Mathematical
Inequalities
& Applications
Volume 27, Number 3 (2024), 647–658 doi:10.7153/mia-2024-27-44

INEQUALITIES FOR DIAGONALLY DOMINANT MATRICES

VINAYAK GUPTA, GARGI LATHER * AND R. BALAJI

(*Communicated by I. Perić*)

Abstract. Let $A = (a_{ij})$ and $H = (h_{ij})$ be positive semidefinite matrices of the same order. If $a_{ij} \geqslant |h_{ij}|$ for all i, j; A is diagonally dominant and all row sums of H are equal to zero, then we show that the sum of all $k \times k$ principal minors of A is greater than or equal to the sum of all $k \times k$ principal minors of H.

1. Introduction

This paper is inspired by the following result in [1].

THEOREM 1. *Let G be a connected graph on n vertices. If $L(G)$ and $|L(G)|$ are the Laplacian and the signless Laplacian matrices of G, then the sum of all $k \times k$ principal minors of $|L(G)|$ is greater than or equal to the sum of all $k \times k$ principal minors of $L(G)$.*

Let $c_j(A)$ denote the sum of all $j \times j$ principal minors of an $n \times n$ matrix A. Then the characteristic polynomial of A can be written as

$$t^n - c_1(A)t^{n-1} + \cdots + (-1)^n c_n(A).$$

Now, the above result says that

$$c_j(L(G)) \leqslant c_j(|L(G)|) \quad j = 1, \dots c, n.$$

Theorem 1 has significance in deriving certain powerful inequalities connecting the eigenvalues of $L(G)$ and $|L(G)|$. These eigenvalue inequalities follow immediately from a remarkable result of Efroymson, Swartz and Wendroff [4] on elementary symmetric functions which in particular says the following.

Mathematics subject classification (2020): 15A45.
Keywords and phrases: Diagonally dominant matrix, weighted Laplacian, signless Laplacian and incidence matrix, Cauchy-Binet formula.
* Corresponding author.

Paper MIA-27-44

THEOREM 2. *Let $n \in \mathbb{N}$, $k \in \{1,\dots,n\}$ and $0 < \alpha \leqslant 1$. For each non-negative vector $x = (x_1,\dots,x_n)'$, define*

$$f_1(x) = \sum_{i=1}^{n} x_i \qquad s_1(x) = \sum_{i=1}^{n} x_i^{\alpha}$$

$$f_2(x) = \sum_{i<j} x_i x_j \qquad s_2(x) = \sum_{i<j} (x_i x_j)^{\alpha}$$

$$\vdots$$

$$f_n(x) = \prod_{j=1}^{n} x_j \qquad s_n(x) = \prod_{j=1}^{n} x_j^{\alpha}.$$

If p and q are non-negative vectors such that

$$f_k(p) \leqslant f_k(q) \quad k = 1,\dots,n,$$

then

$$s_k(p) \leqslant s_k(q) \quad k = 1,\dots,n.$$

In this paper, we obtain a significantly broader result on diagonally dominant matrices encompassing Theorem 1 as a special case.

THEOREM 3. *Let $A = (a_{ij})$ and $H = (h_{ij})$ be $n \times n$ positive semidefinite matrices. If $a_{ij} \geqslant |h_{ij}|$ for all i, j; A is diagonally dominant and all row sums of H are equal to zero, then*

$$c_k(A) \geqslant c_k(H) \quad k = 1,\dots,n.$$

In other words, if

$$t^n - \alpha_1 t^{n-1} + \dots + (-1)^n \alpha_n \ \text{ and } \ t^n - \beta_1 t^{n-1} + \dots + (-1)^n \beta_n$$

are the characteristic polynomials of A and H respectively, then

$$\alpha_k \geqslant \beta_k \quad k = 1,\dots,n.$$

We prove Theorem 3 by employing arguments similar to those in Theorem 1 with necessary modifications, and utilizing standard matrix-theoretic techniques. The consequences of Theorem 3 are discussed in Section 5.

2. Preliminaries

We shall use the following notations and definitions.

(a) We consider only simple graphs. The vertices of a graph G with n vertices will be labelled $1,\dots,n$ and each edge will be denoted by (i, j), where $i < j$. We use $V(G)$ and $E(G)$ to denote the set of all vertices and edges of G.

