Mathematical
Inequalities
& Applications
Volume 27, Number 3 (2024), 627–645 doi:10.7153/mia-2024-27-43

WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED BERGMAN–ORLICZ SPACES WITH DOUBLING WEIGHTS TO WEIGHTED ZYGMUND SPACES

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(*Communicated by I. Perić*)

Abstract. We characterize the boundedness and compactness of weighted composition operators from weighted Bergman-Orlicz spaces with doubling weights to weighted Zygmund spaces on the unit disc.

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let ϕ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator uC_ϕ on $H(\mathbb{D})$ is defined by

$$uC_\phi f(z) = (u \cdot f \circ \phi)(z), \quad z \in \mathbb{D}.$$

When $u \equiv 1$, the operator uC_ϕ is called the composition operator and denoted by C_ϕ. Beside integral type operators (see, e.g., [1, 9, 13, 14, 26, 31] and the references therein) and the differentiation type operators (see, e.g., [30, 33, 34, 36, 37] and the references therein), the weighted composition operators are concrete operators which have been studied on some spaces of analytic functions on various domains in many books and papers (see, e.g., [2, 3, 4, 8, 11, 12, 24, 28, 27, 29, 37] and the references therein).

A growth function is a function $\Psi : [0,\infty) \to [0,\infty)$, non-decreasing, surjective and such that $\Psi(0) = 0, \Psi(x) > 0$ for $x > 0$ and $\lim_{x\to\infty} \Psi(x) = \infty$. It is said that the function Ψ is of upper type s, if there are $s > 0$ and $C > 0$ such that (see, for example, [20, 21])

$$\Psi(rt) \leqslant Ct^s \Psi(r)$$

for every $r > 0$ and $t \geqslant 1$. We use $\mathscr{U}^s$ to denote the family of all growth functions Ψ of upper type $s\,(s \geqslant 1)$ such that the function $\Psi(t)/t$ is non-decreasing on $(0,\infty)$. It is said that the function Ψ is of lower type q, if there are $q > 0$ and $C > 0$ such that

$$\Psi(rt) \leqslant Ct^q \Psi(r)$$

Mathematics subject classification (2020): 30H99, 47B33.
Keywords and phrases: Weighted composition operator, Bergman-Orlicz space, weighted Zygmund space, doubling weight.
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for every $r>0$ and $0<t\leqslant 1$. We use $\mathscr{L}_q$ to denote the family of all growth functions Ψ of lower type $q(0<q\leqslant 1)$ such that the function $\Psi(t)/t$ is non-increasing on $(0,\infty)$. If $\Psi\in\mathscr{U}^s$ (resp. $\mathscr{L}_q$), we will assume that Ψ is convex (resp. concave) on $[0,\infty)$. The relations with convexity and concavity were given and explained in [21].

Let dA be the normalized Lebesgue measure on $\mathbb{D}$. Let $\alpha>-1$ and $dA_\alpha(z)=(\alpha+1)(1-|z|^2)^\alpha dA(z)$ denote the weighted Lebesgue measure on $\mathbb{D}$. Let Ψ be a growth function. The weighted Bergman-Orlicz space $A^\Psi_\alpha(\mathbb{D}):=A^\Psi_\alpha$ consists of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{A^\Psi_\alpha}=\int_{\mathbb{D}}\Psi(|f(z)|)dA_\alpha(z)<\infty.$$

If $\Psi\in\mathscr{U}^s$ or $\Psi\in\mathscr{L}_q$, then the following quasi-norm is finite and called the Luxembourg norm.

$$\|f\|^{lux}_{A^\Psi_\alpha}=\inf\left\{\lambda>0:\int_{\mathbb{D}}\Psi\left(\frac{|f(z)|}{\lambda}\right)dA_\alpha(z)\leqslant 1\right\}.$$

When $\Psi(t)=t^p$, A^Ψ_α is just the standard weighted Bergman space A^p_α [35]. It is well known that A^p_α is a Banach space when $p\geqslant 1$, while it is a translation-invariant metric space when $0<p<1$. See [5, 6, 7, 15, 19, 20, 21, 22, 23, 32] and the references therein for more results of various operators on Bergman-Orlicz spaces.

Let ω be a radial weight, that is, ω is a positive, measurable and integrable function on $[0,1)$ and $\omega(z)=\omega(|z|)$ for all $z\in\mathbb{D}$. We say that ω is a doubling weight, denoted by $\omega\in\hat{\mathscr{D}}$, if (see [16])

$$\hat{\omega}(r)\leqslant C\hat{\omega}\left(\frac{1+r}{2}\right)$$

for some constant $C\geqslant 1$ and for all $r\in[0,1)$, where $\hat{\omega}(r)=\int_r^1\omega(t)dt$. If $0<p<\infty$ and $\omega\in\hat{\mathscr{D}}$, the weighted Bergman space A^p_ω consists of all $f\in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\omega}=\int_{\mathbb{D}}|f(z)|^p\omega(z)dA(z)<\infty.$$

When $\omega(z)=(1-|z|^2)^\alpha$, A^p_ω is just the weighted Bergman space A^p_α. For more study of doubling weights and weighted Bergman space A^p_ω, see [16, 17, 18, 25] and the references therein.

Suppose Ψ is a growth function and $\omega\in\hat{\mathscr{D}}$. Inspired by the definitions of the weighted Bergman space A^p_ω and the weighted Bergman-Orlicz space A^Ψ_α, in this paper we define a Bergman-Orlicz space A^Ψ_ω, called the Bergman-Orlicz space with doubling weights, which consists of all $f\in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}}\Psi(|f(z)|)\omega(z)dA(z)<\infty.$$

The space was previously studied by Stević in [25] for the case when the growth function is subadditive (in this case the last quantity is a metrizable vector space). To avoid the restriction it is usually considered the space with a Luxemburg quasi-norm.

Hence, here we define on A^{Ψ}_{ω} the following quasi-norm:

$$\|f\|^{lux}_{A^{\Psi}_{\omega}} = \inf\left\{\lambda > 0 : \int_{\mathbb{D}} \Psi\left(\frac{|f(z)|}{\lambda}\right)\omega(z)dA(z) \leqslant 1\right\}, \tag{1}$$

slightly changing the standard quasi-norm (see, e.g., [20, 21]) in a natural way. If $\Psi \in \mathscr{U}^s$ or $\Psi \in \mathscr{L}_q$, then the quantity in (1) is finite for every $f \in A^{\Psi}_{\omega}$.

