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PARTITIONING BOUNDED SETS IN SYMMETRIC SPACES INTO SUBSETS WITH REDUCED DIAMETER

XINLING ZHANG* AND CHAN HE

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Abstract. Borsuk's problem on partitioning bounded sets into sets having smaller diameters is considered. For each positive integer m and each n-dimensional Banach space X, let $\beta(X,m)$ be the infimum of $\delta \in (0,1]$ such that each bounded set $A \subseteq X$ with diameter 1 can be partitioned into m subsets whose diameters are at most δ. With the help of characterizations of complete sets in ℓ_1^3, we prove that $\beta(\ell_1^3, 8) \leqslant 0.75$. By using the stability of $\beta(X,m)$ with respect to X in the sense of Banach-Mazur metric and estimations of the Banach-Mazur distance between ℓ_p^n and ℓ_q^n, we show that $\beta(\ell_p^3, 8) \leqslant 0.88185$ holds for each $p \in [1,\infty]$. This improves a recent result of Y. Lian and S. Wu. Furthermore, we prove that $\beta(X, 2^3) < 1$ when X is a three-dimensional Banach space symmetric with the natural basis $\{e_i \mid i \in [3]\}$ and satisfies $\alpha(X) = \left\| \sum_{i\in[3]} e_i \right\| > 9/4$.

1. Introduction

Let $X = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional Banach space with *origin* o and *unit ball* B_X, and $A \subseteq X$ be a nonempty bounded set. Denote by $\delta(A) := \sup\{\|x-y\| \mid x,y \in A\}$ the *diameter* of A. If $x \notin A \Rightarrow \delta(A \cup \{x\}) > \delta(A)$, then A is said to be *complete*. If A^C is a complete set with diameter $\delta(A)$ and containing A, then it is called a *completion* of A. Note that A^C is not unique in general. A compact convex set having interior points is called a *convex body*. Let $\mathbb{E}^n$ be the n-dimensional Euclidean space. Put $[n] := \{i \in \mathbb{Z}^+ \mid 1 \leqslant i \leqslant n\}$, $\forall n \in \mathbb{Z}^+$.

In 1933, K. Borsuk (cf. [3]) proposed the following problem:

PROBLEM 1. (Borsuk's Problem). Is it true that every nonempty bounded set $A \subseteq \mathbb{E}^n$ can be divided into $n+1$ subsets $A_1, A_2, \cdots, A_{n+1}$ such that $\delta(A_i) < \delta(A)$, $\forall i \in [n+1]$?

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* Corresponding author.

For $n = 2$, the affirmative answer was provided by K. Borsuk (cf. [3]). For $n = 3$, the affirmative answer was given by J. Perkal (cf. [10]) and H. G. Eggleston (cf. [4]). For $n \geqslant 64$, the answer is negative (cf. [2], [5], and [6]). Up to now, the answer is not clear for $4 \leqslant n \leqslant 63$. See [14] for more information on this problem and a quantitative program to attack it.

It is natural to consider Borsuk's problem in a finite dimensional Banach space X. For a bounded set $A \subseteq X$, let $b_X(A)$ be the smallest positive integer m such that A can be divided into m subsets whose diameters are strictly smaller than $\delta(A)$. We refer to the book [1] for more information about Borsuk's problem.

For a real number $p \geqslant 1$, ℓ_p^n denotes the space $(\mathbb{R}^n, \|\cdot\|_p)$, where the p-norm $\|\cdot\|_p$ of $x = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ is defined by

$$\|x\|_p = \|(\alpha_1, \cdots, \alpha_n)\|_p = \left(\sum_{i\in[n]} |\alpha_i|^p\right)^{\frac{1}{p}}, \quad \forall p \in [1, \infty),$$

and

$$\|x\|_\infty = \|(\alpha_1, \cdots, \alpha_n)\|_\infty = \max_{i\in[n]} |\alpha_i|.$$

Denote by B_p^n the unit ball of ℓ_p^n. I.e., $B_p^n = \{x \in \mathbb{R}^n \mid \|x\|_p \leqslant 1\}$.

