\right)^2+\frac{4\lambda^2\mu^2}{(\lambda^2+\mu^2)^2}}}{2\max\{\lambda^2,\mu^2\}}\\&\times\left(2\max\{\lambda^2,\mu^2\}+\lambda^2+\mu^2-\frac{(\lambda+\mu)(\lambda^2+\mu^2)}{J_{\lambda,\mu}(X)}\right)\end{aligned}\tag{2.9}$$

*Proof.* Let $\mu'=\frac{\min\{\lambda,\mu\}}{\max\{\lambda,\mu\}}$, then $0<\mu'\leqslant 1$. By Lemma 2.5 and note that $L_{YJ}(\lambda,\mu,X)=L_{YJ}(\mu,\lambda,X)=L_{YJ}(1,\mu',X)$, we have

$$L_{YJ}(\lambda,\mu,X)=L_{YJ}(1,\mu',X)\leqslant\frac{3+\mu'^2}{(1+\mu')(1+\mu'^2)}J_{1,\mu'}(X),$$

that is

$$L_{YJ}(\lambda,\mu,X)\leqslant\frac{3\max\{\lambda^2,\mu^2\}+\min\{\lambda^2,\mu^2\}}{(\lambda+\mu)(\lambda^2+\mu^2)}J_{\lambda,\mu}(X).$$

Similarly, (2.9) can be obtained from (2.6). □

As an application of Theorem 2.1, we have

THEOREM 2.2. *If $\lambda,\mu>0$, then $X$ is uniformly non-square if and only if $J_{\lambda,\mu}(X)<\lambda+\mu$.*

*Proof.* If $X$ is not uniformly non-square, then $L'_{YJ}(\lambda,\mu,X)=1+\frac{2\lambda\mu}{\lambda^2+\mu^2}$, then (2.9) implies

$$\begin{aligned}&1+\frac{2\lambda\mu}{\lambda^2+\mu^2}\\&\leqslant 1+\frac{1-\frac{\lambda+\mu}{J_{\lambda,\mu}(X)}+\sqrt{\left(1-\frac{\lambda+\mu}{J_{\lambda,\mu}(X)}\right)^2+\frac{4\lambda^2\mu^2}{(\lambda^2+\mu^2)^2}}}{2\max\{\lambda^2,\mu^2\}}\\&\quad\times\left(2\max\{\lambda^2,\mu^2\}+\lambda^2+\mu^2-\frac{(\lambda+\mu)(\lambda^2+\mu^2)}{J_{\lambda,\mu}(X)}\right)\\&\leqslant 1+1-\frac{\lambda+\mu}{J_{\lambda,\mu}(X)}+\sqrt{\left(1-\frac{\lambda+\mu}{J_{\lambda,\mu}(X)}\right)^2+\frac{4\lambda^2\mu^2}{(\lambda^2+\mu^2)^2}}\\&\leqslant 1+1-\frac{\lambda+\mu}{J_{\lambda,\mu}(X)}+\left|1-\frac{\lambda+\mu}{J_{\lambda,\mu}(X)}\right|+\frac{2\lambda\mu}{\lambda^2+\mu^2}\\&=1+\frac{2\lambda\mu}{\lambda^2+\mu^2}.\end{aligned}$$

Hence, $J_{\lambda,\mu}(X)=\lambda+\mu$.

On the other hand, if $J_{\lambda,\mu}(X)=\lambda+\mu$, then there exist $x_n.y_n\in B_X$ such that $\|\lambda x_n+\mu y_n\|\to\lambda+\mu$ and $\|\mu x_n-\lambda y_n\|\to\lambda+\mu$. We can easily get $\|x_n\|,\|y_n\|\to 1$ and $\|\lambda\frac{x_n}{\|x_n\|}+\mu\frac{y_n}{\|y_n\|}\|\to\lambda+\mu$ and $\|\mu\frac{x_n}{\|x_n\|}-\lambda\frac{y_n}{\|y_n\|}\|\to\lambda+\mu$. Therefore, $L'_{YJ}(\lambda,\mu,X)$ $=1+\frac{2\lambda\mu}{\lambda^2+\mu^2}$ by (1.3) and this implies that $X$ is not uniformly non-square. □

REMARK 2.1. By the proof of Theorem 2.2 and (2.9), we also have

$$1\leqslant L_{Y,J}(\lambda,\mu,X)\leqslant 1+\frac{2\lambda\mu}{\lambda^2+\mu^2},$$

which is an improvement of the inequality $1\leqslant L_{Y,J}(\lambda,\mu,X)\leqslant 2$.

In addition, similar to the inequality between $C_{NJ}(X)$ and $C'_{NJ}(X)$, there are also the following inequality between $L_{YJ}(\lambda,\mu,X)$ and $L'_{YJ}(\lambda,\mu,X)$.