For $\beta > 0$, the weighted Zygmund space $\mathscr{Z}_\beta$ consists of all $g \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|g''(z)| < \infty.$$

It is a Banach space with the norm

$$\|g\|_{\mathscr{Z}_\beta} = |g(0)| + |g'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|g''(z)|.$$

When $\beta = 1$, this space is well known Zygmund space and is denoted by $\mathscr{Z}$ (see [10]), where $\|g\|_{\mathscr{Z}} = |g(0)| + |g'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|g''(z)|$.

The little weighted Zygmund space $\mathscr{Z}_{\beta,0}$ consists of those functions g in $\mathscr{Z}_\beta$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)^{\beta}|g''(z)| = 0.$$

For a Zygmund type space on the upper half-plane see [27]. For a generalization of the space see [28]. For the corresponding space on the unit ball in $\mathbb{C}^n$ see [13, 14, 35]. For more results about Zygmund type spaces and concrete linear operators on them see also [6, 11, 31, 36, 37] and the related references therein.

Let X and Y be topological vector spaces whose topologies are given by translation invariant metrics d_X and d_Y, respectively. It is said that a linear operator $B : X \to Y$ is metrically bounded if there exists a positive constant C such that

$$d_Y(Bf, 0) \leqslant C d_X(f, 0)$$

for all $f \in X$. When X and Y are Banach spaces, the metrical boundedness coincides with the boundedness. Operator $B : X \to Y$ is said to be metrically compact if it maps bounded sets into relatively compact sets. When X and Y are Banach spaces, the metrical compactness coincides with the compactness of operators between Banach spaces.

Sehba and Stević in [20] investigated the boundedness and compactness of weighted composition operators and a class of integral-type operators from Bergman-Orlicz spaces A^{Ψ}_{α} to a class of weighted-type spaces on the unit ball of $\mathbb{C}^n$ and then continued the investigation in [21]. Jiang in [5] characterized the boundedness and compactness of a product-type operator from Bergman-Orlicz spaces A^{Ψ}_{α} to weighted Bloch spaces and weighted Zygmund spaces on the unit disc $\mathbb{D}$. Jiang in [6] described the boundedness and compactness of a product-type operator from Bergman-Orlicz spaces A^{Ψ}_{α} to weighted Zygmund spaces on $\mathbb{D}$. In [7], Jiang characterized the boundedness and compactness of a generalized product-type operator from Bergman-Orlicz spaces A^{Ψ}_{α}

to Bloch-Orlicz spaces on $\mathbb{D}$. In [32] Stević and Jiang investigated the boundedness and compactness of the weighted iterated radial composition operators from weighted Bergman-Orlicz spaces to weighted-type spaces on the unit ball. It should be said that these investigations are natural continuations of the studies of concrete linear operators from the standard Bergman spaces to some spaces with weights such as are, for example, some ones in [12, 27, 28, 29]. See also [2, 3] for more results of concrete linear operators on Bergman-Orlicz spaces in the unit ball of $\mathbb{C}^n$.

Motivated by the above mentioned papers, here we investigate some properties of functions in the Bergman-Orlicz space A^{Ψ}_{ω} and then study the boundedness and compactness of weighted composition operators from Bergman-Orlicz spaces A^{Ψ}_{ω} to weighted Zygmund spaces $\mathscr{Z}_{\beta}$ on the unit disc $\mathbb{D}$.

In this paper, we denote $E \lesssim F$ means that $E \leqslant CF$ for some positive constant C, which will be a constant not necessary the same at each occurrence. We write $E \asymp F$ when $E \lesssim F$ and $F \lesssim E$.

2. Some lemmas

In this section, we investigate some properties, such as growth properties, of functions in the Bergman-Orlicz space A^{Ψ}_{ω}. First, we need some notations.

The Carleson square $S(I)$ based on an interval I on the boundary $\mathbb{T}$ of $\mathbb{D}$ is the set

$$S(I) = \left\{re^{it} \in \mathbb{D} : e^{it} \in I, 1-|I| \leqslant r < 1\right\},$$

where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{T}$. We associate to each $a \in \mathbb{D}\backslash\{0\}$ the interval

$$I_a = \left\{e^{i\theta} : |\arg(ae^{-i\theta})| \leqslant \frac{1-|a|}{2}\right\},$$

and denote $S(a) = S(I_a)$.

LEMMA 1. [16, Lemma 2.1] *Let ω be a radial weight. Then the following statements are equivalent.*

(i) $\omega \in \hat{\mathscr{D}}$;

(ii) There exists $C = C(\omega) \geqslant 1$ *and* $\beta = \beta(\omega) > 0$ *such that*

$$\hat{\omega}(r) \leqslant C\left(\frac{1-r}{1-t}\right)^{\beta} \hat{\omega}(t),\ \ 0 \leqslant r \leqslant t < 1;$$

(iii) There exists $\lambda = \lambda(\omega) \geqslant 0$ *such that*

$$\int_{\mathbb{D}} \frac{\omega(z)dA(z)}{|1-\overline{a}z|^{\lambda+1}} \asymp \frac{\hat{\omega}(a)}{(1-|a|)^{\lambda}},\quad a \in \mathbb{D};$$

(iv) $\omega(S(z)) \asymp \hat{\omega}(z)(1-|z|),\ |z| \to 1^{-}$.

For a radial weight ω, for each orthonormal basis $\{e_n\}$ of A^2_ω, the kernel has the representation $B^\omega_z(\xi) = \sum \overline{e_n(z)} e_n(\xi)$, and therefore we have

$$B^\omega_z(\xi) = \sum_{n=0}^{\infty} \frac{(\xi\overline{z})^n}{2\omega_{2n+1}}.$$

Here, $\omega_x = \int_0^1 r^x \omega(r)dr$ for all $x \geqslant 0$.

For $\omega \in \hat{\mathscr{D}}$, A^2_ω is a Hilbert space. Therefore there exists reproducing kernel $B^\omega_z \in A^2_\omega$ such that

$$f(z) = \langle f, B^\omega_z \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(w)\overline{B^\omega_z(w)}\omega(w)dA(w), \ z \in \mathbb{D}, \ f \in A^2_\omega.$$

The following lemma is similar to Lemma 6 in [18], and we give the proof for the sake of completeness. Denote $\omega_\alpha(z) = (1-|z|)^\alpha \omega(z)$ for all $\alpha \in \mathbb{R}$ and $z \in \mathbb{D}$. In addition, we denote by $\mathbb{N}$ the set of positive integers.

