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SOME CONVERSES OF FUNCTIONAL HÖLDER–TYPE INEQUALITIES

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Abstract. In this paper we obtain some new converses of the Hölder and Minkowski inequalities for positive linear functionals. Our results also provide new converses in case of sums, integrals and time scales integrals. Further we obtain a converse of integral Minkowski's inequality on time scales.

1. Introduction

Hölder's inequality has numerous applications in various areas, including measure theory, probability theory, Fourier analysis, numerical analysis, social science and cultural science as well as in natural science. It serves as a foundation for other important inequalities, such as the Minkowski inequality and the Young inequality. Extensions and generalizations of Hölder's inequality have been developed to accommodate more general settings and conditions on the exponents, leading to a rich theory with broader applications in mathematics and its applications. In case of positive linear functionals, Hölder's and Minkowski's inequalities and some of their converses, given below, can be found in [11].

DEFINITION 1. Let E be a nonempty set and L be a linear class of real-valued functions $f : E \to R$ having the following properties:

(L_1) If $f, g \in L$ and $a, b \in R$, then $(af + bg) \in L$.

(L_2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

A positive linear functional is a functional $A : L \to R$ having the following properties:

(A_1) If $f, g \in L$ and $a, b \in R$, then $A(af + bg) = aA(f) + bA(g)$.

(A_2) If $f \in L$ and $f(t) \geqslant 0$ for all $t \in E$, then $A(f) \geqslant 0$.

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THEOREM 1. *Let L satisfy conditions L_1, L_2, and A satisfy conditions A_1, A_2 on a base set E. Let $p > 1$ and $p^{-1} + q^{-1} = 1$. If $w, f, g \geqslant 0$ on E such that wf^p, wg^q, $wfg \in L$, then we have*

$$A(wfg) \leqslant A^{1/p}(wf^p) A^{1/q}(wg^q). \tag{1}$$

In the case $0 < p < 1$ and $A(wg^q) > 0$ (or $p < 0$ and $A(wf^p) > 0$), the inequality (1) is reversed.

THEOREM 2. *Let L satisfy conditions L_1, L_2, and A satisfy conditions A_1, A_2 on a base set E. If $p > 1$ and $w, f, g \geqslant 0$ on E such that wf^p, wg^p, $w(f+g)^p \in L$, then we have*

$$A^{\frac{1}{p}}(w(f+g)^p) \leqslant A^{1/p}(wf^p) + A^{1/p}(wg^p). \tag{2}$$

If $0 < p < 1$ or $p < 0$ and $A(wf^p)$, $A(wg^p) > 0$, then the inequality (2) is reversed.

THEOREM 3. *For $p \neq 1$ let $q = p/(p-1)$. Assume w, f, g are nonnegative functions such that wf^p, wg^q, $wfg \in L$. Suppose*

$$0 < m \leqslant f(t) g^{-q/p}(t) \leqslant M \text{ for all } t \in E.$$

If $p > 1$, then

$$A(wfg) \geqslant K(p,m,M) A^{\frac{1}{p}}(wf^p) A^{\frac{1}{q}}(wg^q), \tag{3}$$

where

$$K(p,m,M) = |p|^{1/p} |q|^{1/q} \frac{(M-m)^{1/p} |mM^p - Mm^p|^{1/q}}{|M^p - m^p|}. \tag{4}$$

If $0 < p < 1$ or $p < 0$, then (3) is reversed provided either $A(wf^p) > 0$ or $A(wg^q) > 0$.

THEOREM 4. *For $p \neq 1$ let $q = p/(p-1)$. Assume w, f, g are nonnegative functions such that wf^p, wg^p, $w(f+g)^p \in L$. Let $0 < m < f(f+g)^{\frac{-q}{p}} \leqslant M$ and $0 < m < g(f+g)^{\frac{-q}{p}} \leqslant M$. If $p > 1$, then*

$$A^{\frac{1}{p}}(w(f+g)^p) \geqslant K(p,m,M)\left(A^{\frac{1}{p}}(wf^p) + A^{\frac{1}{p}}(wg^p)\right) \tag{5}$$

holds where $K(p,m,M)$ is defined as in (4).

If $0 < p < 1$ or $p < 0$, then the inequality (5) is reversed provided $A(w(f+g)^p) > 0$ for $p < 0$.