In 2009, L. Yu and C. Zong proved that $b_{\ell_p^3}(A) \leqslant 2^3$ holds for every nonempty bounded set A and each $p \in [1, \infty]$ (cf. [12]). In 2021, Y. Lian and S. Wu studied Borsuk's partition problem in finite dimensional Banach spaces by estimating

$$\beta_X(A, m) = \inf\left\{\frac{1}{\delta(A)} \max\{\delta(A_k) \mid k \in [m]\} \mid A = \bigcup_{k=1}^{m} A_k\right\}$$

for $A \in \mathscr{B}^n$, and

$$\beta(X, m) = \sup\{\beta_X(A, m) \mid A \in \mathscr{B}^n\},$$

where $\mathscr{B}^n = \{A \subseteq X \mid A \text{ is bounded and } \delta(A) > 0\}$. They showed that (cf. [7])

$$\beta(\ell_p^3, 8) \leqslant 0.925, \quad \forall p \in [1, \infty].$$

Later, L. Zhang, L. Meng, and S. Wu (cf. [13]) improved this estimation by showing that

$$\beta(\ell_p^3, 8) \leqslant 0.9, \quad \forall p \in [1, \infty].$$

We will improve this estimation in Section 3 with the help of the following theorem, which is proved in Section 2.

THEOREM 1. *For every complete set K with $\delta(K) = 2$ in ℓ_1^3, we have*

$$\beta_{\ell_1^3}(K, 8) \leqslant \frac{3}{4}.$$

Moreover,

$$\beta(\ell_1^3, 8) \leqslant \frac{3}{4}.$$

In section 4, we show that $\beta(X,2^3)<1$ when X is a three-dimensional Banach space symmetric with $\{e_i \mid i\in[3]\}$ and satisfies $\alpha(X)>9/4$, where $\{e_i \mid i\in[3]\}$ is the natural basis and $\alpha(X)=\left\|\sum\limits_{i\in[3]} e_i\right\|$.

2. Partitions of complete sets in ℓ_1^3

For a convex body K, the closed set Σ between two parallel supporting hyperplanes H and H' of K is called a *supporting slab* of K. The distance between H and H' is called the *width of* Σ. Let M be another convex body and Σ be a supporting slab of K. If the union of the bounding hyperplanes of the supporting slab of M that is parallel to Σ contains a smooth boundary point of M, then Σ is said to be *M-regular*. The following lemma proved by J. P. Moreno and R. Schneider is critical for the proof of Theorem 1.

LEMMA 1. ([9]) *Let $\Sigma_1,\cdots,\Sigma_k$ be the B_X-regular supporting slabs of the polyhedral unit ball B_X. Each complete set K with diameter 2 is of the form*

$$K=\bigcap_{i=1}^{k}(\Sigma_i+t_i)$$

with $t_i\in\mathbb{R}^n$, $\forall i\in[k]$.

For vectors $u\in\mathbb{R}^n\backslash\{o\}$ and $\tau\in\mathbb{R}$, we write

$$H(u,\tau):=\{x\in\mathbb{R}^n \mid \langle x,u\rangle=\tau\},$$

$$H^+(u,\tau):=\{x\in\mathbb{R}^n \mid \langle x,u\rangle\geqslant\tau\},\quad\text{and}\quad H^-(u,\tau):=\{x\in\mathbb{R}^n \mid \langle x,u\rangle\leqslant\tau\},$$

where $\langle\cdot,\cdot\rangle$ is the scalar product.

The unit ball B_1^3 of ℓ_1^3 is a regular octahedron whose vertices are $\pm(0,0,1)$, $\pm(1,0,0)$, and $\pm(0,1,0)$. Clearly, $\pm u_1=\pm(1,1,1)$, $\pm u_2=\pm(-1,1,1)$, $\pm u_3=\pm(1,-1,1)$, and $\pm u_4=\pm(-1,-1,1)$ are the outer normal vectors of the facets of B_1^3. Therefore, the B_1^3-regular supporting slabs of B_1^3 are Σ_1, Σ_2, Σ_3, Σ_4, where, for each $i\in[4]$, Σ_i is the slab bounded by $H(u_i,1)$ and $H(u_i,-1)$. After applying a suitable translation if necessary, every complete set in ℓ_1^3 with diameter 2 has the form

$$\begin{aligned}P(\alpha)&=\Sigma_2\cap\Sigma_3\cap\Sigma_4\cap(\Sigma_1+\alpha u_1)\\&=(B_1^3\cup S_{\pm1})\cap(\Sigma_1+\alpha u_1)\\&=\big(B_1^3\cap(\Sigma_1+\alpha u_1)\big)\cup(S_{\pm1}\cap(\Sigma_1+\alpha u_1)),\end{aligned}$$

where $S_1=\operatorname{conv}((B_1^3\cap H(u_1,1))\cup\{u_1\})$, $S_{-1}=-S_1$, and $|\alpha|\leqslant 2/3$.

It is not difficult to verify that $P(0)=B_1^3$, $S_1=P(2/3)$, and $S_{-1}=P(-2/3)$. These sets are all complete (the completeness of $S_{\pm1}$ follows from Claim 1 in [8]).