LEMMA 2. *Let $\omega \in \hat{\mathscr{D}}$ and $m \in \mathbb{N} \cup \{0\}$. Then*

$$\|(B^\omega_z)^{(m)}\|_{H^\infty} \leqslant \frac{e_m}{\omega(S(z))(1-|z|^2)^m}, \ z \in \mathbb{D},$$

where e_m depends only on m and ω.

Proof. Since

$$B^\omega_z(\xi) = \sum_{n=0}^{\infty} \frac{(\xi\overline{z})^n}{2\omega_{2n+1}}, \ (B^\omega_z)^{(m)}(\xi) = \sum_{n=m}^{\infty} \frac{n(n-1)\cdots(n-m+1)\xi^{n-m}\overline{z}^n}{2\omega_{2n+1}}, \ z, \xi \in \mathbb{D},$$

the estimate (20) of [17],

$$\sum_{n=N}^{\infty} \frac{r^{pn}}{(n+1)^{-(N+1)p+2}\omega^p_{2n+1}} \asymp \int_0^r \frac{dt}{\hat{\omega}_{N+1}(t)^p}, \quad r \to 1^-$$

with $p = 1$, $N = m+1$, and $r = |z|^2$, together with Lemma 1 yields

$$\begin{aligned}
|(B^\omega_z)^{(m)}(z)| &\leqslant \frac{|z|^m}{2} \sum_{n=m}^{\infty} \frac{n(n-1)\cdots(n-m+1)|z|^{2(n-m)}}{\omega_{2n+1}} \\
&\leqslant \frac{1}{2} \cdot c_1 \int_0^{|z|^2} \frac{dt}{\hat{\omega}(t)(1-t)^{m+2}} \\
&\leqslant \frac{1}{2} \cdot c_1 c_2 \frac{1}{\hat{\omega}(|z|^2)(1-|z|^2)^{m+1}} \\
&\leqslant \frac{1}{2} \cdot c_1 c_2 c_3 \frac{1}{\omega(S(z))(1-|z|^2)^m}, \ |z| \to 1^-.
\end{aligned}$$

Take $e_m = \frac{c_1 c_2 c_3}{2}$, where c_1 and c_2 depend only on m and ω, and c_3 depends only on ω. We get the desired result. $\square$

LEMMA 3. *Let $n \in \mathbb{N} \cup \{0\}$, $p \geqslant 1$, $\omega \in \hat{\mathscr{D}}$ and $\Psi \in \mathscr{U}^s$. There exist two constants C_n, D_n independent of $f \in A_\omega^{\Psi_p}$ such that*

$$|f^{(n)}(z)| \leqslant \frac{C_n}{(1-|z|^2)^n} \Psi_p^{-1}\left(\frac{D_n}{\omega(S(z))}\right) \|f\|_{A_\omega^{\Psi_p}}^{lux}, \quad z \in \mathbb{D}. \tag{2}$$

Here, $\Psi_p(t) = \Psi(t^p)$.

Proof. When $\Psi \in \mathscr{U}^s$ and $\omega \in \hat{\mathscr{D}}$, letting $\Psi_p(t) = \Psi(t^p)$, it is easy to check that $A_\omega^{\Psi_p} \subset A_\omega^p$. In fact, if $f \in A_\omega^{\Psi_p}$ and set $\lambda = 2\|f\|_{A_\omega^{\Psi_p}}^{lux}$, we have

$$\int_{\mathbb{D}} \Psi\left(\frac{|f(z)|^p}{\lambda^p}\right) \omega(z) dA(z) < \infty.$$

Since $\Psi \in \mathscr{U}^s$, $\Psi(t)/t$ is non-decreasing. So, when $|f(z)| > \lambda$, we have

$$\Psi\left(\frac{|f(z)|^p}{\lambda^p}\right) \geqslant \Psi(1) \frac{|f(z)|^p}{\lambda^p},$$

which implies $f \in A_\omega^p$. Since $A_\omega^{\Psi_p} \subset A_\omega^p$, for any $f \in A_\omega^{\Psi_p}$ and $z \in \mathbb{D}$, we have

$$f(z) = \int_{\mathbb{D}} f(w) \overline{B_z^\omega(w)} \omega(w) dA(w).$$

Therefore, when $|z| \geqslant \frac{1}{2}$,

$$\begin{aligned} |f^{(n)}(z)| &\leqslant \int_{\mathbb{D}} |f(w)| |(B_w^\omega(z))^{(n)}| \omega(w) dA(w) \\ &\leqslant \frac{1}{|z|^n} \int_{\mathbb{D}} |f(w)| |(B_z^\omega)^{(n)}(w)| \omega(w) dA(w). \end{aligned} \tag{3}$$

When $n \in \mathbb{N} \cup \{0\}$, by [17, Theorem 1], we have

$$\int_{\mathbb{D}} |(B_z^\omega)^{(n)}(w)| \omega(w) dA(w) \approx \frac{1}{(1-|z|)^n}.$$

So, we can choose s_1 such that

$$s_1 (1-|z|^2)^n |(B_z^\omega)^{(n)}(w)| \omega(w) dA(w)$$

is a probability measure on $\mathbb{D}$. From the above and applying Jensen's inequality in (3), we obtain

$$|z|^{np}(1-|z|^2)^{np}|f^{(n)}(z)|^p \leqslant s_1^{1-p} \int_{\mathbb{D}} |f(w)|^p (1-|z|^2)^n |(B_z^\omega)^{(n)}(w)| \omega(w) dA(w).$$

From this, we have

$$s_1^p |z|^{np}(1-|z|^2)^{np}|f^{(n)}(z)|^p \leqslant s_1 \int_{\mathbb{D}} |f(w)|^p (1-|z|^2)^n |(B_z^\omega)^{(n)}(w)| \omega(w) dA(w).$$

Let $\|f\|_{A_\omega^{\Psi_p}}^{lux} \leqslant \lambda \leqslant 2\|f\|_{A_\omega^{\Psi_p}}^{lux}$, such that

$$\int_{\mathbb{D}} \Psi\left(\frac{|f(w)|^p}{\lambda^p}\right)\omega(w)dA(w) \leqslant 1.$$

Since Ψ is convex, by Jensen's inequality and Lemma 2 , we have

$$\begin{aligned}
&\Psi\left(\frac{s_1^p|z|^{np}(1-|z|^2)^{np}|f^{(n)}(z)|^p}{\lambda^p}\right) \\
\leqslant &\int_{\mathbb{D}} \Psi\left(\frac{|f(w)|^p}{\lambda^p}\right) s_1(1-|z|^2)^n|(B_z^\omega)^{(n)}(w)|\omega(w)dA(w) \\
\leqslant & s_1(1-|z|^2)^n\|(B_z^\omega)^{(n)}\|_{H^\infty} \leqslant \frac{s_1 e_n}{\omega(S(z))}.
\end{aligned}$$