I. İşcan in [9] generalized Hölder's inequality in the following way.

THEOREM 5. *Let L satisfy conditions L_1, L_2, and A satisfy conditions A_1, A_2 on a base set E. Let $p > 1$ and $p^{-1} + q^{-1} = 1$. If $\alpha, \beta, w, f, g \geqslant 0$ on E, αwfg, βwfg, αwf^p, αwg^q, βwf^p, βwg^q, $wfg \in L$ and $\alpha + \beta = 1$ on E, then we have*

$$A(wfg) \leqslant A^{1/p}(\alpha wf^p) A^{1/q}(\alpha wg^q) + A^{1/p}(\beta wf^q) A^{1/q}(\beta wg^q) \tag{6}$$

and

$$A^{1/p}(\alpha wf^p) A^{1/q}(\alpha wg^q) + A^{1/p}(\beta wf^p) A^{1/q}(\beta wg^q) \leqslant A^{1/p}(wf^p) A^{1/q}(wg^q). \tag{7}$$

In order to obtain the converses of Hölder's inequality we use Young's inequality [8, 10, 12, 13] with Specht's ratio:

If $a,b>0$, $\frac{1}{p}+\frac{1}{q}>1$ with $p>1$, then the following converse of Young's inequality holds

$$s\left(\frac{a}{b}\right)a^{\frac{1}{p}}b^{\frac{1}{q}} \geqslant \frac{a}{p}+\frac{b}{q}, \tag{8}$$

where s is Specht's ratio defined by

$$s(x)=\frac{x^{\frac{1}{x-1}}}{e\log x^{\frac{1}{x-1}}}$$

for $x>0$, $x\neq 1$, and $s(1)=1$.

REMARK 1. (i) $\lim_{x\to 1} s(x)=1$ and $s(x)=s(\frac{1}{x})$ for $x>0$.

(ii) $s(x)$ is a monotone increasing function on $(1,\infty)$.

(iii) $s(x)$ is a monotone decreasing function on $(0,1)$.

2. Converses of functional Hölder's and Minkowski's inequalities

Our first result provides the converse of functional Hölder's inequality (1).

THEOREM 6. *Let L satisfy conditions L_1, L_2 and A satisfy conditions A_1, A_2. If $p>1$, $q=p/(p-1)$, and w,f,g are positive functions such that wf^p, wg^q, $s\left(\frac{A(wg^q)f^p}{A(wf^p)g^q}\right)wfg\in L$, then we have*

$$A\left(s\left(\frac{A(wg^q)f^p}{A(wf^p)g^q}\right)wfg\right) \geqslant A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q), \tag{9}$$

where s is Specht's ratio.

Proof. By taking $a=\frac{wf^p}{A(wf^p)}$ and $b=\frac{wg^q}{A(wg^q)}$ in (8), we get

$$s\left(\frac{A(wg^q)f^p}{A(wf^p)g^q}\right)\frac{wfg}{A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q)} \geqslant \frac{1}{p}\left(\frac{wf^p}{A(wf^p)}\right)+\frac{1}{q}\left(\frac{wg^q}{A(wg^q)}\right).$$

Now applying the positive linear functional A on both sides, we get

$$\frac{1}{A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q)}A\left(s\left(\frac{A(wg^q)f^p}{A(wf^p)g^q}\right)wfg\right) \geqslant \frac{1}{p}\left(\frac{A(wf^p)}{A(wf^p)}\right)+\frac{1}{q}\left(\frac{A(wg^q)}{A(wg^q)}\right)=1,$$

which leads to the required result. □

When $w(t)=1$ in Theorem 6 we obtain the following result.

COROLLARY 1. *Let L satisfy conditions L_1, L_2 and A satisfy conditions A_1, A_2. If $p>1$, $q=p/(p-1)$, and f,g are positive functions such that f^p, g^q, $s\left(\frac{A(g^q)f^p}{A(f^p)g^q}\right)fg \in L$, then we have*

$$A\left(s\left(\frac{A(g^q)f^p}{A(f^p)g^q}\right)fg\right) \geqslant A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q), \tag{10}$$

where s is Specht's ratio.

Next theorem gives the reverse case for $p<0$ and $0<p<1$.

THEOREM 7. *Let all the assumptions of Theorem 6 are satisfied.*

(i) If $p<0$, then we get

$$A^{\frac{1}{q}}\left(s\left(\frac{A(wfg)f^{p-1}}{A(wf^p)g}\right)wg^q\right)A^{\frac{1}{p}}(wf^p) \geqslant A(wfg).$$

(ii) If $0<p<1$, then we get

$$A^{\frac{1}{p}}\left(s\left(\frac{A(wg^q)f}{A(wfg)g^{q-1}}\right)wf^p\right)A^{\frac{1}{q}}(wg^q) \geqslant A(wfg).$$

Proof.

