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AN ESTIMATE OF THE MAXIMAL OPERATOR OF THE NÖRLUND LOGARITHMIC MEANS WITH RESPECT TO THE WALSH–PALEY SYSTEM ON THE HARDY SPACE H_p

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Abstract. We define a weighted maximal operator $\tilde{L}_p^*$ and prove that it is bounded from $H_p(G)$ to $L_p(G)$, for $p \in (0,1)$.

1. Introduction

Let $\mathbb{Z}_2$ denote the discrete cyclic group $\mathbb{Z}_2 = \{0,1\}$, where the group operation is addition modulo 2.

The dyadic group G is obtained by $G = \prod_{i=0}^{\infty} \mathbb{Z}_2$ (see [15]), where topology and the probability measure $|.|$ are obtained by the product.

Let $x = (x_n)_{n \geqslant 0} \in G$. The sets $I_n(x) := \{y \in G : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$, $n \geqslant 1$ and $I_0(x) := G$ are dyadic intervals of G. Let $I_n = I_n(0)$, and $e_n := (\delta_{in})_i$. It is easily seen that $(I_n)_n$ is a decreasing sequence of subgroups.

Since every nonnegative integer i can be written in the form $i = \sum_{k=0}^{\infty} i_k 2^k$, where $i_k \in \{0,1\}$, we define the sequence $(z_i)_{i \geqslant 0}$ of elements from G by

$$z_i = \sum_{k=0}^{\infty} i_k e_k.$$

It is easily seen that for each positive integer n, the set $\{z_i, i < 2^n\}$ is a set of representatives of I_n-cosets.

The Walsh-Paley system is defined as the set of Walsh-Paley functions:

$$\omega_n(x) = \prod_{k=0}^{|n|} (r_k(x))^{n_k}, \ \ n \in \mathbb{N}, \ \ x \in G,$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$, $n_k \in \{0,1\}$, $|n| = \max\{k, n_k \neq 0\}$ and $r_k(x) = (-1)^{x_k}$.

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If $f \in L_1$, we can define the Fourier coefficients, partial sums of the Fourier series and Dirichlet kernels with respect to the Walsh system as

$$\widehat{f}(k) := \int_G f \omega_k d\mu,$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\,\omega_k, \qquad (S_0 f := 0),$$

$$D_n := \sum_{k=0}^{n-1} \omega_k.$$

It can be easily seen that $S_n f(x) = (D_n * f)(x)$ and $D_{2^n}(x) = 2^n 1_{I_n}(x)$.

Nörlund logarithmic means and kernels are defined by

$$L_n f := \frac{1}{l_n} \sum_{k=0}^{n} \frac{S_k f}{n-k},$$

and

$$F_n := \frac{1}{l_n} \sum_{k=0}^{n} \frac{D_k}{n-k},$$

where

$$l_n = \sum_{k=1}^{n} \frac{1}{k}.$$

It is known that ([14])

$$\|F_n\|_1 \leqslant c \log n,$$

and

$$\sup_n \left\| \frac{F_n}{(n+1)^{1/p-1}} \right\|_1 \leqslant C < \infty. \tag{1}$$

For every positive integer n, the algebra generated by the intervals $\{I_n(x), x \in G\}$ is denoted by F_n. If $f = (f_n)_n$ is a martingale with respect to F_n, then its maximal function f^* is defined by $f^* = \sup_n |f_n|$. If $f \in L_1(G)$, then its maximal function is defined by

$$f^*(x) = \sup_n \frac{1}{|I_n|} \left| \int_{I_n(x)} f(t)dt \right|.$$

For every $p \in (0, \infty)$, the Hardy space $H_p(G)$ consists of all martingales $f = (f_n)_n$ such that $f^* \in L_p(G)$. The norm on $H_p(G)$ is defined by

$$\|f\|_{H_p} := \|f^*\|_p.$$

A bounded measurable function a is a p-atom, if it is supported on some dyadic interval I, such that

$$\int_I a = 0, \; \|a\|_\infty \leqslant | I |^{-\frac{1}{p}}.$$

Throughout the paper C denotes an absolute positive constant which may vary in different contexts.

The a.e. convergence of a subsequence of logarithmic means of Walsh-Fourier series of integrable functions was studied in the works [3], [5] and [6]. Results related to partial sums and Nörlund logarithmic means with respect to unbounded Vilenkin system can be found in [1] and [8]. Nörlund logarithmic means were studied in different contexts in [1], [2], [4], [9], [10], [11], [12], [16], [17], [18], [19] and [20].

Many maximal operators were studied on the Hardy space H_p of general Vilenkin groups, for $p \in (0,1]$ (see for example [1], [12], [13] and [17]).

In [19, Theorem 1] it was proved that the operator

$$\sup_n \frac{|L_n f|}{(n+1)^{\frac{1}{p}-1}}$$

is bounded from the space H_p to the space L_p on general Vilenkin groups. Moreover, for every nondecreasing positive sequence $\{\varphi_n\}$ satisfying

$$\limsup_{n\to\infty} \frac{(n+1)^{\frac{1}{p}-1}}{\log n \varphi_n} = \infty,$$

the maximal operator

$$\sup_n \frac{|L_n f|}{\varphi_n}$$

is not bounded from H_p to L_p.

