Mathematical
Inequalities
& Applications
Volume 27, Number 3 (2024), 659–667 doi:10.7153/mia-2024-27-45

ON THE CALCULATIONS OF THE GENERALIZED VON NEUMANN–JORDAN CONSTANTS $C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu)$ AND $\tilde{C}_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu))$

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(*Communicated by S. Varošanec*)

Abstract. In this paper, we calculate $C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu))$ and $\tilde{C}_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu))$, where (Ω,Σ,μ) is a measure space and $r > 1$.

1. Introduction

Let X be a real normed space with the unit ball $B_X = \{x \in X : \|x\| \leqslant 1\}$ and the unit sphere $S_X = \{x \in X : \|x\| = 1\}$.

It is known that geometric constants play an important role in the description of various geometric structures of Banach spaces. In recent years, many constants in Banach spaces have been defined and studied. The *von Neumann-Jordan constant*,

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \ : \ x,y \in X \text{ not both zero} \right\},$$

was introduced by Clarkson [4] in 1937 and studied intensively by many authors [3, 7, 8, 9, 12, 13, 14]. In [6], as a generalization of the von Neumann-Jordan constant, a new geometric constant called the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ was introduced:

Let $p \in [1,\infty)$.

$$\begin{aligned} C_{NJ}^{(p)}(X) &= \sup\left\{ \frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p+\|y\|^p)} : x,y \in X \text{ not both zero} \right\} \\ &= \sup\left\{ \frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} : x,y \in S_X,\ 0 \leqslant t \leqslant 1 \right\}. \end{aligned} \tag{1}$$

THEOREM 1. ([2, Lemma 2.2.]) *Let X be a Banach space, let $p > 1$ and let p' denotes the conjugate index satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$\left(\frac{C_{NJ}^{(p)}(X)}{2} \right)^{\frac{1}{p}} = \left(\frac{C_{NJ}^{(p')}(X^*)}{2} \right)^{\frac{1}{p'}}.$$

Mathematics subject classification (2020): 46B20, 46E30.
Keywords and phrases: Generalized von Neumann Jordan constant, Lebesgue space.

Paper MIA-27-45

Motivated by constants $C'_{NJ}(X)$ and $C_{NJ}^{(p)}(X)$, the authors in [14] defined the following constant:

$$\tilde{C}_{NJ}^{(p)}(X) = \sup\left\{ \frac{\|x+y\|^p + \|x-y\|^p}{2^p} : x,y \in S_X \right\}.$$

In [5] the authors introduced the constant $A_{2,p}(X)$ for a Banach space X and for $1 \leqslant p < \infty$ as follows:

$$A_{2,p}(X) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} : x,y \in X,\ \|x\|^p + \|y\|^P \leqslant 2 \right\}.$$

They showed that for each $1 \leqslant p < \infty$ [5, Proposition 2.5]

$$\begin{aligned} A_{2,p}(X) &= \sup\left\{ \frac{\|x+ty\| + \|x-ty\|}{2^{1-\frac{1}{p}}(1+t^p)^{\frac{1}{p}}} : x,y \in S_X,\ 0 \leqslant t \leqslant 1 \right\} \\ &= \sup\left\{ \frac{\rho_X(t)+1}{2^{-\frac{1}{p}}(1+t^p)^{\frac{1}{p}}} : x,y \in S_X,\ 0 \leqslant t \leqslant 1 \right\}, \end{aligned} \tag{2}$$

where the *modulus of smoothness* of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{ \frac{\left(\|x+ty\| + \|x-ty\|\right)}{2} - 1 : x,y \in S_X \right\}.$$

In [6], the authors calculated $C_{NJ}^{(p)}(L_r[0,1])$ for $1 < r \leqslant 2$. In this paper, we calculate $C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu))$ and $\tilde{C}_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu))$, where (Ω,Σ,μ) is an arbitrary measure space and $r > 1$.

2. The calculation of $C_{NJ}^{(p)}(L_r)$

In the following result, we calculate $C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu))$, where (Ω,Σ,μ) is an arbitrary measure space and $r > 1$, which improves the main result of [6].