(i) For $p<0$ let $P=\frac{-p}{q}$, $Q=\frac{1}{q}$, $F=f^{-q}$ and $G=f^qg^q$. Then $P,Q>1$, $1/P+1/Q=1$ and F,G are positive functions. By replacing p,q,f,g with P,Q,F,G in (9), we get

$$A\left(s\left(\frac{A(wG^Q)F^P}{A(wF^P)G^Q}\right)wFG\right) \geqslant A^{\frac{1}{P}}(wF^P)A^{\frac{1}{Q}}(wG^Q).$$

By substituting values of P,Q,F and G in the above inequality, we get

$$A\left(s\left(\frac{A(wfg)f^p}{A(wf^p)fg}\right)wg^q\right) \geqslant A^{\frac{-q}{p}}(wf^p)A^q(wfg).$$

Hence

$$A\left(s\left(\frac{f^pA(wfg)}{fgA(wf^p)}\right)wg^q\right)A^{\frac{q}{p}}(wf^p) \geqslant A^q(wfg).$$

Now taking power $\frac{1}{q}$ on both sides we obtain the required result.

(ii) If $0<p<1$, then $q<0$ and we let $P=\frac{1}{p}$, $Q=\frac{-q}{p}$, $F=f^pg^p$ and $G=g^{-p}$. Now by replacing p,q,f,g with P,Q,F,G in (9), and then substituting the values of P,Q,F and G, we get

$$A\left(s\left(\frac{A(wg^q)f}{A(wfg)g^{q-1}}\right)wf^p\right) \geqslant A^p(wfg)A^{\frac{-p}{q}}(wg^q).$$

After some calculation, we obtain the required result. □

Next two results give converses of the inequalities (6) and (7), respectively.

THEOREM 8. *Let all the assumptions of Theorem 6 be satisfied. Further assume that* $\alpha,\beta>0$ *on* E *such that* αwfg, βwfg, αwf^p, αwg^q, βwf^p, $\beta wg^q\in L$ *and* $\alpha+\beta=1$ *on* E. *Then we get*

$$\begin{aligned} & s\left(\frac{A(\alpha wf^p)A(wg^q)}{A(\alpha wg^q)A(wf^p)}\right)A^{\frac{1}{p}}(\alpha wf^p)A^{\frac{1}{q}}(\alpha wg^q) \\ &\quad + s\left(\frac{A(\beta wf^p)A(wg^q)}{A(\beta wg^q)A(wf^p)}\right)A^{\frac{1}{p}}(\beta wf^p)A^{\frac{1}{q}}(\beta wg^q) \\ & \geqslant A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q). \end{aligned} \tag{11}$$

Proof. Using (10), we get

$$s\left(\frac{u_1^p\left(v_1^q+v_2^q\right)}{v_1^q\left(u_1^p+u_2^p\right)}\right)u_1v_1+s\left(\frac{u_2^p\left(v_1^q+v_2^q\right)}{v_2^q\left(u_1^p+u_2^p\right)}\right)u_2v_2\geqslant\left(u_1^p+u_2^p\right)^{\frac{1}{p}}+\left(v_1^q+v_2^q\right)^{\frac{1}{q}}. \tag{12}$$

Let

$$\begin{aligned} u_1 &= A^{\frac{1}{p}}(\alpha wf^p), & v_1 &= A^{\frac{1}{q}}(\alpha wg^q), \\ u_2 &= A^{\frac{1}{p}}(\beta wf^p), & v_2 &= A^{\frac{1}{q}}(\beta wg^q). \end{aligned}$$

Substituting these values in equation (12), we obtain the inequality (11). □

THEOREM 9. *Let all the assumptions of Theorem 8 be satisfied. If*

$$s\left(\frac{A(\alpha wg^q)f^p}{A(\alpha wf^p)g^q}\right)\alpha+s\left(\frac{A(\beta wg^q)f^p}{A(\beta wf^p)g^q}\right)\beta\leqslant s\left(\frac{A(wg^q)f^p}{A(wf^p)g^q}\right), \tag{13}$$

then we have

$$A\left(s\left(\frac{A(wg^q)f^p}{A(wf^p)g^q}\right)wfg\right)\geqslant A^{\frac{1}{p}}(\alpha wf^p)A^{\frac{1}{q}}(\alpha wg^q)+A^{\frac{1}{p}}(\beta wf^p)A^{\frac{1}{q}}(\beta wg^q).$$