In Theorem 1 we prove that the operator

$$\tilde{L}_p^* := \sup_n \frac{\log(n+1)\,|L_n f|}{(n+1)^{\frac{1}{p}-1}}$$

is bounded from the space $H_p(G)$ to the space $L_p(G)$. Of course, the second part of [19, Theorem 1] describes the sharpness of the result obtained in Theorem 1. Similar results were obtained for the Walsh-Kaczmarz system in [7].

The following open problem was stated on pages 476–477 of the recent book [14]:

OPEN PROBLEM. *For any $0 < p < 1$, is it possible to find non-negative, non-decreasing sequence $(\Theta_n, n \in \mathbb{N})$ such that the maximal operator $\overset{\sim *}{L}_p$ defined by*

$$\overset{\sim *}{L}_p f := \sup_{n\in\mathbb{N}} \frac{|L_n f|}{\Theta_{n+1}}$$

is bounded from the Hardy space H_p to the Lebesgue space L_p? Moreover, is it true that the rate of $(\Theta_n, n \in \mathbb{N})$ is sharp, that is, for any non-negative, non-decreasing sequence $(\varphi_n, n \in \mathbb{N})$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\Theta_n}{\varphi_n} = \infty,$$

there exists a martingale $f \in H_p(G)$ *such that the maximal operator*

$$\sup_{n\in\mathbb{N}} \frac{|L_n f|}{\varphi_{n+1}}$$

is not bounded from the Hardy space H_p *to the Lebesgue space* L_p*?*

In [19] Tephnadze and Tutberidze partially answered this open problem and proved that there exist absolute constants C_1 and C_2 such that

$$\frac{C_1 n^{1/p-1}}{\log(n+1)} \leqslant \Theta_n \leqslant C_2 n^{1/p-1}.$$

In this paper we establish an answer for the Walsh system by proving that such optimal weights are

$$\left\{\frac{n^{1/p-1}}{\log(n+1)}\right\}.$$

2. Main results

LEMMA 1. *Let* n *be a positive integer having the dyadic representation* $n = 2^{N_1} + \ldots + 2^{N_t}$, *where* $N_1 < N_2 < \ldots < N_t$ *and* $N_t = |n|$. *Then,* $D_n(x)$ *can be written in the form*

$$D_n(x) = D_{2^{N_t}} + \sum_{j=0}^{2^{N_t}-1} A_{n,j}\left(D_{2^{N_t+1}}(x+z_j) - D_{2^{N_t}}(x+z_j)\right), \tag{2}$$

where

$$A_{n,0} := \sum_{i=1}^{t-1} 2^{N_i - N_t}, \tag{3}$$

$$A_{n,j} := r_{N_{t-1}}(z_j)\ldots r_{N_{i+1}}(z_j)\left[2^{N_i-N_t} + r_{N_i}(z_j)\sum_{s=1}^{i-1} 2^{N_s-N_t}\right], \tag{4}$$

if $j = 0 \pmod{2^{N_i}}$ *and* $j \neq 0 \pmod{2^{N_{i+1}}}$ *for some* $i \in \{1,\ldots,t-1\}$.

For $j \neq 0 \pmod{2^{N_1}}$, $A_{n,j} = 0$.

Proof. It was proved in [15] that

$$D_n = \sum_{i=1}^{t} r_{N_t}\ldots r_{N_{i+1}} D_{2^{N_i}}. \tag{5}$$

Therefore,

$$\begin{aligned} D_n(x) &= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} r_{N_t}(x)\dots r_{N_{i+1}}(x) D_{2^{N_i}}(x) \\ &= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} 2^{N_i - N_t} r_{N_t}(x) D_{2^{N_t}}(x) \\ &\quad + \sum_{i=1}^{t-1} 2^{N_i - N_t} r_{N_t}(x)\dots r_{N_{i+1}}(x) \sum_{\substack{j\in\{1,\dots,2^{N_t}-1\} \\ j=0 \pmod{2^{N_i}}}} D_{2^{N_t}}(x+z_j) \\ &= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} 2^{N_i - N_t} (D_{2^{N_t+1}}(x) - D_{2^{N_t}}(x)) \\ &\quad + \sum_{i=1}^{t-1} 2^{N_i - N_t} \sum_{\substack{j\in\{1,\dots,2^{N_t}-1\} \\ j=0 \pmod{2^{N_i}}}} r_{N_{t-1}}(z_j)\dots r_{N_{i+1}}(z_j) (D_{2^{N_t+1}}(x+z_j) - D_{2^{N_t}}(x+z_j)), \end{aligned}$$

because

$$r_{N_t}(y) D_{2^{N_t}}(y) = D_{2^{N_t+1}}(y) - D_{2^{N_t}}(y), \ \forall y \in G,$$

besides, $r_{N_t}(z_j) = 1$, for every $j < 2^{N_t}$, so that $r_{N_t}(x) = r_{N_t}(x+z_j)$. Therefore, we get