LEMMA 2. *Let T be the regular tetrahedron in ℓ_1^3 with vertices*

$$v_1=\left(1,-\frac{1}{2},-\frac{1}{2}\right),\quad v_2=\left(-\frac{1}{2},1,-\frac{1}{2}\right),\quad v_3=\left(-\frac{1}{2},-\frac{1}{2},1\right),\quad v_4=(1,1,1).$$

Then T can be divided into five parts whose diameters are $3/2$.

Proof. Denote the points

$$c_1=\frac{v_1+v_4}{2}=\left(1,\frac{1}{4},\frac{1}{4}\right),\quad c_2=\frac{v_2+v_4}{2}=\left(\frac{1}{4},1,\frac{1}{4}\right),$$
$$c_3=\frac{v_3+v_4}{2}=\left(\frac{1}{4},\frac{1}{4},1\right),\quad d_1=\frac{v_2+v_3}{2}=\left(-\frac{1}{2},\frac{1}{4},\frac{1}{4}\right),$$
$$d_2=\frac{v_1+v_3}{2}=\left(\frac{1}{4},-\frac{1}{2},\frac{1}{4}\right),\quad d_3=\frac{v_1+v_2}{2}=\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right),$$

and the sets

$$A_1=\operatorname{conv}\{v_1,c_1,d_2,d_3\},\quad A_2=\operatorname{conv}\{v_2,c_2,d_1,d_3\},$$
$$A_3=\operatorname{conv}\{v_3,c_3,d_1,d_2\},\quad A_4=\operatorname{conv}\{v_4,c_1,c_2,c_3\},$$
$$A_5=\operatorname{conv}\{c_1,c_2,c_3,d_1,d_2,d_3\}.$$

Then $T=\cup_{i\in[5]}A_i$, and

$$\delta(A_i)=\max\left\{\|v_i-c_i\|_1,\max_{j\in[3]\setminus\{i\}}\{\|v_i-d_j\|_1,\|c_i-d_j\|_1\}\right\}=\frac{3}{2},\quad i\in[3],$$
$$\delta(A_4)=\max\{\max\{\|v_4-c_j\|_1\mid j\in[3]\},\max\{\|c_i-c_j\|_1\mid i,j\in[3]\}\}=\frac{3}{2},$$
$$\delta(A_5)=\max_{i,j\in[3]}\{\|c_i-c_j\|_1,\|d_i-d_j\|_1,\|c_i-d_j\|_1\}=\frac{3}{2}.$$

This means that A can be divided into five parts whose diameters are $3/2$. □

COROLLARY 1. $\beta_{\ell_1^3}(P(1/3),5)\leqslant 3/4$.

Proof. It follows from Lemma 2 and the fact that $P(1/3)$ is contained in the regular tetrahedron with vertices v_1,v_2,v_3,v_4 as in Lemma 2. □

Proof of Theorem 1. Let K be a complete set in ℓ_1^3 with $\delta(K)=2$. As above, we may assume that K has the following form:

$$P(\alpha)=\Sigma_2\cap\Sigma_3\cap\Sigma_4\cap(\Sigma_1+\alpha u_1),\ \ |\alpha|\leqslant 2/3.$$

We show that $\beta_{\ell_1^3}(P(\alpha),8)\leqslant 3/4$, $\forall\alpha\in[0,2/3]$. The case when $\alpha\in[-2/3,0]$ can be proved in a similar way. We consider the following two cases.

Case 1: If $\alpha \in [0,1/3)$, then $P(\alpha) = A \cup B$, where

$$A = P(\alpha) \cap H^+(u_1, 0), \quad \text{and} \quad B = P(\alpha) \cap H^-(u_1, 0).$$

Notice that A is contained in the regular tetrahedron T defined in Lemma 2. Thus A can be divided into five parts whose diameters are $3/2$.

The bounding hyperplanes of the slab $\Sigma_1 + \alpha u_1$ are $H(u_1, 3\alpha - 1)$ and $H(u_1, 1 + 3\alpha)$. It is clear that $B_1^3 \cap H(u_1, 3\alpha - 1)$ is a convex hexagon with vertices

$$a_1 = \left(0, \frac{3}{2}\alpha - 1, \frac{3}{2}\alpha\right), \quad a_2 = \left(\frac{3}{2}\alpha, \frac{3}{2}\alpha - 1, 0\right), \quad a_3 = \left(\frac{3}{2}\alpha, 0, \frac{3}{2}\alpha - 1\right),$$
$$a_4 = \left(0, \frac{3}{2}\alpha, \frac{3}{2}\alpha - 1\right), \quad a_5 = \left(\frac{3}{2}\alpha - 1, \frac{3}{2}\alpha, 0\right), \quad a_6 = \left(\frac{3}{2}\alpha - 1, 0, \frac{3}{2}\alpha\right).$$