Proof. Multiplying the both sides of (13) by wfg, applying the positive linear functional A on it and then applying reverse Hölder's inequality (9), we obtain

$$\begin{aligned} & A\left(s\left(\frac{A(wg^q)f^p}{A(wf^p)g^q}\right)wfg\right) \\ & \geqslant A\left(\left(s\left(\frac{A(\alpha wg^q)f^p}{A(\alpha wf^p)g^q}\right)\alpha+s\left(\frac{A(\beta wg^q)f^p}{A(\beta wf^p)g^q}\right)\beta\right)wfg\right) \\ & = A\left(s\left(\frac{A(\alpha wg^q)f^p}{A(\alpha wf^p)g^q}\right)\alpha wfg\right)+A\left(s\left(\frac{A(\beta wg^q)f^p}{A(\beta wf^p)g^q}\right)\beta wfg\right) \\ & \geqslant A^{\frac{1}{p}}(\alpha wf^p)A^{\frac{1}{q}}(\alpha wg^q)+A^{\frac{1}{p}}(\beta wf^p)A^{\frac{1}{q}}(\beta wg^q). \end{aligned}$$

Hence the result follows. □

In the following result, we obtain another converse of the improved Hölder inequalities (6) and (7).

THEOREM 10. *Let all the assumptions of Theorem 8 be satisfied. Suppose K be defined as in (4) and*

$$0 < m \leqslant f(t) g^{-q/p}(t) \leqslant M \text{ for all } t \in E.$$

If $p > 1$, then

$$A(wfg) \geqslant K(p,m,M)\left(A^{1/p}(\alpha w f^p) A^{1/q}(\alpha w g^q) + A^{1/p}(\beta w f^p) A^{1/q}(\beta w g^q)\right) \quad (14)$$

and

$$\begin{aligned} &A^{1/p}(\alpha w f^p) A^{1/q}(\alpha w g^q) + A^{1/p}(\beta w f^p) A^{1/q}(\beta w g^q) \\ &\geqslant K(p,m,M) A^{\frac{1}{p}}(w f^p) A^{\frac{1}{q}}(w g^q). \end{aligned} \quad (15)$$

Proof. By using reverse Hölder's inequality (3), we get

$$\begin{aligned} A(wfg) &= A(\alpha wfg + \beta wfg) = A(\alpha wfg) + A(\beta wfg) \\ &\geqslant K(p,m,M)\left(A^{1/p}(\alpha w f^p) A^{1/q}(\alpha w g^q) + A^{1/p}(\beta w f^p) A^{1/q}(\beta w g^q)\right), \end{aligned}$$

which is the required inequality (14).

Now using discrete form of the inequality (3), we get

$$u_1 v_1 + u_2 v_2 \geqslant K(p,m,M)\left(u_1^p + u_2^p\right)^{\frac{1}{p}} \left(v_1^q + v_2^q\right)^{\frac{1}{q}}. \quad (16)$$

Let

$$\begin{aligned} u_1 &= A^{\frac{1}{p}}(\alpha w f^p), & v_1 &= A^{\frac{1}{q}}(\alpha w g^q), \\ u_2 &= A^{\frac{1}{p}}(\beta w f^p), & v_2 &= A^{\frac{1}{q}}(\beta w g^q). \end{aligned}$$

Substituting these values in (16) we obtain

$$\begin{aligned} &A^{1/p}(\alpha w f^p) A^{1/q}(\alpha w g^q) + A^{1/p}(\beta w f^p) A^{1/q}(\beta w g^q) \\ &\geqslant K(p,m,M)\left(A(\alpha w f^p) + A(\beta w f^p)\right)^{\frac{1}{p}} \left(A(\alpha w g^q) + A(\beta w g^q)\right)^{\frac{1}{q}}. \end{aligned}$$

By using the linearity of A and then using $\alpha + \beta = 1$, we get the inequality (15). □

Next result gives the converse of functional Minkowski's inequality.

THEOREM 11. *Let L satisfy conditions L_1, L_2 and A satisfy conditions A_1, A_2. Suppose that $p>1$, $q=p/(p-1)$, and w,f,g are positive functions such that wf^p, wg^q, wg^p, $w(f+g)^p$, $s\left(\frac{A(wg^q)f^p}{A(wf^p)g^q}\right)w(f+g)^p \in L$. If $s_1 \geqslant s_2, s_3$, where*

$$s_1 = s\left(\frac{A\left(wg^q\right)f^p}{A\left(wf^p\right)g^q}\right),$$
$$s_2 = s\left(\frac{A\left(w(f+g)^p\right)f^p}{A\left(wf^p\right)(f+g)^p}\right),$$
$$s_3 = s\left(\frac{A\left(w(f+g)^p\right)g^p}{A\left(wg^p\right)(f+g)^p}\right),$$

then we have

$$A^{\frac{1}{p}}\left(wf^p\right)+A^{\frac{1}{p}}\left(wg^p\right) \leqslant \left[\frac{A^p\left(s_1 w(f+g)^p\right)}{A^{p-1}\left(w(f+g)^p\right)}\right]^{\frac{1}{p}}.$$