(b) Let G be a graph with n vertices and m edges. If each edge $e \in E(G)$ is assigned some positive number $w(e)$, then we say that G is weighted and $w(e)$ is the weight of e. The weighted incidence vector of the edge $e = (i, j)$ is defined by $q(e) := (q_1(e), \ldots c, q_n(e))'$, where

$$q_k(e) := \begin{cases} \sqrt{w(e)} & k = i \\ -\sqrt{w(e)} & k = j \\ 0 & \text{otherwise.} \end{cases}$$

The weighted incidence matrix of G is now constructed by arranging its weighted incidence vectors as distinct columns. If $F \subseteq E(G)$, then $\langle F \rangle$ will denote the subgraph of G with vertex set containing all the end vertices of edges in F and $E(\langle F \rangle) = F$. As usual, K_n will be the complete graph on n vertices.

(c) Let $A = (a_{ij})$ be an $n \times n$ matrix with real entries. We say that A is diagonally dominant if

$$|a_{ii}| \geqslant \sum_{\{j:j \neq i\}} |a_{ij}| \quad i = 1, \ldots c, n.$$

If U is an $n \times n$ matrix, then $c_k(U)$ will denote the sum of all $k \times k$ principal minors of U. If U is symmetric, then we denote and arrange its eigenvalues by

$$\lambda_1(U) \geqslant \cdots \geqslant \lambda_n(U).$$

Let $B = (b_{ij})$ be an $n \times m$ matrix. Then, $|B|$ will denote the matrix $(|b_{ij}|)$. If $T \subseteq \{1, \ldots c, n\}$ and $S \subseteq \{1, \ldots c, m\}$, then $B[T, S]$ will denote the submatrix of B obtained by selecting the rows corresponding to T and the columns corresponding to S.

3. Intermediate results

To prove our main result, we need a weighted version of Theorem 7.4 in [3]. Let G be a weighted graph with n vertices. For non-empty subsets $X \subseteq V(G)$ and $Y \subseteq E(G)$ such that $|X| = |Y|$, we say that the pair (X, Y) has property $(*)$, if the following conditions hold:

(i) Every vertex in X is incident with at least one edge in Y.

(ii) Every component of $\langle Y \rangle$ is a tree.

(iii) If T is a component of $\langle Y \rangle$, then $V(T) \smallsetminus X$ contains exactly one vertex.

We now have the following lemma.

LEMMA 1. *Suppose (X, Y) has property $(*)$. Let T be a component of $\langle Y \rangle$, $\{b\} := V(T) \smallsetminus X$ and $e := (a, b) \in E(T)$. Then, $(X \smallsetminus \{a\}, Y \smallsetminus \{e\})$ will have property $(*)$.*

Proof. Let $T_1,\dots c,T_m$ be the components of $\langle Y\rangle$ and $T := T_1$. Define $\Omega_1 := X \smallsetminus \{a\}$ and $\Omega_2 := Y \smallsetminus \{e\}$. We claim that the pair (Ω_1,Ω_2) satisfy (i), (ii) and (iii). Let $v \in \Omega_1$. To show that v is incident with an edge in Ω_2, it suffices to show that v is not incident with $e = (a,b)$. If this happens, then $v = b$. However, since $b \notin X$, we conclude that $v \notin \Omega_1$. This contradicts $v \in \Omega_1$. Thus, (i) holds. While (ii) follows immediately, to show (iii), we consider the following possibilities.

(I) If both a and b are pendant vertices, then T_1 contains only one edge (a,b). Thus, the components of $\langle\Omega_2\rangle$ are precisely $T_2,\dots,T_m$.

(II) If both a and b are not pendant, then the components of $\langle\Omega_2\rangle$ are $R_a,R_b,T_2,\dots c,T_m$, where R_a and R_b are subtrees of $\langle\Omega_2\rangle$ containing vertices a and b respectively. Moreover,
$$V(R_a)\smallsetminus\Omega_1 = \{a\} \text{ and } V(R_b)\smallsetminus\Omega_1 = \{b\}.$$

(III) If a is pendant and b is not pendant, then the components of $\langle\Omega_2\rangle$ are $R_b,T_2,\dots c,T_m$, where R_b is the subtree of $\langle\Omega_2\rangle$ containing b and $V(R_b)\smallsetminus\Omega_1 = \{b\}$.