THEOREM 2. *Let (Ω,Σ,μ) be a measure space, let $r > 1$ and let $X = L_r(\Omega,\Sigma,\mu)$.*

(i) If $1 < r \leqslant 2$ then,

$$C_{NJ}^{(p)}(X) = \begin{cases} 2^{2-p}, & 1 < p \leqslant r \\ 2^{1+\frac{p}{r}-p}, & r < p \leqslant r' \\ 1, & r' < p. \end{cases} \tag{3}$$

(ii) If $r > 2$ then,

$$C_{NJ}^{(p)}(X) = \begin{cases} 1, & 1 < r \leqslant p \\ 2^{1+\frac{p}{r'}-p}, & r' < p < r \\ 2^{2-p}, & p < r'. \end{cases} \tag{4}$$

Proof. (i). We first show that for $1 < r \leqslant 2$, we have

$$C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu)) \leqslant \begin{cases} 2^{2-p}, & 1 < p \leqslant r \\ 2^{1+\frac{p}{r}-p}, & r < p \leqslant r' \\ 1, & r' < p \end{cases} \tag{5}$$

In virtue of Remark 2.3 from [10], for any $x,y \in S_X$, and any $0 \leqslant t \leqslant 1$, we have

$$\|x+ty\|_r^p + \|x-ty\|_r^p \leqslant \begin{cases} 2(1+t^r)^{\frac{p}{r}}, & 1 \leqslant p \leqslant r' \\ 2^{\frac{p}{r'}}(1+t^r)^{\frac{p}{r}}, & r' < p \end{cases} \tag{6}$$

and so, from the definition of $C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu))$, we have

$$\begin{aligned} C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu)) &= \sup\left\{ \frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} : x,y \in S_X, 0 \leqslant t \leqslant 1 \right\} \\ &\leqslant \begin{cases} 2^{2-p}\sup_{t\in[0,1]} \frac{(1+t^r)^{\frac{p}{r}}}{1+t^p}, & 1 < p \leqslant r' \\ 2^{\frac{p}{r'}-p+1}\sup_{t\in[0,1]} \frac{(1+t^r)^{\frac{p}{r}}}{1+t^p}, & r' < p \end{cases} \\ &= \begin{cases} 2^{2-p}, & 1 < p \leqslant r \\ 2^{1+\frac{p}{r}-p}, & r < p \leqslant r' \\ 1, & r' < p \end{cases} \end{aligned} \tag{7}$$

which proves (5).

To prove the converse inequality of (5), we first show that there exist $A_0, B_0 \in \Sigma$ such that

$$A_0 \cap B_0 = \emptyset \quad \text{and} \quad 0 < \mu(A_0) < \infty, \;\; 0 < \mu(B_0) < \infty.$$

There are two cases:

Case (i). There exist a nonnegative $0 \neq f \in L_r$, and two disjoint bounded half open intervals $I,J \subseteq (0,\infty)$ such that $\mu(f^{-1}(I)) > 0$ and $\mu(f^{-1}(J)) > 0$. In this case, let $A_0 = f^{-1}(I)$ and $B_0 = f^{-1}(J)$. It is clear that $\mu(A_0) > 0$ and $\mu(B_0) > 0$. To show that $\mu(A_0) < \infty$, let $I = (a,b]$ and note that

$$\mu(A_0) = \frac{1}{a^r}\int_{A_0} a^r d\mu \leqslant \frac{1}{a^r}\int_{\Omega} (f(x))^r d\mu \leqslant \frac{1}{a^r}\|f\|_r^r < \infty.$$

Similarly, we can prove that $\mu(B_0) < \infty$.

Case (ii). For each nonnegative $0 \neq f \in L_r$ and each two disjoint bounded intervals $I,J \subseteq (0,\infty)$, we have either $\mu(f^{-1}(I)) = 0$ or $\mu(f^{-1}(J)) = 0$. Since

$$0 < \mu(f^{-1}(0,\infty)) = \mu\left(f^{-1}\left(\cup_{n=-\infty}^{\infty} I_n\right)\right) = \sum_{n=-\infty}^{\infty} \mu(f^{-1}(I_n)),$$

where

$$I_n = \begin{cases} (2^{n-1}, 2^n], & n \leqslant 0 \\ (n, n+1], & n \geqslant 1. \end{cases}$$

for each $n \in \mathbb{Z}$, then there exists a unique $n_0 \in \mathbb{Z}$ such that $\mu(f^{-1}(I_{n_0})) > 0$. Let $J_0 = I_{n_0} = (a_0, b_0]$, then