Proof. $s_1 \geqslant s_2$ implies that

$$s_1 wf(f+g)^{p-1} \geqslant s_2 wf(f+g)^{p-1}.$$

Similarly $s_1 \geqslant s_3$ implies that

$$s_1 wg(f+g)^{p-1} \geqslant s_3 wg(f+g)^{p-1}.$$

By adding the above two inequalities we get

$$s_1 w(f+g)^p \geqslant s_2 wf(f+g)^{p-1} + s_3 wg(f+g)^{p-1}.$$

Applying positive linear functional A and then reverse Hölder's inequality (9), we get

$$\begin{aligned} A\left(s_1 w(f+g)^p\right) &\geqslant A\left(s_2 wf(f+g)^{p-1}\right)+A\left(s_3 wg(f+g)^{p-1}\right) \\ &\geqslant A^{\frac{1}{p}}\left(wf^p\right)A^{\frac{p-1}{p}}\left(w(f+g)^p\right)+A^{\frac{1}{p}}\left(wg^p\right)A^{\frac{p-1}{p}}\left(w(f+g)^p\right). \end{aligned}$$

Dividing both sides with $A^{\frac{p-1}{p}}\left(w(f+g)^p\right)$, we get our required result. □

3. Applications on time scales

In this section we obtain new converses of Hölder's and Minkowski's inequalities on time scales. According to Stefan Hilger's 1988 PhD thesis, the theory of time scales combines the science of differential equations and difference equations, extending to cases "in between", uniting integral and differential calculus with the calculus of finite differences, and providing a paradigm for investigating hybrid discrete-continuous dynamic systems. It can be used in any field that demands the simultaneous modeling of discrete and continuous data. Now that the time scales calculus has been introduced, see [1, 2, 4, 5, 6, 7] for more information.

Let $n \in \mathbb{N}$ be fixed. For time scales, T_i, $i \in \{1,\dots,n\}$, let

$$\Lambda^n = T_1 \times \ldots \times T_n = \{x = (x_1,\dots,x_n) : x_i \in T_i,\ 1 \leqslant i \leqslant n\} \tag{17}$$

an n-dimensional time scale. Suppose that μ_Δ is the σ-additive Lebesgue Δ-measure on Λ^n and M is the collection of Δ-measurable subsets of Λ^n. If $\mathscr{A} \in M$, $(\mathscr{A}, M, \mu_\Delta)$ is a time scale measure space, and $s : \mathscr{A} \to \mathbb{R}$ is a Δ-measurable function, then the corresponding Δ-integral of s over $\mathscr{A}$ is denoted by (see [7, (3.18)])

$$\int_{\mathscr{A}} s(x_1,\dots,x_p)\Delta_1 x_1 \ldots \Delta_p x_p, \quad \int_{\mathscr{A}} s(x)\Delta x, \quad \int_{\mathscr{A}} s d\mu_\Delta, or \int_{\mathscr{A}} s(x) d\mu_\Delta(x).$$

All theorems of the general Lebesgue integration theory also hold for Lebesgue Δ-integrals on Λ^p.

THEOREM 12. *Let (X, M, μ_Δ) be a time scales measure space. If $p > 1$, $q = p/(p-1)$, w, f, g are positive Δ-integrable functions such that wf^p, wg^q, wfg are Δ-integrable, then we have*

$$\int_X s\left(\frac{\int_X w(x)g^q(x)d\mu_\Delta(x) f^p(x)}{\int_X w(x)f^p(x)d\mu_\Delta(x) g^q(x)}\right) w(x)f(x)g(x)d\mu_\Delta(x) \geqslant \left(\int_X w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}} \left(\int_X w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}, \tag{18}$$

where s is Specht's ratio.

Proof. The result follows from Theorem 6 and the fact that delta integral is a positive linear functional. □

REMARK 2. Some specific cases of time scales are taken below to obtain converses, otherwise we can also obtain these inequalities for other important time scales, e.g., for $T = h\mathbb{Z}$ and $T = q^{\mathbb{N}}$.