The intersection of $H(u_1, 3\alpha - 1)$ and the segment between o and $-u_1$ is $o' = (\alpha - 1/3, \alpha - 1/3, \alpha - 1/3)$. Note that $B_1^3 \cap H(u_1, 0)$ is also a convex hexagon which vertices are

$$b_1 = \left(0, -\frac{1}{2}, \frac{1}{2}\right), \quad b_2 = \left(\frac{1}{2}, -\frac{1}{2}, 0\right), \quad b_3 = \left(\frac{1}{2}, 0, -\frac{1}{2}\right),$$
$$b_4 = \left(0, \frac{1}{2}, -\frac{1}{2}\right), \quad b_5 = \left(-\frac{1}{2}, \frac{1}{2}, 0\right), \quad b_6 = \left(-\frac{1}{2}, 0, \frac{1}{2}\right).$$

Let

$$V_1 = \left\{a_1, a_2, \frac{a_1 + a_6}{2}, \frac{a_2 + a_3}{2}, o', o, b_1, b_2, \frac{b_1 + b_6}{2}, \frac{b_2 + b_3}{2}\right\},$$
$$V_2 = \left\{a_3, a_4, \frac{a_2 + a_3}{2}, \frac{a_4 + a_5}{2}, o', o, b_3, b_4, \frac{b_2 + b_3}{2}, \frac{b_4 + b_5}{2}\right\},$$
$$V_3 = \left\{a_5, a_6, \frac{a_4 + a_5}{2}, \frac{a_1 + a_6}{2}, o', o, b_5, b_6, \frac{b_4 + b_5}{2}, \frac{b_1 + b_6}{2}\right\},$$

and B_i be the polyhedron with V_i as the set of vertices, $i \in [3]$. Then $B = B_1 \cup B_2 \cup B_3$, and

$$\delta(B_1) = \max\{\|x - y\|_1 \mid x, y \in V_1\} = \left\|\frac{a_1 + a_6}{2} - \frac{a_2 + a_3}{2}\right\|_1 = \frac{3}{2},$$
$$\delta(B_2) = \max\{\|x - y\|_1 \mid x, y \in V_2\} = \left\|\frac{a_2 + a_3}{2} - \frac{a_4 + a_5}{2}\right\|_1 = \frac{3}{2},$$
$$\delta(B_3) = \max\{\|x - y\|_1 \mid x, y \in V_3\} = \left\|\frac{a_4 + a_5}{2} - \frac{a_1 + a_6}{2}\right\|_1 = \frac{3}{2}.$$

Thus $P(\alpha)$ can be divided into eight parts whose diameters are $3/2$. Hence

$$\beta_{\ell_1^3}(P(\alpha), 8) \leqslant \frac{3}{4}$$

holds for every $\alpha \in [0,1/3)$.

Case 2: If $\alpha \in [1/3,2/3]$, then, since $3\alpha - 1 > 0$, $P(\alpha)$ is completely contained in the regular tetrahedron T defined in Lemma 2, which means that $P(\alpha)$ can be divided into five parts of diameters $3/2$. Thus

$$\beta_{\ell_1^3}(P(\alpha),8) \leqslant \beta_{\ell_1^3}(P(\alpha),5) \leqslant \frac{3}{4}$$

holds for each $\alpha \in [1/3,2/3]$.

By Proposition 1 in [7], we have

$$\beta(\ell_1^3,8) \leqslant \frac{3}{4}.$$

This completes the proof. □

3. An estimation of $\beta(\ell_p^3,8)$

The (multiplicative) Banach-Mazur distance between two isomorphic Banach spaces X and Y is defined by

$$d_{BM}^M(X,Y) = \inf\{\|T\| \cdot \|T^{-1}\| \mid T:\ X \to Y \text{ is an isomorphism}\}.$$

Moreover, if X, Y, and Z are isomorphic Banach spaces, then

$$d_{BM}^M(X,Y) \leqslant d_{BM}^M(X,Z) \cdot d_{BM}^M(Z,Y),$$

see [11]. In this section, we use estimations of the Banach-Mazur distance between ℓ_p^n and ℓ_q^n and the stability of $\beta(X,m)$ with respect to X in the sense of Banach-Mazur distance to get an estimation of $\beta(\ell_p^3,8)$.