$$0 < \mu(f^{-1}(I_{n_0})) = \mu\left(f^{-1}\left(\left(a_0, \frac{a_0+b_0}{2}\right] \cup \left(\frac{a_0+b_0}{2}, b_0\right]\right)\right)$$
$$= \mu\left(f^{-1}\left(\left(a_0, \frac{a_0+b_0}{2}\right]\right)\right) + \mu\left(f^{-1}\left(\left(\frac{a_0+b_0}{2}, b_0\right]\right)\right),$$

and so $\mu(f^{-1}(J_1)) > 0$, where J_1 is either $(a_0, \frac{a_0+b_0}{2}]$ or $(\frac{a_0+b_0}{2}, b_0]$ but not both of them. Proceeding this manner, we find a sequence $\{J_n\}_{n=0}^{\infty}$ of half open intervals such that

$$J_0 \supseteq J_1 \supseteq \dots J_n \supseteq \dots,$$
$$\mu(f^{-1}(0,\infty)) = \mu(f^{-1}(J_n)) > 0, \text{ and } \mu(f^{-1}(J_n^c)) = 0, \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

Since $l(J_n) \to 0$, where $l(J_n)$ denotes the length of J_n, then either $\cap_{n=1}^{\infty} J_n = \emptyset$ or $\cap_{n=1}^{\infty} J_n$ is singleton (note that $\cap_{n=1}^{\infty} J_n$ is an interval). Suppose in the contrary, $\cap_{n=1}^{\infty} J_n = \emptyset$. Then

$$\emptyset = f^{-1}\left(\cap_{n=1}^{\infty} J_n\right) = \cap_{n=1}^{\infty} f^{-1}(J_n),$$

and so

$$0 = \mu(\cap_{n=1}^{\infty} J_n) = \lim_n \mu(f^{-1}(J_n)) = \mu(f^{-1}(0,\infty)),$$

a contradiction and so $\cap_{n=1}^{\infty} J_n$ is singleton. Let $\cap_{n=1}^{\infty} J_n = \{c\}$, then $f = c\chi_A$, where $A = f^{-1}(c)$. Now let $\{f, g\}$ be two nonnegative linearly independent functions in L_r. Then by the above argument, $g = d\chi_B$. Thus $\{\chi_A, \chi_B\}$ is a linearly independent set and so, either $\mu(A \setminus B) > 0$ or $\mu(B \setminus A) > 0$. Without loss of generality, we may assume that $\mu(A \setminus B) > 0$. Let $B_0 = B$ and $A_0 = A \setminus B$. Now, define $x_0, y_0 : \Omega \to \mathbb{R}$ by $x_0 = \frac{1}{(\mu(A_0))^{\frac{1}{r}}} \chi_{A_0}$ and $y_0 = \frac{1}{(\mu(B_0))^{\frac{1}{r}}} \chi_{B_0}$. Then $\|x_0\|_r = \|y_0\|_r = 1$, and so for each $t \in [0,1]$

$$C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu)) \geqslant \sup_{t\in[0,1]} \frac{\|x_0 + ty_0\|_r^p + \|x_0 - ty_0\|_r^p}{2^{p-1}(1+t^p)} \tag{8}$$
$$= \sup\left\{\frac{2(1+t^r)^{\frac{p}{r}}}{2^{p-1}(1+t^p)} : 0 \leqslant t \leqslant 1\right\}$$
$$= \begin{cases} 2^{2-p}, & 1 < p \leqslant r \\ 2^{1+\frac{p}{r}-p}, & r < p \leqslant r' \end{cases}$$

Therefore (3) is proved.