(IV) If b is pendant and a is not pendant, then the components of $\langle\Omega_2\rangle$ are $R_a,T_2,\dots c,T_m$, where R_a is the subtree of $\langle\Omega_2\rangle$ containing a and $V(R_a)\smallsetminus\Omega_1 = \{a\}$.

Since
$$V(T_i)\smallsetminus\Omega_1 = V(T_i)\smallsetminus X \quad i = 2,\dots c,m,$$
and each $V(T_i)\smallsetminus X$ has exactly one vertex, (iii) is satisfied. The proof is complete. □

LEMMA 2. *Let Q be a weighted incidence matrix of G. Then, $Q[X,Y]$ is non-singular if and only if (X,Y) has property $(*)$.*

Proof. Suppose $Q[X,Y]$ is non-singular. Items (i) and (iii) of property $(*)$ are proven in the same way as [3, Theorem 7.4]. To prove (ii), consider a component of $\langle Y\rangle$. Suppose this component contains a cycle $C_m = \{v_1,e_1,v_2,e_2,\dots,e_{m-1},v_m,e_m,v_1\}$, where $1 \leqslant v_1 < \cdots < v_m \leqslant n$ are vertices and $e_1,\dots c,e_m$ are edges. Define
$$(z_1,\dots c,z_m) := \left(\frac{1}{\sqrt{w(e_1)}},\dots c,\frac{1}{\sqrt{w(e_{m-1})}},-\frac{1}{\sqrt{w(e_m)}}\right)'.$$
It is easy to verify that
$$\sum_{j=1}^{m} z_j q(e_j) = 0.$$
Thus, $q(e_1),\dots c,q(e_m)$ are linearly dependent and hence the columns of $Q[X,Y]$ are linearly dependent, which is a contradiction to our assumption. The necessary condition is proved.

Now, consider two non-empty subsets $X \subseteq V(G)$ and $Y \subseteq E(G)$ such that $|X| = |Y|$ and (X,Y) has property $(*)$. Let $\alpha := |X| = |Y|$. By induction on α, we show that $Q[X,Y]$ is non-singular. Suppose $\alpha = 1$. In view of (i), $X = \{a\}$ and $Y = \{e\}$, where

e is incident with the vertex a. In this case, the conclusion follows immediately as $Q[X,Y] = (\pm\sqrt{w(e)})$. Suppose the result is true for all X and Y with $\alpha < k$. Assume, $\alpha = k$. Let $T_1,\ldots,T_m$ be the components of $\langle Y \rangle$. Since we can relabel, by (iii), we may assume $V(T_1) \setminus X = \{2\}$, $e := (1,2) \in E(T_1)$ and $q(e)$ is the first column of $Q[X,Y]$. Define $\Omega_1 := X \setminus \{1\}$ and $\Omega_2 := Y \setminus \{e\}$. As $q(e) = (\sqrt{w(e)}, 0, \ldots c, 0)'$,

$$\det Q[X,Y] = \sqrt{w(e)} \det Q[\Omega_1, \Omega_2]. \tag{3.1}$$

By Lemma 1, (Ω_1, Ω_2) has property $(*)$. Induction hypothesis now implies that $Q[\Omega_1, \Omega_2]$ is non-singular and so is $Q[X,Y]$ by (3.1). The proof is complete. □

LEMMA 3. *Let $X \subseteq V(G)$ and $Y \subseteq E(G)$ be such that $|X| = |Y|$. If $M = |Q|$, where Q is a weighted incidence matrix of G, then*

$$(\det M[X,Y])^2 \geqslant (\det Q[X,Y])^2.$$

Proof. It suffices to show the result when $Q[X,Y]$ is non-singular. In view of previous lemma, (X,Y) will have property $(*)$. Let $\alpha := |X| = |Y|$. We prove by induction on α. If $\alpha = 1$, then by item (i) of property $(*)$, $X = \{a\}$ and $Y = \{e\}$, where e is incident with a; hence

$$Q[X,Y] = (\pm\sqrt{w(e)}) \text{ and } M[X,Y] = (\sqrt{w(e)}).$$

The inequality holds here. Assuming the result for all $\alpha < k$, we now prove for $\alpha = k$. By item (ii) of property $(*)$, all the components of $\langle Y \rangle$ are trees. Let T be a component of $\langle Y \rangle$. Item (iii) of property $(*)$ implies that $V(T) \setminus X$ contains precisely one vertex and let this be equal to $\{r\}$. In T, let r be adjacent to $s \in X$. Put $e := (r,s)$, $u := Q[X,\{e\}]$, $\Omega_1 := X \setminus \{s\}$ and $\Omega_2 := Y \setminus \{e\}$. As