LEMMA 3. ([7]) *If $X = (\mathbb{R}^n, \|\cdot\|_X)$ and $Y = (\mathbb{R}^n, \|\cdot\|_Y)$ are two Banach spaces satisfying $d_{BM}^M(X,Y) \leqslant \gamma$ for some $\gamma \geqslant 1$, then*

$$\beta(X,m) \leqslant \gamma\beta(Y,m), \forall m \in \mathbb{Z}^+.$$

THEOREM 2. ([11]) *Let n be a positive integer and $1 \leqslant p,q \leqslant \infty$.*

- *If $1 \leqslant p,q \leqslant 2$ or $2 \leqslant p,q \leqslant \infty$, then $d_{BM}^M(\ell_p^n,\ell_q^n) = n^{\frac{1}{p}-\frac{1}{q}}$.*
- *If $1 \leqslant p < 2 < q \leqslant \infty$, then $\gamma n^\alpha \leqslant d_{BM}^M(\ell_p^n,\ell_q^n) \leqslant \eta n^\alpha$, where $\alpha = \max\{\frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \frac{1}{q}\}$, and γ and η are universal constants.*

From Theorem 2, it follows that $d_{BM}^M(\ell_p^n,\ell_\infty^n) = n^{\frac{1}{p}}$, $\forall p \in [2,\infty]$.

THEOREM 3. *For* $p \in [1,\infty]$, $\beta(\ell_p^3, 8) \leqslant 0.88185$.

Proof. We distinguish three cases.

Case 1: $p \in [1, 1.1729]$. By Theorem 2,

$$d_{BM}^M(\ell_1^3, \ell_p^3) = 3^{1-\frac{1}{p}} \leqslant 3^{1-\frac{1}{1.1729}} \leqslant 1.1758, \ \forall p \in [1, 1.1729].$$

By Theorem 1 and Lemma 3, we have

$$\beta(\ell_p^3, 8) \leqslant 1.1758 \times 0.75 = 0.88185, \ \forall p \in [1, 1.1729].$$

Case 2: $p \in [1.1729, 1.45]$. By the proof of Lemma 14 in [7],

$$d_{BM}^M(\ell_p^3, \ell_\infty^3) \leqslant \frac{1}{10}(2+4^p)^{\frac{1}{p}} \cdot (2 \cdot 3^{\frac{p}{p-1}} + 1)^{\frac{p-1}{p}}, \ \forall p \in (1, 2]. \tag{1}$$

For $p \in (1, 2]$, set

$$f(p) = \ln(2+4^p) + (p-1)\ln\left(2 \cdot 3^{\frac{p}{p-1}} + 1\right),$$

$r(p) = f(p)/p$, and $w(p) = pf'(p) - f(p)$. We have $r'(p) = w(p)/p^2$ and $w'(p) = pf''(p)$. By the proof of Lemma 8 in [13], $w(p)$ is strictly increasing on $(1, 2]$. Meanwhile, since $w(1.1729) < 0$ and $w(1.45) > 0$, there exists a unique point $p_0 \in (1.1729, 1.45)$ such that $w(p_0) = 0$. Therefore, $r'(p) \leqslant 0$ for $p \in [1.1729, p_0]$ and $r'(p) > 0$ for $p \in [p_0, 1.45]$. Hence $r(p)$ decreases on $[1.1729, p_0]$ and increases on $[p_0, 1.45]$. Since $2.8682 \approx r(1.45) < r(1.1729) \approx 2.8700$, we have $r(p) \leqslant r(1.1729) \approx 2.8700$, $\forall p \in [1.1729, 1.45]$. By (1),

$$d_{BM}^M(\ell_p^3, \ell_\infty^3) \leqslant \frac{e^{r(p)}}{10} \leqslant \frac{e^{r(1.1729)}}{10} \leqslant 1.7637.$$

By Lemma 3 and Proposition 4 in [7], we have

$$\beta(\ell_p^3, 8) \leqslant 1.7637 \times 0.5 = 0.88185, \ \forall p \in [1.1729, 1.45].$$