$$\begin{aligned} D_n(x) &= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} 2^{N_i - N_t} (D_{2^{N_t+1}}(x) - D_{2^{N_t}}(x)) \\ &\quad + \sum_{i=1}^{t-1} \sum_{\substack{j=0 \pmod{2^{N_i}} \\ j\neq 0 \pmod{2^{N_{i+1}}}}} r_{N_{t-1}}(z_j)\dots r_{N_{i+1}}(z_j) \\ &\quad \times \left[2^{N_i - N_t} + r_{N_i}(z_j) 2^{N_{i-1} - N_t} + \dots + r_{N_i}(z_j)\dots r_{N_2}(z_j) 2^{N_1 - N_t}\right] \\ &\quad \times (D_{2^{N_t+1}}(x+z_j) - D_{2^{N_t}}(x+z_j)) \\ &= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} 2^{N_i - N_t} (D_{2^{N_t+1}}(x) - D_{2^{N_t}}(x)) \\ &\quad + \sum_{i=1}^{t-1} \sum_{\substack{j=0 \pmod{2^{N_i}} \\ j\neq 0 \pmod{2^{N_{i+1}}}}} r_{N_{t-1}}(z_j)\dots r_{N_{i+1}}(z_j) \\ &\quad \times \left[2^{N_i - N_t} + r_{N_i}(z_j) 2^{N_{i-1} - N_t} + \dots + r_{N_i}(z_j) 2^{N_1 - N_t}\right] (D_{2^{N_t+1}}(x+z_j) - D_{2^{N_t}}(x+z_j)) \\ &= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} 2^{N_i - N_t} (D_{2^{N_t+1}}(x) - D_{2^{N_t}}(x)) \\ &\quad + \sum_{i=1}^{t-1} \sum_{\substack{j=0 \pmod{2^{N_i}} \\ j\neq 0 \pmod{2^{N_{i+1}}}}} r_{N_{t-1}}(z_j)\dots r_{N_{i+1}}(z_j) \left[2^{N_i - N_t} + r_{N_i}(z_j) \sum_{s=1}^{i-1} 2^{N_s - N_t}\right] \\ &\quad \times (D_{2^{N_t+1}}(x+z_j) - D_{2^{N_t}}(x+z_j)), \end{aligned}$$

where the previous sums are obviously taken over $j \in \{0,\dots,2^{N_t}-1\}$. □

The following remark is a direct consequence of formula (4).

REMARK 1. let $j = 2^{l_1} + 2^{l_2} + 2^{l_3}M$, where $l_1 < l_2 < l_3$ and M is a nonnegative integer. If $n = \sum_{i=0}^{|n|} n_i 2^i$ is larger than $2^{|j|+1}$ and such that $n_{l_2} = 0$, then

$$A_{n,j} = -A_{n+2^{l_2},j}.$$

LEMMA 2. *Let* $n = 2^{N_1} + \dots + 2^{N_t}$*, where* $N_1 < N_2 < \dots < N_t$ *and let* $j = 2^{l_1} + 2^{l_2}M$*, where* $l_1 < N_t$ *and* $M \geqslant 0$*. Then,*

$$\left| \sum_{k=2^{N_t-1}}^{2^{N_t}-1} \frac{A_{k,j}}{n-k} \right| \leqslant C \min\{l_2, N_t\} 2^{l_1-N_t} \tag{6}$$

and

$$\left| \sum_{k=2^{N_t}}^{n-1} \frac{A_{k,j}}{n-k} \right| \leqslant C \min\{l_2, N_t\} 2^{l_1-N_t}. \tag{7}$$

Since the notation $A_{k,j}$ *wasn't defined in Lemma 1 for* $k < 2^{|j|+1}$*, we may consider in this case that* $A_{k,j} := A_{k,j-2^{|k|}\lfloor \frac{j}{2^{|k|}} \rfloor}$.

Proof. We consider the following cases

1. $l_1 + 1 = N_t$,
2. $l_1 + 1 < N_t$ and $l_2 \geqslant N_t - 1$,
3. $l_1 + 1 < N_t$ and $N_{t-1} \leqslant l_2 < N_t - 1$,
4. $l_2 < N_{t-1}$.

In the first case we have $N_t \leqslant l_2$. Besides, it can be seen from formulae (4) and (3) that $|A_{k,j}| < 2^{l_1-|k|+1}$. Hence,

$$\sum_{k=2^{N_t-1}}^{n-1} \left| \frac{A_{k,j}}{n-k} \right| \leqslant \sum_{k=2^{N_t-1}}^{n-1} \frac{C}{n-k} \leqslant C \sum_{t=1}^{n} \frac{1}{t} \leqslant C \cdot N_t.$$

In the second case, estimates (6) and (7) can be deduced from

$$\sum_{k=2^{N_t-1}}^{n-1} \left| \frac{A_{k,j}}{n-k} \right| \leqslant 2^{l_1-N_t+1} \sum_{k=2^{N_t-1}}^{n-1} \frac{1}{n-k} \leqslant C \cdot N_t 2^{l_1-N_t} \leqslant C \cdot \min\{l_2, N_t\} 2^{l_1-N_t},$$

because $N_t \leqslant 2\min\{l_2, N_t\}$.