$$u_v = \begin{cases} -\sqrt{w(e)} & v = s \\ 0 & v \in X \setminus \{s\}, \end{cases}$$

we see that

$$(\det Q[X,Y])^2 = w(e)(\det Q[\Omega_1, \Omega_2])^2. \tag{3.2}$$

Similarly,

$$(\det M[X,Y])^2 = w(e)(\det M[\Omega_1, \Omega_2])^2. \tag{3.3}$$

In view of Lemma 1, (Ω_1, Ω_2) has property $(*)$. By induction hypothesis,

$$(\det Q[\Omega_1, \Omega_2])^2 \leqslant (\det M[\Omega_1, \Omega_2])^2. \tag{3.4}$$

Now (3.2), (3.3) and (3.4) imply $(\det Q[X,Y])^2 \leqslant (\det M[X,Y])^2$. The proof is complete. □

REMARK 1. A parallel step of Lemma 3 in [1] uses a result of Poincaré [3, Proposition 5.3] which asserts that every square submatrix of an incidence matrix has determinant equal to 0 or ± 1. This property does not extend to weighted incidence matrices. Hence, the proof of Lemma 3 is completed by an induction argument.

The previous lemmas imply the following.

LEMMA 4. *Let $S = (s_{ij})$ be an $n \times n$ symmetric matrix such that all off-diagonal entries are negative and the row sums are all equal to zero. Then*

$$c_k(|S|) \geqslant c_k(S) \quad k = 1, \dots, n.$$

Proof. Consider the complete graph K_n with the edge set $E(K_n) := \{(i,j) : 1 \leqslant i < j \leqslant n\}$. To each edge $e := (i,j) \in E(K_n)$, assign the weight $w(e) := |s_{ij}|$. Then, $S = QQ'$, where each column of Q is a weighted incidence vector of some edge in K_n. Corresponding to an edge $e = (i,j)$, define $M(e) = (p_1, \dots c, p_n)'$ where

$$p_k := \begin{cases} \sqrt{w(e)} & k = i, j \\ 0 & \text{otherwise.} \end{cases}$$

Then, $|S| = MM'$, where each column of M is given by $M(f)$ for some $f \in E(K_n)$. For $1 \leqslant k \leqslant n$, define

$$\Omega := \{X \subseteq V(K_n) : |X| = k\} \ \text{ and } \ \Delta := \{Y \subseteq E(K_n) : |Y| = k\}.$$

As $c_k(S)$ is the sum of all $k \times k$ principal minors of S, we have

$$c_k(S) = \sum_{X \in \Omega} \det S[X,X].$$

Since $S = QQ'$, by Cauchy-Binet formula,

$$c_k(S) = \sum_{X \in \Omega, Y \in \Delta} (\det Q[X,Y])^2.$$

Similarly,

$$c_k(|S|) = \sum_{X \in \Omega, Y \in \Delta} (\det M[X,Y])^2.$$

In view of Lemma 3, if $X \in \Omega$ and $Y \in \Delta$, then

$$(\det M[X,Y])^2 \geqslant (\det Q[X,Y])^2.$$

Therefore, $c_k(|S|) \geqslant c_k(S)$. □

The following result is well-known (see [5, Corollary 4.3.12]).

THEOREM 4. *Let A and B be $n \times n$ symmetric matrices. If B is positive semidefinite, then*

$$\lambda_k(A) \leqslant \lambda_k(A+B) \quad k = 1, \dots c, n.$$

4. Main result

We recall the main result that needs to be proved.

THEOREM 5. *Let $A = (a_{ij})$ and $H = (h_{ij})$ be $n \times n$ positive semidefinite matrices. If A is diagonally dominant, all row sums of H are equal to zero, and*

$$a_{ij} \geqslant |h_{ij}| \quad i, j = 1, \ldots c, n$$

then

$$c_k(A) \geqslant c_k(H) \quad k = 1, \ldots c, n.$$

Proof. We first prove the result by assuming that all off-diagonal entries of H are non-zero. Define $L := (l_{ij})$, where

$$l_{ij} = \begin{cases} -|h_{ij}| & i \neq j \\ \sum\limits_{\{k:k\neq i\}} |h_{ik}| & i = j. \end{cases}$$