Case 3: $p \in [1.45, 2]$. Let $\mathrm{GL}_n(\mathbb{R})$ denote the set of all nonsingular $n \times n$ matrices of real number. Here we assume that $A = (a_{ij})_{3\times 3} \in \mathrm{GL}_3(\mathbb{R})$ and A_{ij} is the cofactor of a_{ij}, $\forall i, j \in [3]$. Let x_1, x_2, x_3 be the column vectors of A and set $y_i = (A_{1i}, A_{2i}, A_{3i})^T$, $\forall i \in [3]$. For $p \in [1, \infty]$, put

$$g_p(A) = \frac{1}{|\det A|} \max\{\|y_i\|_q \|\sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3\|_p \mid i \in [3], \sigma_1, \sigma_2, \sigma_3 \in \{-1, 1\}\}, \tag{2}$$

where q is the conjugate of p. By Lemma 5 in [13], $d_{BM}^M(\ell_p^3, \ell_\infty^3) \leqslant g_p(A)$. Set

$$A_1 = \begin{pmatrix} 1 & -1.71 & 1.71 \\ -1.71 & 1 & 1.71 \\ 1.71 & 1.71 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 17 & -32.9 & 32.9 \\ -32.9 & 17 & 32.9 \\ 32.9 & 32.9 & 17 \end{pmatrix}.$$

By (2), we have

$$d_{BM}^{M}(\ell_{1.5}^{3},\ell_{\infty}^{3}) \leqslant g_{1.5}(A_1) \leqslant 1.6732 \quad \text{and} \quad d_{BM}^{M}(\ell_{1.9}^{3},\ell_{\infty}^{3}) \leqslant g_{1.9}(A_2) \leqslant 1.7135.$$

By Theorem 2,

$$d_{BM}^{M}(\ell_{1.5}^{3},\ell_{p}^{3}) = 3^{\frac{1}{p}-\frac{1}{1.5}} \leqslant 3^{\frac{1}{1.45}-\frac{1}{1.5}} \leqslant 1.0256, \ \forall p \in [1.45, 1.5],$$

$$d_{BM}^{M}(\ell_{1.5}^{3},\ell_{p}^{3}) = 3^{\frac{1}{1.5}-\frac{1}{p}} \leqslant 3^{\frac{1}{1.5}-\frac{1}{1.6}} \leqslant 1.0468, \ \forall p \in [1.5, 1.6],$$

$$d_{BM}^{M}(\ell_{1.6}^{3},\ell_{p}^{3}) = 3^{\frac{1}{1.6}-\frac{1}{p}} \leqslant 3^{\frac{1}{1.6}-\frac{1}{1.7}} \leqslant 1.0412, \ \forall p \in [1.6, 1.7],$$

$$d_{BM}^{M}(\ell_{1.7}^{3},\ell_{p}^{3}) = 3^{\frac{1}{1.7}-\frac{1}{p}} \leqslant 3^{\frac{1}{1.7}-\frac{1}{1.8}} \leqslant 1.0366, \ \forall p \in [1.7, 1.8],$$

$$d_{BM}^{M}(\ell_{1.8}^{3},\ell_{p}^{3}) = 3^{\frac{1}{1.8}-\frac{1}{p}} \leqslant 3^{\frac{1}{1.8}-\frac{1}{1.9}} \leqslant 1.0327, \ \forall p \in [1.8, 1.9],$$

$$d_{BM}^{M}(\ell_{1.9}^{3},\ell_{p}^{3}) = 3^{\frac{1}{1.9}-\frac{1}{p}} \leqslant 3^{\frac{1}{1.9}-\frac{1}{1.96}} \leqslant 1.0179, \ \forall p \in [1.9, 1.96],$$

$$d_{BM}^{M}(\ell_{2}^{3},\ell_{p}^{3}) = 3^{\frac{1}{p}-\frac{1}{2}} \leqslant 3^{\frac{1}{1.96}-\frac{1}{2}} \leqslant 1.0113, \ \forall p \in [1.96, 2].$$

It follows from Table 1 and the proof of Theorem 2 in [13] that

$$d_{BM}^{M}(\ell_{p}^{3},\ell_{\infty}^{3}) \leqslant d_{BM}^{M}(\ell_{1.5}^{3},\ell_{p}^{3}) \cdot d_{BM}^{M}(\ell_{1.5}^{3},\ell_{\infty}^{3}) \leqslant 1.7161, \ \forall p \in [1.45, 1.5],$$

$$d_{BM}^{M}(\ell_{p}^{3},\ell_{\infty}^{3}) \leqslant d_{BM}^{M}(\ell_{1.5}^{3},\ell_{p}^{3}) \cdot d_{BM}^{M}(\ell_{1.5}^{3},\ell_{\infty}^{3}) \leqslant 1.7516, \ \forall p \in [1.5, 1.6],$$

$$d_{BM}^{M}(\ell_{p}^{3},\ell_{\infty}^{3}) \leqslant d_{BM}^{M}(\ell_{1.6}^{3},\ell_{p}^{3}) \cdot d_{BM}^{M}(\ell_{1.6}^{3},\ell_{\infty}^{3}) \leqslant 1.7451, \ \forall p \in [1.6, 1.7],$$