(i) Let $n = 1$ in (17). If $\Lambda^1 = T_1 = [a,b] \subseteq \mathbb{R}$ and $L = L[a,b]$, then the inequality (18) becomes

$$\int_a^b s\left(\frac{\int_a^b w(x)g^q(x)d\mu(x) f^p(x)}{\int_a^b w(x)f^p(x)d\mu(x) g^q(x)}\right) w(x)f(x)g(x)d\mu(x) \geqslant \left(\int_a^b w(x)f^p(x)d\mu(x)\right)^{\frac{1}{p}} \left(\int_a^b w(x)g^q(x)d\mu(x)\right)^{\frac{1}{q}}.$$

(ii) Let $n = 2$ in (17). If $\Lambda^2 = T_1 \times T_2 = [a,b] \times [c,d] \subseteq \mathbb{R}^2$ and $L = L([a,b] \times [c,d])$, then the inequality (18) becomes

$$\int_a^b \int_c^d s\left(\frac{\int_a^b \int_c^d w(x,y)g^q(x,y)d\mu(x)d\mu(y)f^p(x,y)}{\int_a^b \int_c^d w(x,y)f^p(x,y)d\mu(x)d\mu(y)g^q(x,y)}\right)\times$$
$$\times w(x,y)f(x,y)g(x,y)d\mu(x)d\mu(y)$$
$$\geqslant \left(\int_a^b \int_c^d w(x,y)f^p(x,y)d\mu(x)d\mu(y)\right)^{\frac{1}{p}}\times$$
$$\times\left(\int_a^b \int_c^d w(x,y)g^q(x,y)d\mu(x)d\mu(y)\right)^{\frac{1}{q}}.$$

(iii) Let $n=1$ in (17). If $T_1=\{1,2,\ldots,n\}$, $f(r)=f_r$, $g(r)=g_r$, and $w(r)=w_r$ where $r=1,\ldots,n$, then the inequality (18) becomes

$$\sum_{r=1}^n s\left(\frac{\left(\sum_{r=1}^n w_r g_r^q\right) f_r^p}{\left(\sum_{r=1}^n w_r f_r^p\right) g_r^q}\right) w_r f_r g_r \geqslant \left(\sum_{r=1}^n w_r f_r^p\right)^{\frac{1}{p}} \left(\sum_{r=1}^n w_r g_r^q\right)^{\frac{1}{q}}.$$

THEOREM 13. *Let all the assumptions of Theorem 12 are satisfied.*

(i) If $p<0$, then we get

$$\left(\int_X s\left(\frac{\int_X w(x)f(x)g(x)d\mu_\Delta(x)f^{p-1}(x)}{\int_X w(x)f^p(x)d\mu_\Delta(x)g(x)}\right) w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}\times$$
$$\times\left(\int_X w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}} \geqslant \int_X w(x)f(x)g(x)d\mu_\Delta(x).$$

(ii) If $0<p<1$, then we get

$$\left(\int_X s\left(\frac{\int_X w(x)g^q(x))d\mu_\Delta(x)f(x)}{\int_X w(x)f(x)g(x)d\mu_\Delta(x)g^{q-1}(x)}\right) w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\times$$
$$\times\left(\int_X w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}} \geqslant \int_X w(x)f(x)g(x)d\mu_\Delta(x).$$

Proof. The result follows from Theorem 7 and the fact that delta integral is a positive linear functional. □

THEOREM 14. *Let all the assumptions of Theorem 12 be satisfied. Further assume that $\alpha,\beta>0$ on X such that αwfg, βwfg, αwf^p, αwg^q, βwf^p, βwg^q are*

Δ-integrable and $\alpha+\beta=1$ on X. Then we get

$$s\left(\frac{\left(\int_X \alpha(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}}{\left(\int_X w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \alpha(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}}\right)\times$$
$$\times\left(\int_X \alpha(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \alpha(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}$$
$$+s\left(\frac{\left(\int_X \beta(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}}{\left(\int_X w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \beta(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}}\right)\times$$
$$\times\left(\int_X \beta(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \beta(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}$$
$$\geqslant\left(\int_X w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}.$$

Proof. The result follows from Theorem 8 and the fact that delta integral is a positive linear functional. □

THEOREM 15. *Let all the assumptions of Theorem 14 be satisfied. If*

$$s\left(\frac{\int_X \alpha(x)w(x)g^q(x)d\mu_\Delta(x)f^p(x)}{\int_X \alpha(x)w(x)f^p(x)d\mu_\Delta(x)g^q(x)}\right)\alpha(x)+s\left(\frac{\int_X \beta(x)w(x)g^q(x)d\mu_\Delta(x)f^p(x)}{\int_X \beta(x)w(x)f^p(x)d\mu_\Delta(x)g^q(x)}\right)\beta(x)$$
$$\leqslant s\left(\frac{\int_X w(x)g^q(x)d\mu_\Delta(x)f^p(x)}{\int_X w(x)f^p(x)d\mu_\Delta(x)g^q(x)}\right),$$

then we have

$$\int_X s\left(\frac{\int_X w(x)g^q(x)d\mu_\Delta(x)f^p(x)}{\int_X w(x)f^p(x)d\mu_\Delta(x)g^q(x)}\right)w(x)f(x)g(x)d\mu_\Delta(x)$$
$$\geqslant\left(\int_X \alpha(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \alpha(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}$$
$$+\left(\int_X \beta(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \beta(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}.$$