Therefore, we choose C_n and D_n independent of z and f, such that when $n \in \mathbb{N}\cup\{0\}$, for all $|z| > \frac{1}{2}$ and $f \in A_\omega^{\Psi_p}$,

$$(1-|z|^2)^n|f^{(n)}(z)| \leqslant C_n\left(\Psi^{-1}\left(\frac{D_n}{\omega(S(z))}\right)\right)^{\frac{1}{p}}\|f\|_{A_\omega^{\Psi_p}}^{lux} = C_n\Psi_p^{-1}\left(\frac{D_n}{\omega(S(z))}\right)\|f\|_{A_\omega^{\Psi_p}}^{lux},$$

where $C_n = 2^n s_1^{-1}$ and $D_n = s_1 e_n$. The last equality is obtained by

$$\Psi_p^{-1}(t) = (\Psi^{-1}(t))^{\frac{1}{p}}.$$

So (2) holds for all $z \in \mathbb{D}$ and $f \in A_\omega^{\Psi_p}$ when $n \in \mathbb{N}\cup\{0\}$. $\square$

LEMMA 4. *Let $p \geqslant 1$, $\omega \in \hat{\mathscr{D}}$ and $\Psi \in \mathscr{U}^s$. Then, if γ is large enough, for any given $M > 0$ and every $t \geqslant 0$, the function*

$$f_{a,\gamma}(z) = \Psi_p^{-1}\left(\frac{M}{\omega(S(a))}\right)\left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\frac{\gamma+1}{p}+t}$$

belongs to $A_\omega^{\Psi_p}$ and

$$\sup_{a\in\mathbb{D}}\|f_{a,\gamma}\|_{A_\omega^{\Psi_p}}^{lux} \lesssim 1.$$

Proof. Let

$$g_{a,\gamma}(z) = \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\frac{\gamma+1}{p}+t}.$$

Since $\Psi \in \mathscr{U}^s$ $(s \geqslant 1)$, $\frac{\Psi(t)}{t}$ is non-decreasing and since $\Psi(t_2) = \Psi\left(\frac{t_2}{t_1}\cdot t_1\right) \leqslant C\frac{t_2^s}{t_1^s}\Psi(t_1)$ $(t_1 \leqslant t_2)$, $\frac{\Psi(t)}{t^s}$ is essentially non-increasing on $(0,\infty)$ (due to the constant C). Thus for any $\lambda > 0$, we have

$$\Psi\left(\frac{\Psi^{-1}\left(\frac{M}{\omega(S(a))}\right)|g_{a,\gamma}(z)|^p}{\lambda^p}\right) \lesssim \begin{cases} \frac{M|g_{a,\gamma}(z)|^p}{\lambda^p\omega(S(a))}, & \text{when } |g_{a,\gamma}(z)| \leqslant \lambda, \\ \frac{M|g_{a,\gamma}(z)|^{ps}}{\lambda^{ps}\omega(S(a))}, & \text{when } |g_{a,\gamma}(z)| > \lambda. \end{cases}$$

Therefore, from Lemma 1, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} \Psi_p\left(\frac{|f_{a,\gamma}(z)|}{\lambda}\right)\omega(z)dA(z)\\
=&\int_{\mathbb{D}} \Psi\left(\frac{|f_{a,\gamma}(z)|^p}{\lambda^p}\right)\omega(z)dA(z)\\
=&\int_{\mathbb{D}} \Psi\left(\Psi^{-1}\left(\frac{M}{\omega(S(a))}\right)\frac{|g_{a,\gamma}(z)|^p}{\lambda^p}\right)\omega(z)dA(z)\\
\lesssim&\frac{1}{\lambda^p}\frac{M}{\omega(S(a))}\int_{\mathbb{D}}|g_{a,\gamma}(z)|^p\omega(z)dA(z)\\
&+\frac{1}{\lambda^{ps}}\frac{M}{\omega(S(a))}\int_{\mathbb{D}}|g_{a,\gamma}(z)|^{ps}\omega(z)dA(z)\\
\lesssim&\frac{1}{\lambda^p}\frac{1}{\omega(S(a))}\int_{\mathbb{D}}\left|\frac{1-|a|^2}{1-\overline{a}z}\right|^{\gamma+1+pt}\omega(z)dA(z)\\
&+\frac{1}{\lambda^{ps}}\frac{1}{\omega(S(a))}\int_{\mathbb{D}}\left|\frac{1-|a|^2}{1-\overline{a}z}\right|^{(\gamma+1)s+pst}\omega(z)dA(z)\\
\lesssim&\frac{1}{\lambda^p}\frac{(1-|a|^2)^{\gamma+1+pt}}{\omega(S(a))}\int_{\mathbb{D}}\frac{\omega(z)dA(z)}{|1-\overline{a}z|^{\gamma+1+pt}}\\
&+\frac{1}{\lambda^{ps}}\frac{(1-|a|^2)^{(\gamma+1)s+pst}}{\omega(S(a))}\int_{\mathbb{D}}\frac{\omega(z)dA(z)}{|1-\overline{a}z|^{(\gamma+1)s+pst}}\\
\lesssim&\frac{1}{\lambda^p}\frac{(1-|a|^2)^{\gamma+1+pt}}{\omega(S(a))}\frac{\hat{\omega}(a)}{(1-|a|)^{\gamma+pt}}\\
&+\frac{1}{\lambda^{ps}}\frac{(1-|a|^2)^{(\gamma+1)s+pst}}{\omega(S(a))}\frac{\hat{\omega}(a)}{(1-|a|)^{(\gamma+1)s+pst-1}}\\
\lesssim&\frac{1}{\lambda^p}+\frac{1}{\lambda^{ps}}\\
\lesssim&1.
\end{aligned}$$

Thus, we have $f_{a,\gamma}\in A_\omega^{\Psi_p}$ and $\sup_{a\in\mathbb{D}}\|f_{a,\gamma}\|^{lux}_{A_\omega^{\Psi_p}}\lesssim 1$. □

The following lemma can be proved by a similar argument as [4, Proposition 3.11], thus we omit the details.