In case (3) we have

$$\begin{aligned}
&\sum_{k=2^{N_t-1}}^{2^{N_t}-1}\frac{A_{k,j}}{n-k}\\
&=\sum_{k=0}^{2^{N_t-1}-1}\frac{A_{2^{N_t-1}+k,j}}{n-2^{N_t-1}-k}\\
&=\sum_{r=0}^{2^{N_t-1-l_2-1}-1}\sum_{i=0}^{2^{l_2}-1}\left(\frac{A_{2^{N_t-1}+2^{l_2+1}r+2^{l_2}+i,j}}{n-2^{N_t-1}-2^{l_2+1}r-2^{l_2}-i}+\frac{A_{2^{N_t-1}+2^{l_2+1}r+i,j}}{n-2^{N_t-1}-2^{l_2+1}r-i}\right)\\
&=\sum_{r=0}^{2^{N_t-1-l_2-1}-1}\sum_{i=0}^{2^{l_2}-1}A_{2^{N_t-1}+2^{l_2+1}r+2^{l_2}+i,j}\\
&\quad\times\left(\frac{1}{n-2^{N_t-1}-2^{l_2+1}r-2^{l_2}-i}-\frac{1}{n-2^{N_t-1}-2^{l_2+1}r-i}\right)\\
&=\sum_{r=0}^{2^{N_t-1-l_2-1}-1}\sum_{i=0}^{2^{l_2}-1}A_{2^{N_t-1}+2^{l_2+1}r+2^{l_2}+i,j}\\
&\quad\times\frac{2^{l_2}}{(n-2^{N_t-1}-2^{l_2+1}r-2^{l_2}-i)(n-2^{N_t-1}-2^{l_2+1}r-i)},
\end{aligned}$$

where the third inequality is deduced from Remark 1. Therefore,

$$\begin{aligned}
\left|\sum_{k=2^{N_t-1}}^{2^{N_t}-1}\frac{A_{k,j}}{n-k}\right| &\leqslant 2^{l_1-N_t+1}\sum_{r=0}^{2^{N_t-1-l_2-1}-2}\frac{2^{2l_2}}{n-2^{N_t-1}-2^{l_2+1}(r+1)}\\
&\qquad\times\frac{1}{n-2^{N_t-1}-2^{l_2+1}(r+1)+2^{l_2}}\\
&\qquad+2^{l_1-N_t+1}\sum_{i=0}^{2^{l_2}-1}\frac{2^{l_2}}{(n-2^{N_t}+2^{l_2}-i)(n-2^{N_t}+2^{l_2+1}-i)}\\
&\leqslant 2^{l_1-N_t-1}\sum_{r=1}^{2^{N_t-1-l_2-1}-1}\frac{1}{(\frac{n-2^{N_t-1}}{2^{l_2+1}}-r)^2}+2^{l_1-N_t+1}\sum_{i=0}^{2^{l_2}-1}\frac{1}{2^{l_2}-i}\\
&\leqslant C\cdot l_2 2^{l_1-N_t},
\end{aligned}$$

because $n-2^{N_t}+2^{l_2+1}-i\geqslant 2^{l_2}$, for all $i\in\{0,\dots,2^{l_2}-1\}$. Hence, (6) is proved for the case (3).

To establish (7) for the case (3), notice that $n-2^{N_t}\leqslant 2^{N_{t-1}+1}\leqslant 2^{l_2+1}$, it follows that

$$\sum_{k=2^{N_t}}^{n-1}\left|\frac{A_{k,j}}{n-k}\right|\leqslant 2^{l_1-N_t+1}\sum_{t=1}^{n-2^{N_t}}\frac{1}{t}\leqslant 2^{l_1-N_t+1}\sum_{t=1}^{2^{l_2+1}}\frac{1}{t}\leqslant C\cdot l_2 2^{l_1-N_t}.$$

Now we study the fourth case where it is clear that $l_2<N_t-1$, which means that estimate (6) can be proved as done in the previous case. It only remains to prove

estimate (7) in the fourth case. Assume that $j=2^{l_1}+2^{l_2}+\ldots+2^{l_s}+2^{N_{t-1}}M$, where $s\geqslant 2$, $l_1<l_2<\ldots<l_s<N_{t-1}$ and M is a nonnegative integer. We first assume that $l_2\neq N_i$ for all $i\in\{1,\ldots,t-2\}$.

First we have

$$\sum_{k=2^{N_t}}^{n-2^{l_s}-1}\frac{A_{k,j}}{n-k}=\sum_{k=2^{N_t}}^{2^{N_t}+2^{l_2+1}\lfloor\frac{n-2^{l_s}-2^{N_t}}{2^{l_2+1}}\rfloor-1}\frac{A_{k,j}}{n-k}+\sum_{k=2^{N_t}+2^{l_2+1}\lfloor\frac{n-2^{l_s}-2^{N_t}}{2^{l_2+1}}\rfloor}^{n-2^{l_s}-1}\frac{A_{k,j}}{n-k}$$
$$=:I+II.$$

$$|II|\leqslant 2^{l_1-N_t+1}\sum_{k=2^{N_t}+2^{l_2+1}\lfloor\frac{n-2^{l_s}-2^{N_t}}{2^{l_2+1}}\rfloor}^{n-2^{l_s}-1}\frac{1}{n-k}\tag{8}$$
$$\leqslant 2^{l_1-N_t+1}\sum_{m=2^{l_s}}^{2^{l_s}+2^{l_2+1}}\frac{1}{m}\leqslant C\cdot l_2 2^{l_1-N_t}.$$