(ii). Since $r > 2$, then $1 < r' < 2$, and so by Theorem 1.1, we have (note that $X^* = C_{NJ}^{(p)}(L_{r'}(\Omega,\Sigma,\mu))$

$$\left(\frac{C_{NJ}^{(p)}(X)}{2}\right)^{\frac{1}{p}} = \left(\frac{C_{NJ}^{(p')}(X^*)}{2}\right)^{\frac{1}{p'}}$$

$$\left(\frac{C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu))}{2}\right)^{\frac{1}{p}} = \left(\frac{C_{NJ}^{(p')}(L_{r'}(\Omega,\Sigma,\mu))}{2}\right)^{\frac{1}{p'}}$$

hence thanks to (3), we obtain

$$\begin{aligned} C_{NJ}^{(p)}(L_r(\Omega,\Sigma,\mu)) &= 2^{1-\frac{p}{p'}}\left(C_{NJ}^{(p')}(L_{r'}(\Omega,\Sigma,\mu))\right)^{\frac{p}{p'}} \\ &= \begin{cases} 1, & 1 < r \leqslant p \\ 2^{1+\frac{p}{r'}-p}, & r' < p < r \\ 2^{2-p}, & p < r'. \end{cases} \end{aligned} \tag{9}$$

□

3. The calculation of $\tilde{C}_{NJ}^{(p)}(L_r)$

We begin with the following improvement of Theorem 2.8 in [1].

THEOREM 3. *Let X be a Banach space and suppose that $p > 1$. Then*

$$\tilde{C}_{NJ}^{(p)}(X) = 2^{1-p}(A_{2,p'}(X^*))^p.$$

Proof. If $0 \leqslant \beta \leqslant \alpha$, then

$$\max_{t\in[0,1]} \frac{\alpha+\beta t}{(1+t^p)^{1/p}} = \left(\alpha^{p'}+\beta^{p'}\right)^{\frac{1}{p'}}. \tag{10}$$

From the above, we have

$$\begin{aligned} \tilde{C}_{NJ}^{(p)}(X)^{\frac{1}{p}} &= \frac{1}{2} \sup_{\substack{x,y\in S_X: \\ \|x-y\|\leqslant\|x+y\|}} \left(\|x+y\|^p+\|x-y\|^p\right)^{\frac{1}{p}} \\ &= \frac{1}{2}\sup_{x,y\in S_X}\sup_{0\leqslant t\leqslant 1} \frac{\|x+y\|+t\|x-y\|}{(1+t^{p'})^{\frac{1}{p'}}} \\ &= \frac{1}{2}\sup_{x,y\in S_X}\sup_{0\leqslant t\leqslant 1}\sup_{x^*,y^*\in S_{X^*}} \frac{x^*(x+y)+ty^*(x-y)}{(1+t^{p'})^{\frac{1}{p'}}} \\ &= \frac{1}{2}\sup_{x^*,y^*\in S_{X^*}}\sup_{0\leqslant t\leqslant 1}\sup_{x,y\in S_X} \frac{(x^*+ty^*)(x)+(x^*-ty^*)(y)}{(1+t^{p'})^{\frac{1}{p'}}} \\ &= \frac{1}{2}\sup_{\substack{x^*,y^*\in S_X \\ 0\leqslant t\leqslant 1}} \frac{\|x^*+ty^*\|+\|x^*-ty^*\|}{(1+t^{p'})^{\frac{1}{p'}}} \\ &= 2^{\frac{-1}{p'}}A_{2,p'}(X^*) \end{aligned}$$

which proves the result. □

THEOREM 4. *Let $X = L_r(\Omega,\Sigma,\mu)$, where (Ω,Σ,μ) is an arbitrary measure space, and let $r > 1$. Then*

$$\tilde{C}_{NJ}^{(p)}(X) = \begin{cases} 2^{1+\frac{p}{r}-p}, & 1 < r < 2,\ \ p \leqslant r', \\ 2^{1+\frac{p}{r'}-p}, & 2 \leqslant r,\ \ p \leqslant r, \\ 1, & \textit{otherwise.} \end{cases} \tag{11}$$

Proof. We recall that, for each $t \in [0,1]$ [11]

$$\rho_X(t) = \begin{cases} (1+t^r)^{\frac{1}{r}} - 1, & 1 \leqslant r \leqslant 2, \\ (\frac{(1+t)^r+(1-t)^r}{2})^{\frac{1}{r}} - 1, & 2 < r. \end{cases} \tag{12}$$

Then from (2) and (12), and for $1 \leqslant r \leqslant 2$, we have

$$A_{2,p}(L_r) = \sup_{t\in[0,1]} \frac{2^{\frac{1}{p}}(1+t^r)^{\frac{1}{r}}}{(1+t^p)^{\frac{1}{p}}} = \begin{cases} 2^{\frac{1}{r}}, & 1 \leqslant r \leqslant p, \\ 2^{\frac{1}{p}}, & p < r. \end{cases}$$