LEMMA 5. *Let $p\geqslant 1$, $\omega\in\hat{\mathscr{D}}$, $\beta>0$ and $\Psi\in\mathscr{U}^s$ be such that $\Psi_p\in\mathscr{L}_q$. Then the operator $uC_\phi : A_\omega^{\Psi_p}\to\mathscr{Z}_\beta$ is metrically compact if and only if $uC_\phi : A_\omega^{\Psi_p}\to\mathscr{Z}_\beta$ is metrically bounded and, for any bounded sequence $\{f_m\}_{m\in\mathbb{N}}$ in $A_\omega^{\Psi_p}$ such that $f_m\to 0$ uniformly on any compact subset of $\mathbb{D}$ as $m\to\infty$, it follows that*

$$\lim_{m\to\infty}\|uC_\phi f_m\|_{\mathscr{Z}_\beta}=0.$$

3. The operator $uC_\phi : A^{\Psi_p}_\omega \to \mathscr{Z}_\beta$

First we describe the boundedness of operator $uC_\phi : A^{\Psi_p}_\omega \to \mathscr{Z}_\beta$. We assume that $\Psi \in \mathscr{U}^s$ is such that $\Psi_p \in \mathscr{L}_q$. Under this assumption, $A^{\Psi_p}_\omega$ is a complete metric space (see, for example, [20]).

THEOREM 1. *Let $p \geqslant 1$, $\omega \in \hat{\mathscr{D}}$, $\beta > 0$ and $\Psi \in \mathscr{U}^s$ be such that $\Psi_p \in \mathscr{L}_q$. Let $u \in H(\mathbb{D})$ and ϕ be an analytic self-map of $\mathbb{D}$. Then the operator $uC_\phi : A^{\Psi_p}_\omega \to \mathscr{Z}_\beta$ is metrically bounded if and only if*

$$K_1 = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |u''(z)| \Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(z)))}\right) < \infty,$$

$$K_2 = \sup_{z\in\mathbb{D}} \frac{|2u'(z)\phi'(z) + u(z)\phi''(z)|(1-|z|^2)^\beta}{1-|\phi(z)|^2} \Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z)))}\right) < \infty,$$

and

$$K_3 = \sup_{z\in\mathbb{D}} \frac{|u(z)||\phi'(z)|^2(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^2} \Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z)))}\right) < \infty.$$

Moreover, if the operator $uC_\phi : A^{\Psi_p}_\omega \to \mathscr{Z}_\beta$ is nonzero and metrically bounded, then

$$\|uC_\phi\| \asymp 1 + K_1 + K_2 + K_3.$$

Proof. Necessity. Suppose that the operator $uC_\phi : A^{\Psi_p}_\omega \to \mathscr{Z}_\beta$ is metrically bounded. For $a \in \mathbb{D}$, set

$$\begin{aligned} g(z) = &-\frac{\gamma+1+2p}{\gamma+1} \Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(a)))}\right)\left(\frac{1-|\phi(a)|^2}{1-\overline{\phi(a)}z}\right)^{\frac{\gamma+1}{p}} \\ &+ \frac{2(\gamma+1+2p)}{\gamma+1+p} \Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(a)))}\right)\left(\frac{1-|\phi(a)|^2}{1-\overline{\phi(a)}z}\right)^{\frac{\gamma+1}{p}+1} \\ &- \Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(a)))}\right)\left(\frac{1-|\phi(a)|^2}{1-\overline{\phi(a)}z}\right)^{\frac{\gamma+1}{p}+2}. \end{aligned}$$

After a calculation, we obtain

$$g'(\phi(a)) = g''(\phi(a)) = 0. \tag{4}$$

Obviously,

$$g(\phi(a)) = C\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(a)))}\right), \tag{5}$$

where $C=-\frac{\gamma+1+2p}{\gamma+1}+\frac{2(\gamma+1+2p)}{\gamma+1+p}-1\neq 0$. By Lemma 4, $g\in A_{\omega}^{\Psi_p}$ and $\|g\|_{A_{\omega}^{\Psi_p}}^{lux}\lesssim 1$. By the boundedness of $uC_{\phi}:A_{\omega}^{\Psi_p}\to\mathscr{Z}_{\beta}$, we obtain

$$(1-|a|^2)^{\beta}|u''(a)|\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(a)))}\right)\leqslant\|uC_{\phi}g\|_{\mathscr{Z}_{\beta}}\leqslant C\|uC_{\phi}\|,$$

which means

$$K_1=\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u''(z)|\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(z)))}\right)\leqslant C\|uC_{\phi}\|<\infty. \tag{6}$$

Next, we prove $K_2<\infty$. For this we consider the functions $f(z)=1$ and $f(z)=z$, respectively. Since $uC_{\phi}:A_{\omega}^{\Psi_p}\to\mathscr{Z}_{\beta}$ is metrically bounded, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u''(z)|\leqslant\|uC_{\phi}1\|_{\mathscr{Z}_{\beta}}\leqslant C\|uC_{\phi}\| \tag{7}$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u''(z)\phi(z)+2u'(z)\phi'(z)+u(z)\phi''(z)|\leqslant\|uC_{\phi}z\|_{\mathscr{Z}_{\beta}}\leqslant C\|uC_{\phi}\|. \tag{8}$$

By (7), (8) and the boundedness of ϕ, we have

$$I_1=\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|2u'(z)\phi'(z)+u(z)\phi''(z)|\leqslant C\|uC_{\phi}\|. \tag{9}$$

For $a\in\mathbb{D}$, we choose the function

$$\begin{aligned}h(z)=&-\frac{\gamma+1+2p}{\gamma+1+p}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(a)))}\right)\left(\frac{1-|\phi(a)|^2}{1-\overline{\phi(a)}z}\right)^{\frac{\gamma+1}{p}}\\&+\frac{2\gamma+2+3p}{\gamma+1+p}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(a)))}\right)\left(\frac{1-|\phi(a)|^2}{1-\overline{\phi(a)}z}\right)^{\frac{\gamma+1}{p}+1}\\&-\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(a)))}\right)\left(\frac{1-|\phi(a)|^2}{1-\overline{\phi(a)}z}\right)^{\frac{\gamma+1}{p}+2}.\end{aligned}$$

By a simple calculation, we have

$$h(\phi(a))=h''(\phi(a))=0$$

and

$$h'(\phi(a))=\frac{p}{\gamma+1+p}\frac{\overline{\phi(a)}}{1-|\phi(a)|^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(a)))}\right).$$