On the other hand,

$$I=\sum_{k=0}^{2^{l_2+1}\lfloor\frac{n-2^{l_s}-2^{N_t}}{2^{l_2+1}}\rfloor-1}\frac{A_{2^{N_t}+k,j}}{n-2^{N_t}-k}$$
$$=\sum_{r=0}^{\lfloor\frac{n-2^{l_s}-2^{N_t}}{2^{l_2+1}}\rfloor-1}\sum_{i=0}^{2^{l_2}-1}\left(\frac{A_{2^{N_t}+2^{l_2+1}r+i,j}}{n-2^{N_t}-2^{l_2+1}r-i}+\frac{A_{2^{N_t}+2^{l_2+1}r+2^{l_2}+i,j}}{n-2^{N_t}-2^{l_2+1}r-2^{l_2}-i}\right)$$
$$=\sum_{r=0}^{\lfloor\frac{n-2^{l_s}-2^{N_t}}{2^{l_2+1}}\rfloor-1}\sum_{i=0}^{2^{l_2}-1}A_{2^{N_t}+2^{l_2+1}r+2^{l_2}+i,j}$$
$$\times\left(\frac{1}{n-2^{N_t}-2^{l_2+1}r-2^{l_2}-i}-\frac{1}{n-2^{N_t}-2^{l_2+1}r-i}\right).$$

It follows that

$$|I|\leqslant 2^{l_1-N_t+1}\sum_{r=0}^{\lfloor\frac{n-2^{l_s}-2^{N_t}}{2^{l_2+1}}\rfloor-1}\sum_{i=0}^{2^{l_2}-1}\frac{2^{l_2}}{(n-2^{N_t}-2^{l_2+1}r-2^{l_2}-i)(n-2^{N_t}-2^{l_2+1}r-i)}$$
$$\leqslant 2^{l_1-N_t+1}\sum_{r=0}^{\lfloor\frac{n-2^{l_s}-2^{N_t}}{2^{l_2+1}}\rfloor-1}\frac{2^{2l_2}}{(n-2^{N_t}-2^{l_2+1}(r+1))(n-2^{N_t}-2^{l_2+1}(r+1)+2^{l_2})}$$
$$=2^{l_1-N_t-1}\sum_{r=1}^{\lfloor\frac{n-2^{l_s}-2^{N_t}}{2^{l_2+1}}\rfloor}\frac{1}{(\frac{n-2^{N_t}}{2^{l_2+1}}-r)(\frac{n-2^{N_t}}{2^{l_2+1}}-r+\frac{1}{2})}\tag{9}$$
$$\leqslant 2^{l_1-N_t-1}\sum_{t=2^{l_s-l_2-1}}^{\lfloor\frac{n-2^{N_t}}{2^{l_2+1}}\rfloor}\frac{1}{t^2}.$$

Then, if $l_2+1<l_s$ we have

$$\begin{aligned}
&\sum_{k=n-2^{l_s}}^{n-2^{l_2+1}-1}\frac{A_{k,j}}{n-k}\\
&=\sum_{k=0}^{2^{l_s}-2^{l_2+1}-1}\frac{A_{n-2^{l_s}+k,j}}{2^{l_s}-k}\\
&=\sum_{r=0}^{\frac{2^{l_s}-2^{l_2+1}}{2^{l_2+1}}-1}\sum_{i=0}^{2^{l_2}-1}\left(\frac{A_{n-2^{l_s}+2^{l_2+1}r+i,j}}{2^{l_s}-2^{l_2+1}r-i}+\frac{A_{n-2^{l_s}+2^{l_2+1}r+2^{l_2}+i,j}}{2^{l_s}-2^{l_2+1}r-2^{l_2}-i}\right)\\
&=\sum_{r=0}^{\frac{2^{l_s}-2^{l_2+1}}{2^{l_2+1}}-1}\sum_{i=0}^{2^{l_2}-1}A_{n-2^{l_s}+2^{l_2+1}r+2^{l_2}+i,j}\left(\frac{1}{2^{l_s}-2^{l_2+1}r-2^{l_2}-i}-\frac{1}{2^{l_s}-2^{l_2+1}r-i}\right)\\
&=\sum_{r=0}^{\frac{2^{l_s}-2^{l_2+1}}{2^{l_2+1}}-1}\sum_{i=0}^{2^{l_2}-1}A_{n-2^{l_s}+2^{l_2+1}r+2^{l_2}+i,j}\frac{2^{l_2}}{(2^{l_s}-2^{l_2+1}r-2^{l_2}-i)(2^{l_s}-2^{l_2+1}r-i)},
\end{aligned}$$