Now, assume that $2 < r$. Then by (2) and (12), we get

$$A_{2,p}(L_r) = 2^{\frac{1}{p}-\frac{1}{r}} \sup_{t\in[0,1]} \frac{\left((1+t)^r + (1-t)^r\right)^{\frac{1}{r}}}{(1+t^p)^{\frac{1}{p}}}. \tag{13}$$

Define the function $f(t) = \frac{\left((1+t)^r+(1-t)^r\right)^{\frac{1}{r}}}{(1+t^p)^{\frac{1}{p}}}$ for $t \in [0,1]$. We first assume that $2 \leqslant p$. Since the function $t \to \frac{1+t}{1-t}$ is increasing on $[0,1)$, $2 \leqslant p$ and $2 < r$ then

$$\left(\frac{1+t}{1-t}\right)^{r-1} \geqslant \frac{1+t}{1-t} \geqslant \frac{1+t^{p-1}}{1-t^{p-1}},$$

and so

$$f'(t) = \frac{((1+t)^r+(1-t)^r)^{\frac{1}{r}-1}(1-t^{p-1})(1-t)^{r-1}}{(1+t^p)^{\frac{1}{p}+1}} \left(\left(\frac{1+t}{1-t}\right)^{r-1} - \frac{1+t^{p-1}}{1-t^{p-1}}\right) \geqslant 0.$$

Thus $\sup_{t\in[0,1]} f(t) = f(1)$ and so

$$A_{2,p}(L_r) = 2^{\frac{1}{r'}},$$

where $2 \leqslant p$ and $2 < r$.

Now, suppose that $1 < p < 2$ and $2 < r$. We will show that

(i) $f'(t) < 0$ for $0 < t < 1$ sufficiently close to 0,

(ii) $f'(t) > 0$ for $0 < t < 1$ sufficiently close to 1, and

(iii) the equation $(\frac{1+t}{1-t})^{r-1} - \frac{1+t^{p-1}}{1-t^{p-1}} = 0$ has only one root in $(0,1)$

To prove (i), let $0 < r_0 < \frac{1}{r-1}$. Since $1 < p < 2$, we have $\lim_{t\to 0^+} \frac{\frac{1+t}{1-t}-1}{\frac{1+t^{p-1}}{1-t^{p-1}}-1} = 0$ and so

$$\frac{1+t}{1-t} < 1 - r_0 + r_0\left(\frac{1+t^{p-1}}{1-t^{p-1}}\right),$$

for sufficiently small $0 < t < 1$.

Now, let $k(z) = (1-r_0+r_0 z)^{r-1} - z : [0,\infty) \to \mathbb{R}$. Then by our assumption $k'(1) = r_0(r-1)-1 < 0$ and so

$$k(z) = (1-r_0+r_0 z)^{r-1} - z < k(1) = 0 \text{ for } 1 < z \text{ sufficiently close to } 1.$$

Then from the above and for sufficiently small $0 < t$

$$\left(\frac{1+t}{1-t}\right)^{r-1} - \frac{1+t^{p-1}}{1-t^{p-1}} < \left(1-r_0+r_0\left(\frac{1+t^{p-1}}{1-t^{p-1}}\right)\right)^{r-1} - \frac{1+t^{p-1}}{1-t^{p-1}} < 0,$$

and so $f'(t) < 0$ for sufficiently small $0 < t$.

To prove (ii), note that

$$\lim_{t\to 1^-} \frac{(\frac{1+t}{1-t})^{r-1}}{\frac{1+t^{p-1}}{1-t^{p-1}}} = +\infty.$$

Hence

$$\lim_{t\to 1^-}\left(\left(\frac{1+t}{1-t}\right)^{r-1} - \frac{1+t^{p-1}}{1-t^{p-1}}\right) = +\infty,$$

and so $f'(t) > 0$ for $0 < t < 1$ sufficiently close to 1.