Also by Lemma 4, we get $h \in A_\omega^{\Psi_p}$ and $\|h\|^{lux}_{A_\omega^{\Psi_p}} \lesssim 1$. From these facts and the boundedness of $uC_\phi : A_\omega^{\Psi_p} \to \mathscr{Z}_\beta$, we obtain

$$\begin{aligned}&\frac{|2u'(a)\phi'(a)+u(a)\phi''(a)|(1-|a|^2)^\beta|\phi(a)|}{1-|\phi(a)|^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(a)))}\right)\\ \leqslant&\|uC_\phi h\|_{\mathscr{Z}_\beta} \leqslant C\|uC_\phi\|,\end{aligned}$$

which implies that

$$I_2 = \sup_{z\in\mathbb{D}} \frac{|2u'(z)\phi'(z)+u(z)\phi''(z)|(1-|z|^2)^\beta|\phi(z)|}{1-|\phi(z)|^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z)))}\right) \leqslant C\|uC_\phi\|. \tag{10}$$

Let $\delta \in (0,1)$ be fixed. From (9) it follows that

$$\begin{aligned}&\sup_{\{z:|\phi(z)|\leqslant\delta\}} \frac{|2u'(z)\phi'(z)+u(z)\phi''(z)|(1-|z|^2)^\beta}{1-|\phi(z)|^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z)))}\right)\\ \leqslant&\frac{I_1}{1-\delta^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z)))}\right) \leqslant C\|uC_\phi\|,\end{aligned} \tag{11}$$

and from (10) it follows that

$$\begin{aligned}&\sup_{\{z:|\phi(z)|>\delta\}} \frac{|2u'(z)\phi'(z)+u(z)\phi''(z)|(1-|z|^2)^\beta}{1-|\phi(z)|^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z)))}\right)\\ \leqslant&\frac{I_2}{\delta} \leqslant C\|uC_\phi\|.\end{aligned} \tag{12}$$

Therefore, from (11) and (12), we get

$$K_2 = \sup_{z\in\mathbb{D}} \frac{|2u'(z)\phi'(z)+u(z)\phi''(z)|(1-|z|^2)^\beta}{1-|\phi(z)|^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z)))}\right) \leqslant C\|uC_\phi\| < \infty. \tag{13}$$

Finally, we prove $K_3 < \infty$. Taking the function $f(z) = z^2$, we obtain

$$\begin{aligned}&\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u''(z)\phi(z)^2+4u'(z)\phi(z)\phi'(z)+2u(z)\phi'(z)^2+2u(z)\phi(z)\phi''(z)|\\ \leqslant&\|uC_\phi z^2\|_{\mathscr{Z}_\beta} \leqslant C\|uC_\phi\|.\end{aligned} \tag{14}$$

By (7) and the boundedness of ϕ, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u''(z)||\phi(z)|^2 \leqslant C\|uC_\phi\|. \tag{15}$$

From (9), (15) and the boundedness of ϕ, we obtain

$$J_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)||\phi'(z)|^2 \leqslant C\|uC_\phi\|. \tag{16}$$

For $a \in \mathbb{D}$, set

$$\begin{aligned} k(z) = &-\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(a)))}\right)\left(\frac{1-|\phi(a)|^2}{1-\overline{\phi(a)}z}\right)^{\frac{\gamma+1}{p}} \\ &+2\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(a)))}\right)\left(\frac{1-|\phi(a)|^2}{1-\overline{\phi(a)}z}\right)^{\frac{\gamma+1}{p}+1} \\ &-\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(a)))}\right)\left(\frac{1-|\phi(a)|^2}{1-\overline{\phi(a)}z}\right)^{\frac{\gamma+1}{p}+2}. \end{aligned}$$

After a calculation, it follows that

$$k(\phi(a)) = k'(\phi(a)) = 0$$

and

$$k''(\phi(a)) = -2\frac{\overline{\phi(a)}^2}{(1-|\phi(a)|^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(a)))}\right).$$

Also by Lemma 4, we get $k \in A_\omega^{\Psi_p}$ and $\|k\|_{A_\omega^{\Psi_p}}^{lux} \lesssim 1$. From these facts and the boundedness of the operator $uC_\phi : A_\omega^{\Psi_p} \to \mathscr{Z}_\beta$, we obtain

$$\frac{|u(a)||\phi'(a)|^2(1-|a|^2)^\beta|\phi(a)|^2}{(1-|\phi(a)|^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(a)))}\right) \leqslant \|uC_\phi k\|_{\mathscr{Z}_\beta} \leqslant C\|uC_\phi\|.$$

This yields

$$J_2 = \sup_{z\in\mathbb{D}} \frac{|u(z)||\phi'(z)|^2(1-|z|^2)^\beta|\phi(z)|^2}{(1-|\phi(z)|^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z)))}\right) \leqslant C\|uC_\phi\|. \tag{17}$$

Therefore, for a fixed $\delta \in (0,1)$, by (16)

$$\begin{aligned} &\sup_{\{z:|\phi(z)|\leqslant\delta\}} \frac{|u(z)||\phi'(z)|^2(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z)))}\right) \\ &\leqslant \frac{J_1}{1-\delta^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z)))}\right) \leqslant C\|uC_\phi\|, \end{aligned} \tag{18}$$

and by (17), we obtain

$$\sup_{\{z:|\phi(z)|>\delta\}} \frac{|u(z)||\phi'(z)|^2(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z)))}\right) \leqslant \frac{J_2}{\delta^2} \leqslant C\|uC_\phi\|. \tag{19}$$

Hence, from (18) and (19), it follows that

$$K_3 = \sup_{z\in\mathbb{D}} \frac{|u(z)||\phi'(z)|^2(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z)))}\right) \leqslant C\|uC_\phi\| < \infty. \tag{20}$$

Sufficiency. By Lemma 3, for all $f \in A_\omega^{\Psi_p}$ we have

$$\begin{aligned}
\|uC_\phi f\|_{\mathscr{Z}_\beta} =& |(uf\circ\phi)(0)| + |(uf\circ\phi)'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(uf\circ\phi)''(z)| \\
\leqslant& |(uf\circ\phi)(0)| + |(uf\circ\phi)'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u''(z)||f(\phi(z))| \\
&+ \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |2u'(z)\phi'(z)+u(z)\phi''(z)||f'(\phi(z))| \qquad (21)\\
&+ \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u(z)||\phi'(z)|^2|f''(\phi(z))| \\
\leqslant& C(1+K_1+K_2+K_3)\|f\|^{lux}_{A_\omega^{\Psi_p}}.
\end{aligned}$$