$$d_{BM}^{M}(\ell_{p}^{3},\ell_{\infty}^{3}) \leqslant d_{BM}^{M}(\ell_{1.7}^{3},\ell_{p}^{3}) \cdot d_{BM}^{M}(\ell_{1.7}^{3},\ell_{\infty}^{3}) \leqslant 1.7588, \ \forall p \in [1.7, 1.8],$$

$$d_{BM}^{M}(\ell_{p}^{3},\ell_{\infty}^{3}) \leqslant d_{BM}^{M}(\ell_{1.8}^{3},\ell_{p}^{3}) \cdot d_{BM}^{M}(\ell_{1.8}^{3},\ell_{\infty}^{3}) \leqslant 1.7529, \ \forall p \in [1.8, 1.9],$$

$$d_{BM}^{M}(\ell_{p}^{3},\ell_{\infty}^{3}) \leqslant d_{BM}^{M}(\ell_{1.9}^{3},\ell_{p}^{3}) \cdot d_{BM}^{M}(\ell_{1.9}^{3},\ell_{\infty}^{3}) \leqslant 1.7442, \ \forall p \in [1.9, 1.96],$$

$$d_{BM}^{M}(\ell_{p}^{3},\ell_{\infty}^{3}) \leqslant d_{BM}^{M}(\ell_{2}^{3},\ell_{p}^{3}) \cdot d_{BM}^{M}(\ell_{2}^{3},\ell_{\infty}^{3}) \leqslant 1.7517, \ \forall p \in [1.96, 2].$$

Thus $d_{BM}^{M}(\ell_{p}^{3},\ell_{\infty}^{3}) \leqslant 1.7588 < 1.7637, \ \forall p \in [1.45, 2]$.

By Lemma 3 and Proposition 4 in [7], we have

$$\beta(\ell_{p}^{3}, 8) < 1.7637 \times 0.5 = 0.88185, \ \forall p \in [1.45, 2]. \quad \square$$

4. Borsuk's problem in n-dimensional symmetric space

Let X be an n-dimensional real Banach space. If there exists a basis $\{u_i \mid i \in [n]\}$ of X such that the equality

$$\left\| \sum_{i\in[n]} \sigma_i \alpha_i u_{\pi(i)} \right\| = \|x\|$$

holds for each $x = \sum_{i\in[n]} \alpha_i u_i$, any set of numbers $\{\sigma_i \mid |\sigma_i| = 1,\ i \in [n]\}$, and any permutation π of $[n]$, then X is said to be *symmetric* with the *symmetric basis* $\{u_i \mid i \in [n]\}$.

Obviously, for each n-dimensional symmetric space X, there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that X is isometric to $(\mathbb{R}^n, \|\cdot\|)$ and that the symmetric basis of $\mathbb{R}^n$ is also a symmetric basis of $(\mathbb{R}^n, \|\cdot\|)$.

In the following, denote by $\{e_i \mid i \in [n]\}$ the natural basis of $\mathbb{R}^n$. Set

$$\zeta^n = \{(\mathbb{R}^n, \|\cdot\|) \mid (\mathbb{R}^n, \|\cdot\|) \text{ is symmetric with the natural basis } \{e_i \mid i \in [n]\}\},$$

and, for any $X \in \zeta^n$, let

$$\alpha(X) = \left\| \sum_{i\in[n]} e_i \right\|.$$

LEMMA 4. *For any $X \in \zeta^n$, we have*

$$d_{BM}^M(X, \ell_\infty^n) \leqslant \alpha(X).$$

Proof. First, we show that $B_X \subseteq B_\infty^n$. Let $x = \sum_{i\in[n]} \alpha_i e_i \in B_X$. We only need to show that $\alpha_i \in [-1,1]$ holds for all $i \in [n]$. Take α_1 as an example. Since

$$\begin{aligned} \|\alpha_1 e_1\| &= \left\| \frac{1}{2}\left(\alpha_1 e_1 + \sum_{i=2}^n \alpha_i e_i\right) + \frac{1}{2}\left(\alpha_1 e_1 - \sum_{i=2}^n \alpha_i e_i\right) \right\| \\ &\leqslant \max\left\{ \left\|\alpha_1 e_1 + \sum_{i=2}^n \alpha_i e_i\right\|, \left\|\alpha_1 e_1 - \sum_{i=2}^n \alpha_i e_i\right\| \right\} = \|x\| \leqslant 1, \end{aligned}$$

we have $\alpha_1 \in [-1,1]$. In a similar way, we can prove that $\alpha_i \in [-1,1]$, $\forall i = 2, \cdots, n$.