where the third equality is obtained from Remark 1 because if $l_2\neq N_i$ for all $i\in\{1,\ldots,t-2\}$, then the integers $n-2^{l_s}+2^{l_2+1}r+i$, where r is a nonnegative integer and $i\in\{0,\ldots,2^{l_2}-1\}$, satisfy the condition mentioned in Remark 1. It follows that

$$\begin{aligned}
\left|\sum_{k=n-2^{l_s}}^{n-2^{l_2+1}-1}\frac{A_{k,j}}{n-k}\right| &\leqslant 2^{l_1-N_t+1}\sum_{r=0}^{\frac{2^{l_s}-2^{l_2+1}}{2^{l_2+1}}-1}\frac{2^{2l_2}}{(2^{l_s}-2^{l_2+1}(r+1))(2^{l_s}-2^{l_2+1}(r+1)+2^{l_2})}\\
&=2^{l_1-N_t-1}\sum_{r=1}^{\frac{2^{l_s}-2^{l_2+1}}{2^{l_2+1}}}\frac{1}{(2^{l_s-l_2-1}-r)(2^{l_s-l_2-1}-r+\frac{1}{2})} \qquad (10)\\
&\leqslant 2^{l_1-N_t-1}\sum_{t=1}^{2^{l_s-l_2-1}-1}\frac{1}{t^2}.
\end{aligned}$$

It remains to estimate the sum

$$\begin{aligned}
\left|\sum_{k=n-2^{l_2+1}}^{n-1}\frac{A_{k,j}}{n-k}\right| &= \left|\sum_{k=0}^{2^{l_2+1}-1}\frac{A_{n-2^{l_2+1}+k,j}}{2^{l_2+1}-k}\right| \qquad (11)\\
&\leqslant 2^{l_1-N_t+1}\sum_{t=1}^{2^{l_2+1}}\frac{1}{t}\leqslant C\cdot l_2 2^{l_1-N_t}.
\end{aligned}$$

Summing the terms in (8), (9), (10) and (11) proves estimate (7) in case (4) under the assumption that $l_2\neq N_i$ for all $i\in\{1,\ldots,t-2\}$. Now, assume that $l_2=N_i$ for some $i\in\{1,\ldots,t-2\}$. It is clear that the sums in (8) and (9) can be estimated in the same way as in the previous case. In this case we first estimate

$$\left|\sum_{k=n-2^{l_s}}^{n-2^{l_s}+2^{l_2}-1}\frac{A_{k,j}}{n-k}\right|\leqslant 2^{l_1-N_t+1}\sum_{t=2^{l_s}-2^{l_2}+1}^{2^{l_s}}\frac{1}{t}\leqslant C2^{l_1-N_t}. \tag{12}$$

Then we have if $l_2+2<l_s$

$$\begin{aligned}\sum_{k=n-2^{l_s}+2^{l_2}}^{n-2^{l_2+1}-2^{l_2}-1}\frac{A_{k,j}}{n-k}&=\sum_{k=0}^{2^{l_s}-2^{l_2+2}-1}\frac{A_{n-2^{l_s}+2^{l_2}+k,j}}{2^{l_s}-2^{l_2}-k}\\&=\sum_{r=0}^{2^{l_s-l_2-1}-3}\sum_{i=0}^{2^{l_2}-1}\left(\frac{A_{n-2^{l_s}+2^{l_2}+2^{l_2+1}r+i,j}}{2^{l_s}-2^{l_2+1}r-2^{l_2}-i}+\frac{A_{n-2^{l_s}+2^{l_2+1}+2^{l_2+1}r+i,j}}{2^{l_s}-2^{l_2+1}r-2^{l_2+1}-i}\right)\\&=\sum_{r=0}^{2^{l_s-l_2-1}-3}\sum_{i=0}^{2^{l_2}-1}A_{n-2^{l_s}+2^{l_2+1}(r+1)+i,j}\\&\qquad\times\left(\frac{1}{2^{l_s}-2^{l_2+1}(r+1)-i}+\frac{1}{2^{l_s}-2^{l_2+1}(r+1)+2^{l_2}-i}\right),\end{aligned}$$

where the last equality is obtained from Remark 1 because the numbers $n-2^{l_s}+2^{l_2}+2^{l_2+1}r+i$, where r is a nonnegative integer and $i\in\{0,\dots,2^{l_2}-1\}$, satisfy the condition mentioned there. It follows that

$$\begin{aligned}\left|\sum_{k=n-2^{l_s}+2^{l_2}}^{n-2^{l_2+1}-2^{l_2}-1}\frac{A_{k,j}}{n-k}\right|&\leqslant 2^{l_1-N_t+1}\sum_{r=0}^{2^{l_s-l_2-1}-3}\sum_{i=0}^{2^{l_2}-1}\frac{2^{l_2}}{2^{l_s}-2^{l_2+1}(r+1)-i}\\&\qquad\times\frac{1}{2^{l_s}-2^{l_2+1}(r+1)+2^{l_2}-i}\\&\leqslant 2^{l_1-N_t+1}\sum_{r=0}^{2^{l_s-l_2-1}-3}\frac{2^{2l_2}}{(2^{l_s}-2^{l_2+1}(r+1)-2^{l_2})(2^{l_s}-2^{l_2+1}(r+1))}\\&\leqslant 2^{l_1-N_t-1}\sum_{r=1}^{2^{l_s-l_2-1}-2}\frac{1}{(2^{l_s-l_2-1}-r-\frac{1}{2})(2^{l_s-l_2-1}-r)}\\&\leqslant 2^{l_1-N_t-1}\sum_{t=1}^{2^{l_s-l_2-1}-1}\frac{1}{t^2}\leqslant C\cdot 2^{l_1-N_t}.\end{aligned} \tag{13}$$