From the assumed conditions and (21), it follows that $uC_\phi : A_\omega^{\Psi_p} \to \mathscr{Z}_\beta$ is bounded. Moreover,

$$\|uC_\phi\| \lesssim 1+K_1+K_2+K_3. \qquad (22)$$

Since the operator $uC_\phi : A_\omega^{\Psi_p} \to \mathscr{Z}_\beta$ is nonzero, we have $\|uC_\phi\| > 0$. From this, we can find a positive constant C such that $1 \leqslant C\|uC_\psi\|$. This means that

$$1 \lesssim \|uC_\phi\|. \qquad (23)$$

Since the operator $uC_\phi : A_\omega^{\Psi_p} \to \mathscr{Z}_\beta$ is metrically bounded, from (6), (13) and (20), we obtain

$$K_1+K_2+K_3 \lesssim \|uC_\phi\|. \qquad (24)$$

Therefore, combining (23) and (24) gives

$$1+K_1+K_2+K_3 \lesssim \|uC_\phi\|. \qquad (25)$$

Hence, from (22) and (25), we obtain $\|uC_\phi\| \asymp 1+K_1+K_2+K_3$. The proof is complete. □

THEOREM 2. *Let $p \geqslant 1$, $\omega \in \hat{\mathscr{D}}$, $\beta > 0$ and $\Psi \in \mathscr{U}^s$ be such that $\Psi_p \in \mathscr{L}_q$. Let $u \in H(\mathbb{D})$ and ϕ be an analytic self-map of $\mathbb{D}$. The operator $uC_\phi : A_\omega^{\Psi_p} \to \mathscr{Z}_\beta$ is metrically compact if and only if functions u and ϕ satisfy the following conditions:*

$$\begin{aligned}
&N_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u''(z)| < \infty, \\
&N_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |2u'(z)\phi'(z)+u(z)\phi''(z)| < \infty, \\
&N_3 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u(z)||\phi'(z)|^2 < \infty, \\
&\lim_{|\phi(z)|\to 1}(1-|z|^2)^\beta |u''(z)|\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(z)))}\right) = 0,
\end{aligned}$$

$$\lim_{|\phi(z)|\to 1} \frac{(1-|z|^2)^{\beta}|2u'(z)\phi'(z)+u(z)\phi''(z)|}{1-|\phi(z)|^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z)))}\right)=0,$$
$$\lim_{|\phi(z)|\to 1} \frac{(1-|z|^2)^{\beta}|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z)))}\right)=0.$$

Proof. Necessity. By the assumption that the operator $uC_\phi : A_\omega^{\Psi_p} \to \mathscr{Z}_\beta$ is metrically bounded and the proof of Theorem 1, we obtain $N_1, N_2, N_3 < \infty$.

Let $\{\phi(z_m)\}_{m\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{m\to\infty}|\phi(z_m)|=1$. If such a sequence does not exist, then the result obviously holds. Using this sequence, we define

$$\begin{aligned} g_m(z) = &-\frac{\gamma+1+2p}{\gamma+1}\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(z_m)))}\right)\left(\frac{1-|\phi(z_m)|^2}{1-\overline{\phi(z_m)}z}\right)^{\frac{\gamma+1}{p}} \\ &+\frac{2(\gamma+1+2p)}{\gamma+1+p}\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(z_m)))}\right)\left(\frac{1-|\phi(z_m)|^2}{1-\overline{\phi(z_m)}z}\right)^{\frac{\gamma+1}{p}+1} \\ &-\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(z_m)))}\right)\left(\frac{1-|\phi(z_m)|^2}{1-\overline{\phi(z_m)}z}\right)^{\frac{\gamma+1}{p}+2}. \end{aligned}$$

By Lemma 4, we know that the sequence $\{g_m\}_{m\in\mathbb{N}}$ is uniformly bounded in $A_\omega^{\Psi_p}$. Moreover, it is easy to check that the sequence $\{g_m\}_{m\in\mathbb{N}}$ uniformly converges to zero on any compact subset of $\mathbb{D}$ as $m\to\infty$. Therefore, we have $\lim_{m\to\infty}\|uC_\phi g_m\|_{\mathscr{Z}_\beta}=0$ by Lemma 5. From this, (4) and (5), we obtain

$$\lim_{m\to\infty}(1-|z_m|^2)^{\beta}|u''(z_m)|\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(z_m)))}\right)=0.$$

Next, we set

$$\begin{aligned} h_m(z) = &-\frac{\gamma+1+2p}{\gamma+1+p}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z_m)))}\right)\left(\frac{1-|\phi(z_m)|^2}{1-\overline{\phi(z_m)}z}\right)^{\frac{\gamma+1}{p}} \\ &+\frac{2\gamma+2+3p}{\gamma+1+p}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z_m)))}\right)\left(\frac{1-|\phi(z_m)|^2}{1-\overline{\phi(z_m)}z}\right)^{\frac{\gamma+1}{p}+1} \\ &-\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z_m)))}\right)\left(\frac{1-|\phi(z_m)|^2}{1-\overline{\phi(z_m)}z}\right)^{\frac{\gamma+1}{p}+2}. \end{aligned}$$

Similar to the above proof, we have

$$\lim_{m\to\infty}\frac{(1-|z_m|^2)^{\beta}|2u'(z_m)\phi'(z_m)+u(z_m)\phi''(z_m)|}{1-|\phi(z_m)|^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z_m)))}\right)=0.$$

Also, by defining

$$\begin{aligned}k_m(z) = &-\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z_m)))}\right)\left(\frac{1-|\phi(z_m)|^2}{1-\overline{\phi(z_m)}z}\right)^{\frac{\gamma+1}{p}}\\&+2\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z_m)))}\right)\left(\frac{1-|\phi(z_m)|^2}{1-\overline{\phi(z_m)}z}\right)^{\frac{\gamma+1}{p}+1}\\&-\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z_m)))}\right)\left(\frac{1-|\phi(z_m)|^2}{1-\overline{\phi(z_m)}z}\right)^{\frac{\gamma+1}{p}+2},\end{aligned}$$

we obtain

$$\lim_{m\to\infty}\frac{(1-|z_m|^2)^\beta|u(z_m)||\phi'(z_m)|^2}{(1-|\phi(z_m)|^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z_m)))}\right)=0.$$

The last three limits imply the desired result.