We begin by proving that

$$c_k(A) \geqslant c_k(|L|) \quad k = 1, \ldots c, n.$$

Let $W := A - |L|$ with $(i, j)^{\text{th}}$ entry equal to w_{ij}. Since $a_{ij} \geqslant |h_{ij}|$, it follows that $a_{ij} > 0$, $a_{ij} \geqslant |l_{ij}|$ and therefore, $w_{ij} \geqslant 0$ for all i, j. As $A = (a_{ij})$ is diagonally dominant with positive entries,

$$a_{ii} - \sum_{\{j:j\neq i\}} a_{ij} \geqslant 0 \quad i = 1, \ldots c, n.$$

Since $a_{ij} = w_{ij} + |l_{ij}|$ for all i, j,

$$l_{ii} + w_{ii} - \sum_{\{j:i\neq j\}} (|l_{ij}| + w_{ij}) \geqslant 0 \quad i = 1, \ldots c, n.$$

Because $l_{ii} = \sum\limits_{\{j:i\neq j\}} |l_{ij}|$, from the above inequality, we get

$$w_{ii} - \sum_{\{j:i\neq j\}} w_{ij} \geqslant 0 \quad i = 1, \ldots c, n.$$

Moreover, each $w_{ij} \geqslant 0$. So, W is diagonally dominant. To this end, we have $A = |L| + W$, where $|L|$ and W are diagonally dominant. Since diagonally dominant matrices with non-negative diagonal entries are positive semidefinite, $|L|$ and W are positive semidefinite. By Theorem 4,

$$\lambda_j(A) \geqslant \lambda_j(|L|) \quad j = 1, \ldots c, n. \tag{4.1}$$

We recall that if S is an $n\times n$ matrix, then

$$\begin{aligned} c_1(S) &= \sum_{j=1}^{n}\lambda_j(S)\\ c_2(S) &= \sum_{i<j}\lambda_i(S)\lambda_j(S)\\ &\vdots\\ c_n(S) &= \prod_{j=1}^{n}\lambda_j(S). \end{aligned} \tag{4.2}$$

Since A and L are positive semidefinite, (4.1) and (4.2) imply $c_k(A)\geqslant c_k(|L|)$ and hence by Lemma 4, we get $c_k(A)\geqslant c_k(L)$. Now, we show that $c_k(L)\geqslant c_k(H)$. Define $B:=L-H$. All row sums of H and L are zero. Hence, each row sum of B is zero. Since the off-diagonal entries of B are non-positive, we see that the diagonal entries of B are non-negative and B is diagonally dominant. Therefore, B is positive semidefinite. Applying Theorem 4 to $L=H+B$, we have $\lambda_j(L)\geqslant\lambda_j(H)$ for all $j=1,\dots c,n$. Since L and H are positive semidefinite, by (4.2), it now follows that $c_k(L)\geqslant c_k(H)$, and therefore, $c_k(A)\geqslant c_k(H)$.

Suppose some off-diagonal entries of H are zero. For each $m\in\mathbb{N}$, define

$$\beta_{ij}^{(m)}:=\begin{cases} -\frac{1}{m} & l_{ij}=0 \text{ and } i\neq j\\ 0 & l_{ij}\neq 0 \text{ and } i\neq j\\ \sum\limits_{\{k:i\neq k\}}|\beta_{ik}^{(m)}| & i=j, \end{cases}$$

$A_m=(a_{ij}^{(m)}):=(a_{ij}+|\beta_{ij}^{(m)}|)$ and $H_m=(h_{ij}^{(m)}):=(h_{ij}+\beta_{ij}^{(m)})$. Then, A_m is diagonally dominant, sum of all the entries in any row of H_m is zero, each off-diagonal entry of H_m is negative and $a_{ij}^{(m)}\geqslant|h_{ij}^{(m)}|$. Therefore, $c_k(A_m)\geqslant c_k(H_m)$ for all $m\in\mathbb{N}$. By continuity, $c_k(A)\geqslant c_k(H)$. The proof is complete. □

EXAMPLE 1. In general, positive semidefinite matrices are not diagonally dominant. The conclusion of Theorem 5 does not hold if A is only assumed to be positive semidefinite. For example, if

$$A=\begin{bmatrix}3&1&2&1\\1&13&4&8\\2&4&6&1\\1&8&1&10\end{bmatrix}\quad\text{and}\quad H=\begin{bmatrix}3&-1&-1&-1\\-1&13&-4&-8\\-1&-4&6&-1\\-1&-8&-1&10\end{bmatrix},$$

then $c_2(A)=268$, whereas $c_2(H)=271$.