Now, we consider the sum

$$\begin{aligned}\left|\sum_{k=n-2^{l_2+1}-2^{l_2}}^{n-1}\frac{A_{k,j}}{n-k}\right|&=\left|\sum_{k=0}^{2^{l_2+1}+2^{l_2}-1}\frac{A_{n-2^{l_2+1}-2^{l_2}+k,j}}{2^{l_2+1}+2^{l_2}-k}\right|\\&\leqslant 2^{l_1-N_t+1}\sum_{t=1}^{2^{l_2+1}+2^{l_2}}\frac{1}{t}\leqslant C\cdot l_2 2^{l_1-N_t}.\end{aligned} \tag{14}$$

Summing the terms in (8), (9), (12), (13) and (14) proves estimate (7) in case (4) under the assumption that $l_2=N_i$ for some $i\in\{1,\dots,t-2\}$. □

THEOREM 1. *Let* $0 < p < 1$, *then the maximal operator*

$$\tilde{L}_p^* := \sup_n \frac{\log(n+1) \mid L_n f \mid}{(n+1)^{\frac{1}{p}-1}}$$

is bounded from the space $H_p(G)$ *to the space* $L_p(G)$.

Proof. Since $\tilde{L}_p^*$ is bounded (see (1)) according to [17, Lemma 1], it suffices to prove that

$$\int_{\overline{I}_N} \left|\tilde{L}_p^* a\right|^p \leqslant C, \tag{15}$$

for every atom a supported on I_N, where $\overline{I}_N := G \setminus I_N$.

Let a be an atom supported on the interval I_N. We have according to Lemma 5 that

$$\int_{\overline{I}_N} \sup_n \left|\frac{\log(n+1) L_n a}{(n+1)^{\frac{1}{p}-1}}\right|^p dx = \int_{\overline{I}_N} \sup_n \frac{\log^p(n+1)}{(n+1)^{1-p}} \frac{1}{l_n^p} \left|\sum_{k=1}^{n-1} \frac{S_k a}{n-k}\right|^p dx$$

$$= \int_{\overline{I}_N} \sup_n \frac{\log^p(n+1)}{(n+1)^{1-p}} \frac{1}{l_n^p} \left|\sum_{s=1}^{|n|-2} \sum_{k=2^s}^{2^{s+1}-1} \frac{S_k a}{n-k} + \sum_{k=2^{|n|-1}}^{2^{|n|}-1} \frac{S_k a}{n-k} + \sum_{k=2^{|n|}}^{n-1} \frac{S_k a}{n-k}\right|^p dx$$

$$\leqslant C \int_{\overline{I}_N} \sup_n \frac{1}{(n+1)^{1-p}} \left|\sum_{s=1}^{|n|-2} \sum_{k=2^s}^{2^{s+1}-1} \sum_{j=0}^{2^s-1} \frac{A_{k,j}}{n-k} (S_{2^{s+1}} a(x+z_j) - S_{2^s} a(x+z_j))\right|^p dx$$

$$+ C \int_{\overline{I}_N} \sup_n \frac{1}{(n+1)^{1-p}} \left|\sum_{j=0}^{2^{|n|-1}-1} \sum_{k=2^{|n|-1}}^{2^{|n|}-1} \frac{A_{k,j}}{n-k} (S_{2^{|n|}} a(x+z_j) - S_{2^{|n|-1}} a(x+z_j))\right|^p dx$$

$$+ C \int_{\overline{I}_N} \sup_n \frac{1}{(n+1)^{1-p}} \left|\sum_{j=0}^{2^{|n|}-1} \sum_{k=2^{|n|}}^{n-1} \frac{A_{k,j}}{n-k} (S_{2^{|n|+1}} a(x+z_j) - S_{2^{|n|}} a(x+z_j))\right|^p dx$$

$$=: \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3.$$

Notice that for all $s \leqslant |n| - 2$,

$$\sum_{k=2^s}^{2^{s+1}-1} \frac{1}{n-k} \leqslant \sum_{t=n-2^{s+1}}^{n-2^s} \frac{1}{t} \leqslant C \log \frac{n-2^s}{n-2^{s+1}} \leqslant C \log \frac{\frac{n}{2^s} - 1}{\frac{n}{2^s} - 2} \leqslant C. \tag{16}$$