Sufficiency. We first show that $uC_\phi : A_\omega^{\Psi_P} \to \mathscr{Z}_\beta$ is metrically bounded. For this we observe that the assumed conditions imply that for any given $\varepsilon > 0$, there is an $\eta \in (0,1)$ such that

$$H_1(z) := (1-|z|^2)^\beta|u''(z)|\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(z)))}\right) < \varepsilon, \tag{26}$$

$$H_2(z) := \frac{(1-|z|^2)^\beta|2u'(z)\phi'(z)+u(z)\phi''(z)|}{1-|\phi(z)|^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z)))}\right) < \varepsilon, \tag{27}$$

and

$$H_3(z) := \frac{(1-|z|^2)^\beta|u(z)||\phi'(z)|^2}{(1-|\phi(z)|^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z)))}\right) < \varepsilon \tag{28}$$

for $z \in K = \{z \in \mathbb{D} : |\phi(z)| > \eta\}$. From $N_1 < \infty$ and (26), we get

$$\begin{aligned}K_1 = \sup_{z\in\mathbb{D}} H_1(z) &\leqslant \sup_{z\in\mathbb{D}\setminus K} H_1(z) + \sup_{z\in K} H_1(z)\\&\leqslant N_1\Psi_p^{-1}\left(\frac{D_1}{\omega(S(\phi(z)))}\right)+\varepsilon.\end{aligned} \tag{29}$$

From $N_2 < \infty$ and (27), we have

$$\begin{aligned}K_2 = \sup_{z\in\mathbb{D}} H_2(z) &\leqslant \sup_{z\in\mathbb{D}\setminus K} H_2(z) + \sup_{z\in K} H_2(z)\\&\leqslant \frac{N_2}{1-\eta^2}\Psi_p^{-1}\left(\frac{D_2}{\omega(S(\phi(z)))}\right)+\varepsilon.\end{aligned} \tag{30}$$

From $N_3 < \infty$ and (28), we obtain

$$\begin{aligned} K_3 = \sup_{z\in\mathbb{D}} H_3(z) &\leqslant \sup_{z\in\mathbb{D}\setminus K} H_3(z) + \sup_{z\in K} H_3(z) \\ &\leqslant \frac{N_3}{(1-\eta^2)^2}\Psi_p^{-1}\left(\frac{D_3}{\omega(S(\phi(z)))}\right) + \varepsilon. \end{aligned} \tag{31}$$

Hence, the operator $uC_\phi : A_\omega^{\Psi_p} \to \mathscr{Z}_\beta$ is metrically bounded by Theorem 1.

To prove that the operator $uC_\phi : A_\omega^{\Psi_p} \to \mathscr{Z}_\beta$ is metrically compact, by Lemma 5, we only need to prove that, if $\{f_m\}_{m\in\mathbb{N}}$ is a sequence in $A_\omega^{\Psi_p}$ such that $\|f_m\|^{lux}_{A_\omega^{\Psi_p}} \lesssim 1$ and $f_m \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $m\to\infty$, then

$$\lim_{m\to\infty} \|uC_\phi f_m\|_{\mathscr{Z}_\beta} = 0.$$

For the above $\varepsilon > 0$ and η, by using Lemma 3, we get

$$\begin{aligned} &\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(uf_m\circ\phi)''(z)| \\ \leqslant &\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u''(z)||f_m(\phi(z))| \\ &+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |2u'(z)\phi'(z)+u(z)\phi''(z)||f_m'(\phi(z))| \\ &+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u(z)||\phi'(z)|^2|f_m''(\phi(z))| \\ \leqslant &\sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)^\beta |u''(z)||f_m(\phi(z))| + \sup_{z\in K}(1-|z|^2)^\beta |u''(z)||f_m(\phi(z))| \\ &+\sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)^\beta |2u'(z)\phi'(z)+u(z)\phi''(z)||f_m'(\phi(z))| \\ &+\sup_{z\in K}(1-|z|^2)^\beta |2u'(z)\phi'(z)+u(z)\phi''(z)||f_m'(\phi(z))| \\ &+\sup_{z\in\mathbb{D}\setminus K}(1-|z|^2)^\beta |u(z)||\phi'(z)|^2|f_m''(\phi(z))| \\ &+\sup_{z\in K}(1-|z|^2)^\beta |u(z)||\phi'(z)|^2|f_m''(\phi(z))| \\ \leqslant &N_1 \sup_{\{z:|z|\leqslant\eta\}} |f_m(z)| + C\sup_{z\in K} H_1(z) + N_2 \sup_{\{z:|z|\leqslant\eta\}} |f_m'(z)| + C\sup_{z\in K} H_2(z) \\ &+N_3 \sup_{\{z:|z|\leqslant\eta\}} |f_m''(z)| + C\sup_{z\in K} H_3(z) \\ \leqslant &N_1 \sup_{\{z:|z|\leqslant\eta\}} |f_m(z)| + N_2 \sup_{\{z:|z|\leqslant\eta\}} |f_m'(z)| + N_3 \sup_{\{z:|z|\leqslant\eta\}} |f_m''(z)| + 3C\varepsilon. \end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned}
&\|uC_\phi f_m\|_{\mathscr{Z}_\beta}\\
\leqslant&|(uf_m\circ\phi)(0)|+|(uf_m\circ\phi)'(0)|+N_1\sup_{\{z:|z|\leqslant\eta\}}|f_m(z)|+N_2\sup_{\{z:|z|\leqslant\eta\}}|f_m'(z)|\\
&+N_3\sup_{\{z:|z|\leqslant\eta\}}|f_m''(z)|+3C\varepsilon\\
\leqslant&|u(0)f_m(\phi(0))|+|u'(0)f_m(\phi(0))+u(0)f_m'(\phi(0))\phi'(0)|+N_1\sup_{\{z:|z|\leqslant\eta\}}|f_m(z)|\\
&+N_2\sup_{\{z:|z|\leqslant\eta\}}|f_m'(z)|+N_3\sup_{\{z:|z|\leqslant\eta\}}|f_m''(z)|+3C\varepsilon.
\end{aligned}\tag{33}$$

It is easy to see that, when $\{f_m\}_{m\in\mathbb{N}}$ uniformly converges to zero on any compact subset of $\mathbb{D}$, then $\{f_m\}_{m\in\mathbb{N}}$, $\{f_m'\}_{m\in\mathbb{N}}$, $\{f_m''\}_{m\in\mathbb{N}}$ also do as $m\to\infty$. Since $\{z:|z|\leqslant\eta\}$ and $\{\phi(0)\}$ are compact subsets of $\mathbb{D}$, letting $m\to\infty$ in (33) gives

$$\lim_{m\to\infty}\|uC_\phi f_m\|_{\mathscr{Z}_\beta}=0.$$

By Lemma 5, it follows that the operator $uC_\phi : A_\omega^{\Psi_p}\to\mathscr{Z}_\beta$ is metrically compact. The proof is complete. □

Data Availability. No data were used to support this study.

Conflicts of Interest. The authors declare that they have no conflicts of interest.

Acknowledgements. The work is supported by Guangdong Basic and Applied Basic Research Foundation (no. 2023A1515010614). The authors thank the referee for his/her numerous helpful suggestions and comments which led to the improvement of the original manuscript of this paper.

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(Received October 5, 2023)

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