5. Corollary

The following is an immediate consequence of Theorem 2 and Theorem 5.

COROLLARY 1. *Let* $A=(a_{ij})$ *and* $L=(l_{ij})$ *be* $n\times n$ *positive semidefinite matrices. Suppose* A *is diagonally dominant, all row sums of* L *are equal to zero and* $a_{ij}\geqslant |l_{ij}|$ *for all* i,j. *Let* $x_i:=\lambda_i(A)$ *and* $y_i:=\lambda_i(L)$. *Then, for any* $\alpha\in(0,1]$,

$$\sum_{i=1}^{n} x_i^{\alpha} \geqslant \sum_{i=1}^{n} y_i^{\alpha}$$

$$\sum_{i<j}(x_ix_j)^{\alpha} \geqslant \sum_{i<j}(y_iy_j)^{\alpha}$$

$$\vdots$$

$$\prod_{j=1}^{n} x_j^{\alpha} \geqslant \prod_{j=1}^{n} y_j^{\alpha}.$$

Let G be a weighted graph on n vertices with weight w_{ij} on the edge (i,j). Then the weighted Laplacian matrix $L(G)=(l_{ij})$ is the $n\times n$ symmetric matrix such that

$$l_{ij}=\begin{cases} -w_{ij} & \text{if } i\neq j \text{ and } (i,j)\in E(G) \\ 0 & \text{if } i\neq j \text{ and } (i,j)\notin E(G) \\ \sum\limits_{\{s:s\neq i\}} w_{is} & i=j \end{cases}$$

The weighted signless Laplacian matrix is $|L(G)|$. The following result extends Theorem 1 to the weighted case.

COROLLARY 2. *If* $L:=L(G)$ *is an* $n\times n$ *weighted Laplacian matrix of* G, *then*

$$c_k(|L|)\geqslant c_k(L) \quad k=1,\dots c,n.$$

In particular, if $a_i:=\lambda_i(L)$ *and* $b_i:=\lambda_i(|L|)$, *then for any* $\alpha\in(0,1]$,

$$\sum_{i=1}^{n} a_i^{\alpha} \leqslant \sum_{i=1}^{n} b_i^{\alpha}$$

$$\sum_{i<j}(a_ia_j)^{\alpha} \leqslant \sum_{i<j}(b_ib_j)^{\alpha}$$

$$\vdots$$

$$\prod_{j=1}^{n} a_j^{\alpha} \leqslant \prod_{j=1}^{n} b_j^{\alpha}.$$

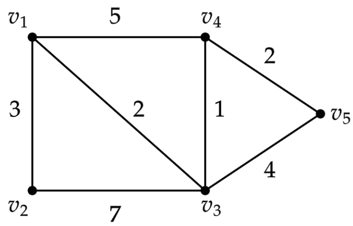

Figure 1: G

EXAMPLE 2. To illustrate Corollary 2, consider G. The weighted Laplacian is

$$L = \begin{bmatrix} 10 & -3 & -2 & -5 & 0 \\ -3 & 10 & -7 & 0 & 0 \\ -2 & -7 & 14 & -1 & -4 \\ -5 & 0 & -1 & 8 & -2 \\ 0 & 0 & -4 & -2 & 6 \end{bmatrix}.$$

The characteristic polynomials of L and $|L|$ are respectively,

$t^5 - 48t^4 + 796t^3 - 5348t^2 + 12520t$ and $t^5 - 48t^4 + 796t^3 - 5588t^2 + 16152t - 16064$.