COROLLARY 2.1. *For any Banach space $X$ and $\lambda,\mu>0$, we have*

$$L_{YJ}(\lambda,\mu,X)\leqslant 2\Big(1+L'_{YJ}(\lambda,\mu,X)-\sqrt{2L'_{YJ}(\lambda,\mu,X)}\Big).$$

*Proof.* By the definition of $L'_{YJ}(\lambda,\mu,X)$, we can get

$$\sup\big\{\|\lambda x+\mu y\|^2+\|\mu x-\lambda y\|^2 : x,y\in S_X\big\}\leqslant 2(\lambda^2+\mu^2)L'(\lambda,\mu,X)$$

and

$$\sup\{\lambda\|\lambda x+\mu y\|+\mu\|\mu x-\lambda y\| : x,y\in S_X\}\leqslant(\lambda^2+\mu^2)\sqrt{2L'(\lambda,\mu,X)}.$$

Now applying Lemma2.1, we have

$$\begin{aligned}
&L_{YJ}(\lambda,\mu,X)\\
&\leqslant 1+L'_{YJ}(\lambda,\mu,X)-\sqrt{2L'_{YJ}(\lambda,\mu,X)}\\
&\quad+\sqrt{[L'_{YJ}(\lambda,\mu,X)-\sqrt{2L'_{YJ}(\lambda,\mu,X)}]^2+[\sqrt{2L'_{YJ}(\lambda,\mu,X)}-1]^2}\\
&=1+L'_{YJ}(\lambda,\mu,X)-\sqrt{2L'_{YJ}(\lambda,\mu,X)}\\
&\quad+\sqrt{[L'_{YJ}(\lambda,\mu,X)-\sqrt{2L'_{YJ}(\lambda,\mu,X)}]^2+2L'_{YJ}(\lambda,\mu,X)-2\sqrt{2L'_{YJ}(\lambda,\mu,X)}+1}\\
&=2[1+L'_{YJ}(\lambda,\mu,X)-\sqrt{2L'_{YJ}(\lambda,\mu,X)}].
\end{aligned}$$

□

## REFERENCES


[1] J. ALONSO, P. MARTÍN AND P. L. PAPINI, *Wheeling around von Neumann-Jordan constant in Banach spaces*, Studia Math. **188** (2008), 135–150.

[2] M. CIESIELSKI AND R. PLUCIENNIK, *On some modifications of n-th von Neumann-Jordan constant for Banach spaces*, Banach J. Math. Anal. **14** (2020), 650–673.

[3] J. A. CLARKSON, *The von Neumann-Jordan constant for the Lebesgue space*, Ann. Math. **38** (1937), 114–115.

[4] Y. CUI, W. HUANG, H. HUDZIK AND R. KACZMAREK, *Generalized von Neumann-Jordan constant and its relationship to the fixed point property*, Fixed Point Theory Appl. **40** (2015), 1–11.

[5] J. GAO AND K. S. LAU, *On the geometry of spheres in normed linear spaces*, J. Austr. Math. Soc. Ser. A. **48** (1990), 101–112.

[6] M. KATO, L. MALIGRANDA AND Y. TAKAHASHI, *On James and Jordan-von Neumann constant and the normal structure coefficient of Banach spaces*, Studia Math. **144** (2001), 275–295.

[7] Q. LIU AND Y.J. LI, *On a new geometric constant related to the Euler-Langrange type identity in Banach spaces*, Mathematics **116** (2021), 1–12.

[8] Q. LIU, C. J. ZHOU, M. SARFRAZ AND Y. J. LI, *On new moduli related to the generalization of the Parallelogram law*, Bull. Malays. Math. Sci. Soc. **45** (2022), 307–321.

[9] M. S. MOSLEHIAN AND J. M. RASSIAS, *A characterization of the inner product spaces concerning an Euler-Lagrange identity*, Commun. Math. Anal. **8** (2010), 16–21.

[10] K. NAOTO, K. S. SAITO AND K. I. MITANI, *Extremal structure of the set of absolute norms on $R^2$ and the von Neumann-Jordan constant*, J. Math. Anal. Appl. **370** (2010), 101–106.

[11] Y. TAKAHASHI AND M. KATO, *A simple inequality for the von Neumann-Jordan and James constants of a Banach space*, J. Math. Anal. Appl. **359** (2009), 602–609.

[12] F. WANG, *On the James and von Neumann-Jordan constant in Banach spaces*, Proc. Amer. Math. Soc., **138** (2009), 695–701.

[13] C. YANG AND H. LI, *An inequality between Jordan-von Neumann constant and James constant*, Appl. Math. Lett. **23** (2010), 277–281.

[14] C. YANG AND H. LI, *Generalized von Neumann-Jordan constant for the Banas-Fraczek space*, Colloq. Math. **154** (2018), 149–156.


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