Proof. The result follows from Theorem 9 and the fact that delta integral is a positive linear functional. □

THEOREM 16. *Let all the assumptions of Theorem 14 be satisfied. Suppose*

$$0<m\leqslant f(t)g^{-q/p}(t)\leqslant M \text{ for all } t\in X.$$

If $p > 1$, *then*

$$\begin{aligned}
&\int_x w(x)f(x)g(x)d\mu_\Delta(x) \qquad (19)\\
&\geqslant K(p,m,M)\left(\left(\int_X \alpha(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \alpha(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_X \beta(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \beta(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_X \alpha(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \alpha(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}} \qquad (20)\\
&\quad+\left(\int_X \beta(x)w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X \beta(x)w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}\\
&\geqslant K(p,m,M)\left(\int_X w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X w(x)g^q(x)d\mu_\Delta(x)\right)^{\frac{1}{q}}
\end{aligned}$$

hold where $K(p,m,M)$ *is defined as in* (4).

Proof. The result follows from Theorem 10 and the fact that delta integral is a positive linear functional. □

THEOREM 17. *Let all the assumptions of Theorem 12 be satisfied. If* $s_1 \geqslant s_2, s_3$, *where*

$$\begin{aligned}
s_1 &= s\left(\frac{\int_X w(x)g^q(x)d\mu_\Delta(x)f^p(x)}{\int_X w(x)f^p(x)d\mu_\Delta(x)g^q(x)}\right),\\
s_2 &= s\left(\frac{\int_X w(x)(f(x)+g(x))^p d\mu_\Delta(x)f^p(x)}{\int_X w(x)f^p(x)d\mu_\Delta(x)(f(x)+g(x))^p}\right),\\
s_3 &= s\left(\frac{\int_X w(x)(f(x)+g(x))^p d\mu_\Delta(x)g^p(x)}{\int_X w(x)g^p(x)d\mu_\Delta(x)(f(x)+g(x))^p}\right),
\end{aligned}$$

then we have

$$\begin{aligned}
&\left(\int_X w(x)f^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}+\left(\int_X w(x)g^p(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\\
&\qquad\leqslant\left[\frac{(\int_X s_1 w(x)(f(x)+g(x))^p d\mu_\Delta(x))^p}{(\int_X s_1 w(x)(f(x)+g(x))^p d\mu_\Delta(x))^{p-1}}\right]^{\frac{1}{p}}.
\end{aligned}$$

Proof. The result follows from Theorem 11 and the fact that delta integral is a positive linear functional. □

REMARK 3. In a similar way as in Remark 2, we can also obtain the specific cases of all results of this section. These reverses are new even in the case of sums and integrals.

4. Reverse of integral Minkowski's inequality

In this section we obtain a converse of improved integral Minkowski's inequality (see [3]) on time scale.

THEOREM 18. *Let (X,M,μ_Δ) and $(Y,L,d\nu_\Delta)$ be time scale measure spaces and let u,v, and f be nonnegative functions on X,Y, and $X\times Y$, respectively. Suppose*

$$0<m\leqslant \frac{f(x,y)}{\int_X f(x,y)v(y)d\nu_\Delta(y)}\leqslant M.$$

If $p\geqslant 1$, then

$$\begin{aligned}
&\left(\int_X\left(\int_Y f(x,y)v(y)d\nu_\Delta(y)\right)^p u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}} \\
&\geqslant \left(\int_X\left(\int_Y f(x,y)v(y)d\nu_\Delta(y)\right)^p u(x)d\mu_\Delta(x)\right)^{\frac{p-1}{p}} B \\
&\geqslant K^2(p,m,M)\int_Y\left(\int_X f^p(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)d\nu_\Delta(y)
\end{aligned} \tag{21}$$

holds provided all integrals in (21) exists, where

$$\begin{aligned}
B=K(p,m,M)\int_Y\Bigg(&\left(\int_X \alpha(x)f^p(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\times \\
&\times\left(\int_X \alpha(x)H^p(x)u(x)d\mu_\Delta(x)\right)^{\frac{p-1}{p}}+\left(\int_X \beta(x)f^p(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\times \\
&\times\left(\int_X \beta(x)H^p(x)u(x)d\mu_\Delta(x)\right)^{\frac{p-1}{p}}\Bigg)v(y)d\nu_\Delta(y).
\end{aligned}$$