Setting $\alpha = \frac{1}{2}$ in the previous corollary, we note the following inequalities:

$$\sum_{i=1}^{5} \sqrt{a_i} < 14 < \sum_{i=1}^{5} \sqrt{b_i}, \quad \sum_{i<j} \sqrt{a_i a_j} < 67 < \sum_{i<j} \sqrt{b_i b_j},$$

$$\sum_{i<j<k} \sqrt{a_i a_j a_k} < 143 < \sum_{i<j<k} \sqrt{b_i b_j b_k}$$

and

$$\sum_{i<j<k<l} \sqrt{a_i a_j a_k a_l} < 112 < \sum_{i<j<k<l} \sqrt{b_i b_j b_k b_l}.$$

Let G be a connected graph on n vertices. The distance between any two vertices i and j is the length of the shortest path between them in G. Let this be d_{ij}. Then, $D(G) = (d_{ij})$ is the distance matrix of G. The distance Laplacian matrix $D_L(G) := (\theta_{ij})$ is the $n \times n$ symmetric matrix such that

$$\theta_{ij} = \begin{cases} -d_{ij} & \text{if } i \neq j \\ \sum_{s=1}^{n} d_{is} & i = j. \end{cases}$$

The signless distance Laplacian matrix is then $|D_L(G)|$. Distance Laplacian matrices are introduced in [2]. We have the following result on distance Laplacians.

COROLLARY 3. *If G is a connected graph on n vertices, then*

$$c_k(|D_L(G)|) \geqslant c_k(D_L(G)) \quad k = 1, \ldots c, n.$$

In particular, if $p_i := \lambda_i(D_L(G))$ and $q_i := \lambda_i(|D_L(G)|)$, then for any $\alpha \in (0,1]$,

$$\sum_{i=1}^{n} p_i^{\alpha} \leqslant \sum_{i=1}^{n} q_i^{\alpha}$$
$$\sum_{i<j} (p_i p_j)^{\alpha} \leqslant \sum_{i<j} (q_i q_j)^{\alpha}$$
$$\vdots$$
$$\prod_{j=1}^{n} p_j^{\alpha} \leqslant \prod_{j=1}^{n} q_j^{\alpha}.$$

Proof. To each edge (i,j) of the complete graph K_n on n vertices, assign the weight d_{ij}, which is the distance between i and j in G. The weighted Laplacian of K_n is then $D_L(G)$. By the previous corollary, we get the desired inequalities. □

EXAMPLE 3. Consider G in Figure 1. The distance Laplacian matrix is then

$$D_L(G) = \begin{bmatrix} 5 & -1 & -1 & -1 & -2 \\ -1 & 6 & -1 & -2 & -2 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -2 & -1 & 5 & -1 \\ -2 & -2 & -1 & -1 & 6 \end{bmatrix}.$$

Then,

$$t^5 - 26t^4 + 250t^3 - 1054t^2 + 1645t \text{ and } t^5 - 26t^4 + 250t^3 - 1138t^2 + 2485t - 2100$$

are the characteristic polynomials of $D_L(G)$ and $|D_L(G)|$, respectively. Setting $\alpha = \frac{1}{2}$, we note the following inequalities:

$$\sum_{i=1}^{5} \sqrt{p_i} < 11 < \sum_{i=1}^{5} \sqrt{q_i}, \quad \sum_{i<j} \sqrt{p_i p_j} < 39 < \sum_{i<j} \sqrt{q_i q_j},$$

$$\sum_{i<j<k} \sqrt{p_i p_j p_k} < 65 < \sum_{i<j<k} \sqrt{q_i q_j q_k}$$

and

$$\sum_{i<j<k<l} \sqrt{p_i p_j p_k p_l} < 41 < \sum_{i<j<k<l} \sqrt{q_i q_j q_k q_l}.$$

REFERENCES

[1] S. AKBARI, E. GHORBANI, J. H. KOOLEN, AND M. R. OBOUDI, *A relation between the Laplacian and signless Laplacian eigenvalues of a graph*, J. Algebraic Comb., **32**, (2010), 459–464.
[2] M. AOUCHICHE AND P. HANSEN, *Two Laplacians for the distance matrix of a graph*, Linear Algebra Appl., **439**, (2013), 21–33.
[3] N. BIGGS, *Algebraic Graph Theory*, Cambridge University Press, New York, 1993.
[4] G. A. EFROYMSON, B. SWARTZ, AND B. WENDROFF, *A new inequality for symmetric functions*, Adv. Math., **38**, (1980), 109–127.
[5] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 2013.

(Received October 6, 2023)

Vinayak Gupta
Department of Mathematics
IIT Madras, Chennai-600036
e-mail: vinayakgupta1729v@gmail.com

Gargi Lather
Department of Mathematics
IIT Madras, Chennai-600036
e-mail: gargilather@gmail.com

R. Balaji
Department of Mathematics
IIT Madras, Chennai-600036
e-mail: balaji5@smail.iitm.ac.in