To prove (iii), we show equivalently that the function

$$g(t) = (r-1)\ln\left(\frac{1+t}{1-t}\right) - \ln\left(\frac{1+t^{p-1}}{1-t^{p-1}}\right) : [0,1) \to \mathbb{R}$$

has only one root in $[0,1)$. On the contrary, assume that g has more than one root in $[0,1)$ then the equation (note that by the above $g(t) > 0$, $0 < t < 1$ sufficiently close to 1)

$$g'(t) = \frac{2(r-1)}{1-t^2} - \frac{2(p-1)t^{p-2}}{1-t^{2p-2}} = 0$$

has at least two roots in $(0,1)$. Thus the equation

$$h(t) = \frac{t^{p-2}-t^p}{1-t^{2p-2}} = \frac{r-1}{p-1}$$

has at least two roots in $(0,1)$. We will show that

$$h'(t) = \frac{t^{p-3}\left((p-2)-pt^2+pt^{2p-2}+(2-p)t^{2p}\right)}{(1-t^{2p-2})^2} < 0 \text{ for } 0 < t < 1,$$

a contradiction.

Let $t^2 = s$, $\beta(s) = (p-2) - ps + ps^{p-1} + (2-p)s^p$. Then we show that $\beta(s) < 0$ for $0 < s < 1$

$$\beta'(s) = -p + p(p-1)s^{p-2} + p(2-p)s^{p-1} for \ \ 0 < s < 1,$$

and so

$$\begin{aligned}\beta''(s) &= p(p-1)(p-2)s^{p-3} + p(2-p)(p-1)s^{p-2}\\ &= p(2-p)(p-1)(s^{p-2} - s^{p-3}) < 0, \quad 0 < s < 1.\end{aligned}$$

Hence $\beta'(s)$ is decreasing and $\beta'(1) = -p + p(p-1) + p(2-p) = 0$. So $\beta'(s) > 0$, $\beta(s)$ is increasing and $\beta(1) = p-2-p+p+p(2-p) = 0$. hence $\beta(s) < 0$, $0 < s < 1$ and we have

$$h'(t) = \frac{t^{p-3}\beta(s)}{(1-t^{2p-2})^2} < 0 \quad \text{for} \quad 0 < t < 1, \ 0 < s < 1.$$

From (i), (ii), and (iii), we obtain

$$\sup_{t\in[0,1]} f(t) = \max\{f(0), f(1)\} = \max\{2^{\frac{1}{r}}, 2^{\frac{1}{p'}}\},$$

and so for $2 < r$ and $1 < p < 2$, we have

$$A_{2,p}(L_r) = \begin{cases} 2^{1-\frac{1}{r}}, & p' \leqslant r,\\ 2^{\frac{1}{p}}, & r < p'.\end{cases}$$

Summarizing all the above,

$$A_{2,p}(L_r) = \begin{cases} 2^{\frac{1}{r}}, & 1 \leqslant r \leqslant 2, \ r \leqslant p,\\ 2^{\frac{1}{p}}, & 1 \leqslant r \leqslant 2, \ p < r,\\ 2^{1-\frac{1}{r}}, & 2 < r, \ p' \leqslant r,\\ 2^{\frac{1}{p}}, & 2 < r, \ r < p'.\end{cases} \tag{14}$$

Now from Theorem 3,

$$\begin{aligned}\tilde{C}_{NJ}^{(p)}(X) &= 2^{1-p}(A_{2,p'}(X^*))^p \qquad (15)\\ &= 2^{1-p}(A_{2,p'}(L_{r'}))^p = \begin{cases} 2^{1+\frac{p}{r'}-p}, & 1 \leqslant r' \leqslant 2, \ r' \leqslant p',\\ 1, & 1 \leqslant r' \leqslant 2, \ p' < r',\\ 2^{1-\frac{p}{r'}}, & 2 < r', \ p \leqslant r',\\ 1, & 2 < r', \ r' < p\\ 1, & \text{otherwise.}\end{cases}\end{aligned}$$

$$= \begin{cases} 2^{1+\frac{p}{r'}-p}, & 1 \leqslant r' \leqslant 2, \ \ r' \leqslant p', \\ 2^{1-\frac{p}{r'}}, & 2 < r', \ \ p \leqslant r', \\ 1, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 2^{1+\frac{p}{r}-p}, & 1 < r < 2, \ \ p \leqslant r', \\ 2^{1+\frac{p}{r'}-p}, & 2 \leqslant r, \ \ p \leqslant r, \\ 1, & \text{otherwise.} \end{cases} \qquad \square$$

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(Received October 7, 2023)

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