Proof. Let $H(x)=\int_Y f(x,y)v(y)d\nu_\Delta(y)$. By using Fubini's theorem and inequalities (19) and (20), we get

$$\begin{aligned}
&\int_X H^p(x)u(x)d\mu_\Delta(x) \\
&=\int_X H(x)H^{p-1}(x)u(x)d\mu_\Delta(x) \\
&=\int_X\left(\int_Y f(x,y)v(y)d\nu_\Delta(y)\right)H^{p-1}(x)u(x)d\mu_\Delta(x) \\
&=\int_Y\left(\int_X f(x,y)H^{p-1}(x)u(x)d\mu_\Delta(x)\right)v(y)d\nu_\Delta(y)
\end{aligned}$$

$$\geqslant K(p,m,M)\int_Y\left(\left(\int_X\alpha(x)f^p(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\times\right.$$
$$\times\left(\int_X\alpha(x)H^p(x)u(x)d\mu_\Delta(x)\right)^{\frac{p-1}{p}}+\left(\int_X\beta(x)f^p(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\times$$
$$\left.\times\left(\int_X\beta(x)H^p(x)u(x)d\mu_\Delta(x)\right)^{\frac{p-1}{p}}\right)v(y)d\nu_\Delta(y)$$
$$\geqslant K^2(p,m,M)\int_Y\left(\int_X f^p(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}\left(\int_X H^p(x)u(x)d\mu_\Delta(x)\right)^{\frac{p-1}{p}}v(y)d\nu_\Delta(y)$$
$$=K^2(p,m,M)\int_Y\left(\int_X f^p(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)d\nu_\Delta(y)\left(\int_X H^p(x)u(x)d\mu_\Delta(x)\right)^{\frac{p-1}{p}}.$$

Now dividing by $\left(\int_X H^p(x)u(x)d\mu_\Delta(x)\right)^{\frac{p-1}{p}}$, we obtain the required result. □

REMARK 4. (i) If $X\subseteq[a,b]$, $Y\subseteq[c,d]$, then the inequality (21) becomes

$$\left(\int_X\left(\int_Y f(x,y)v(y)d\nu(y)\right)^p u(x)d\mu(x)\right)^{\frac{1}{p}}$$
$$\geqslant\left(\int_X\left(\int_Y f(x,y)v(y)d\nu(y)\right)^p(x)u(x)d\mu(x)\right)^{\frac{p-1}{p}}B$$
$$\geqslant K^2(p,m,M)\int_Y\left(\int_X f^p(x,y)u(x)d\mu(x)\right)^{\frac{1}{p}}v(y)d\nu(y),$$

where

$$B=K(p,m,M)\int_Y\left(\left(\int_X\alpha(x)f^p(x,y)u(x)d\mu(x)\right)^{\frac{1}{p}}\times\right.$$
$$\times\left(\int_X\alpha(x)H^p(x)u(x)d\mu(x)\right)^{\frac{p-1}{p}}+\left(\int_X\beta(x)f^p(x,y)u(x)d\mu(x)\right)^{\frac{1}{p}}\times$$
$$\left.\times\left(\int_X\beta(x)H^p(x)u(x)d\mu(x)\right)^{\frac{p-1}{p}}\right)v(y)d\nu(y).$$

(ii) If $X,Y\subseteq\mathbb{N}$ such that $w(r)=w_r$, $g(r)=g_r$ and $f(r,s)=f_{r,s}$, $r,s\in\{1,2,\dots,n\}$, then the inequality (21) becomes

$$\left(\sum_{r=1}^n\left(\sum_{s=1}^n f_{r,s}v_s\right)^p u_r\right)^{\frac{1}{p}}\geqslant\left(\sum_{r=1}^n\left(\sum_{s=1}^n f_{r,s}v_s\right)^p u_r\right)^{\frac{p-1}{p}}B$$
$$\geqslant K^2(p,m,M)\sum_{s=1}^n\left(\sum_{r=1}^n f_{r,s}^p u_r\right)^{\frac{1}{p}}v_s,$$

where

$$B = K(p,m,M)\sum_{s=1}^{n}\left(\left(\sum_{r=1}^{n}\alpha(x)f_{r,s}^{p}u_r\right)^{\frac{1}{p}}\left(\sum_{r=1}^{n}\alpha(x)H_r^{p}u_r\right)^{\frac{p-1}{p}}\right.$$
$$\left.+\left(\sum_{r=1}^{n}\beta(x)f_{r,s}^{p}u_r\right)^{\frac{1}{p}}\left(\sum_{r=1}^{n}\beta(x)H_r^{p}u_r\right)^{\frac{p-1}{p}}\right)v_s.$$

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