Next, by the definition of $\alpha(X)$, we have $(1/\alpha(X))B_\infty^n \subseteq B_X$. It follows that

$$\frac{1}{\alpha(X)} B_\infty^n \subseteq B_X \subseteq B_\infty^n.$$

Thus $d_{BM}^M(X, \ell_\infty^n) \leqslant \alpha(X)$. □

PROPOSITION 1. Let $X \in \zeta^n$. If $\alpha(X) < 2$, then $\beta(X, 2^n) < 1$.

Proof. It follows from Lemma 3, Lemma 4, and Proposition 4 in [7]. □

LEMMA 5. *For any $X \in \zeta^n$, we have*

$$d_{BM}^M(X, \ell_1^n) \leqslant \frac{n}{\alpha(X)}.$$

Proof. The unit ball of ℓ_1^n is

$$B_1^n = \mathrm{conv}\{\pm e_i \mid i \in [n]\}.$$

Then $B_1^n \subseteq B_X$. Next we will show that $B_X \subseteq (n/\alpha(X))B_1^n$. It is sufficient to show that

$$\sum_{i\in[n]} |\alpha_i| \leqslant \frac{n}{\alpha(X)}$$

holds for each $x = \sum_{i\in[n]} \alpha_i e_i \in B_X$. By the symmetry of X, we only consider the case when $\alpha_i \geqslant 0$, $i \in [n]$.

The case when $x = o$ is obvious. Suppose that $x \in B_X \setminus \{o\}$. Then there exists $i_0 \in [n]$ such that $\alpha_{i_0} > 0$. Let

$$\gamma_i = \frac{\alpha_i}{\sum\limits_{j\in[n]} \alpha_j}.$$

Then $\gamma_i \geqslant 0$, $\forall i \in [n]$. Let

$$\begin{aligned}
u_1 &= \gamma_1 e_1 + \gamma_2 e_2 + \cdots + \gamma_{n-1} e_{n-1} + \gamma_n e_n, \\
u_2 &= \gamma_2 e_1 + \gamma_3 e_2 + \cdots + \gamma_n e_{n-1} + \gamma_1 e_n, \\
u_3 &= \gamma_3 e_1 + \gamma_4 e_2 + \cdots + \gamma_n e_{n-2} + \gamma_1 e_{n-1} + \gamma_2 e_n, \\
&\cdots \\
u_n &= \gamma_n e_1 + \gamma_1 e_2 + \cdots + \gamma_{n-2} e_{n-1} + \gamma_{n-1} e_n.
\end{aligned}$$

Since $\{e_i \mid i \in [n]\}$ is a symmetric basis of X, we have $\|u_1\| = \cdots = \|u_n\|$. Note that

$$\frac{1}{n}\sum_{i\in[n]} u_i = \frac{1}{n}\left(\sum_{i\in[n]} \gamma_i\right)\sum_{i\in[n]} e_i = \frac{1}{n}\sum_{i\in[n]} e_i.$$

Then

$$\left\|\frac{1}{n}\sum_{i\in[n]} e_i\right\| \leqslant \frac{1}{n}\sum_{i\in[n]} \|u_i\| = \|u_1\| = \left\|\sum_{i\in[n]} \gamma_i e_i\right\|.$$

It follows that

$$\begin{aligned}
1 \geqslant \left\|\sum_{i\in[n]} \alpha_i e_i\right\| &= \left(\sum_{i\in[n]} \alpha_i\right)\left\|\sum_{i\in[n]} \frac{\alpha_i}{\sum\limits_{i\in[n]} \alpha_i} e_i\right\| \\
&= \left(\sum_{i\in[n]} \alpha_i\right)\left\|\sum_{i\in[n]} \gamma_i e_i\right\| \\
&\geqslant \left(\sum_{i\in[n]} \alpha_i\right)\left\|\frac{1}{n}\sum_{i\in[n]} e_i\right\| = \frac{\alpha(X)}{n}\sum_{i\in[n]} \alpha_i,
\end{aligned}$$

from which the proof is complete. □

PROPOSITION 2. Let $X \in \zeta^3$. If $\alpha(X) > 9/4$, then $\beta(X, 2^3) < 1$.

Proof. It follows from Theorem 1, Lemma 3, and Lemma 5. □

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Xinling Zhang
School of Mathematics
Harbin Institute of Technology
Harbin, Heilongjiang, 150001, China
e-mail: zhangxinling2014@126.com

Chan He
School of Mathematics
North University of China
Taiyuan, Shanxi, 030051, China
e-mail: hechan@nuc.edu.cn