Since $S_{2^s} a(x) = 0$ for all $s \leqslant N$ and $x \in G$, we have from the fact that a is supported on I_N, Remark 1 and (16) that

$$\mathfrak{I}_1 \leqslant C \sum_{i=1}^{2^N-1} \int_{I_N(z_i)} \sup_{n \geqslant 2^N} \frac{1}{(n+1)^{1-p}}$$

$$\times \left|\sum_{s=N}^{|n|-2} \sum_{k=2^s}^{2^{s+1}-1} \sum_{\substack{j \in \{0,\ldots,2^s-1\}, \\ j = i \pmod{2^N}}} \frac{A_{k,j}}{n-k} (S_{2^{s+1}} a(x+z_j) - S_{2^s} a(x+z_j))\right|^p dx$$

$$\leqslant C\|a\|_\infty^p \sum_{i=1}^{2^N-1} \sup_{n\geqslant 2^N} \frac{1}{(n+1)^{1-p}} \left(\sum_{s=N}^{|n|-2} \sum_{k=2^s}^{2^{s+1}-1} \sum_{\substack{j\in\{0,\dots,2^s-1\},\\ j=i \pmod{2^N}}} \frac{|A_{k,j}|}{n-k} \right)^p \int_{I_N(z_i)} dx$$

$$\leqslant C \sum_{r=0}^{N-1} \sum_{\substack{i=0 \pmod{2^r}\\ i\neq 0 \pmod{2^{r+1}}}} \sup_{n\geqslant 2^N} \frac{1}{(n+1)^{1-p}} \left(\sum_{s=N}^{|n|-2} \sum_{k=2^s}^{2^{s+1}-1} 2^{s-N} \frac{2^{r-s}}{n-k} \right)^p$$

$$\leqslant C \sum_{r=0}^{N-1} 2^{(r-N)p} 2^{N-r} \sup_{n\geqslant 2^N} \frac{1}{(n+1)^{1-p}} (|n|-N)^p$$

$$\leqslant C 2^{N(1-p)} \sup_{n\geqslant 2^N} \frac{1}{(n+1)^{1-p}} (|n|-N)^p \leqslant C,$$

where in the third inequality we used the fact that from Remark 1, $|A_{k,j}| \leqslant 2^{r-s+1}$, if $k\in\{2^s,\dots,2^{s+1}-1\}$ and $j\neq 0 \pmod{2^{r+1}}$, while the fourth inequality is deduced from (16).

We use (6) to estimate $\mathfrak{I}_2$. We have

$$\mathfrak{I}_2 \leqslant C\|a\|_\infty^p \sum_{i=1}^{2^N-1} \sup_{n\geqslant 2^N} \frac{1}{(n+1)^{1-p}} \left(\sum_{\substack{j\in\{0,\dots,2^{|n|-1}-1\},\\ j=i \pmod{2^N}}} \left| \sum_{k=2^{|n|-1}}^{2^{|n|}-1} \frac{A_{k,j}}{n-k} \right| \right)^p \int_{I_N(z_i)} dx$$

$$\leqslant C \sum_{l_1=0}^{N-2} \sum_{l_2=l_1+1}^{N-1} \sum_{\substack{i\in\{0,\dots,2^N-1\},\\ i=2^{l_1}+2^{l_2}M,\\ M \ odd}} \sup_{n\geqslant 2^N} \frac{1}{(n+1)^{1-p}} \left(\sum_{\substack{j\in\{0,\dots,2^{|n|-1}-1\},\\ j=i \pmod{2^N}}} \left| \sum_{k=2^{|n|-1}}^{2^{|n|}-1} \frac{A_{k,j}}{n-k} \right| \right)^p$$

$$+C \sum_{l_1=0}^{N-1} \sup_{n\geqslant 2^N} \frac{1}{(n+1)^{1-p}} \left(\sum_{\substack{j\in\{0,\dots,2^{|n|-1}-1\},\\ j=2^{l_1} \pmod{2^N}}} \left| \sum_{k=2^{|n|-1}}^{2^{|n|}-1} \frac{A_{k,j}}{n-k} \right| \right)^p := \mathfrak{I}_{2,1} + \mathfrak{I}_{2,2}.$$

According to (6), for all $j = 2^{l_1} + 2^{l_2}\tilde{M}$, where $\tilde{M}$ is some odd number, $l_1 \in \{0,\dots,N-2\}$ and $l_2 \in \{l_1+1,\dots,N-1\}$, we have that

$$\left| \sum_{k=2^{|n|-1}}^{2^{|n|}-1} \frac{A_{k,j}}{n-k} \right| \leqslant C\cdot l_2 2^{l_1-|n|}.$$

Hence,

$$\mathfrak{I}_{2,1} \leqslant C \sum_{l_1=0}^{N-2} \sum_{l_2=l_1+1}^{N-1} 2^{N-l_2} \sup_{n\geqslant 2^N} \frac{1}{(n+1)^{1-p}} \left(2^{|n|-N} l_2 2^{l_1-|n|} \right)^p$$

$$\leqslant C \sup_{n\geqslant 2^N} \frac{2^{N(1-p)}}{(n+1)^{1-p}} \sum_{l_1=0}^{N-2} 2^{l_1 p} \sum_{l_2=l_1+1}^{N-1} 2^{-l_2} l_2^p.$$

Let $p' \in (p,1)$, we have

$$\begin{aligned}\sum_{l_1=0}^{N-1} 2^{l_1 p} \sum_{l_2=l_1+1}^{N} 2^{-l_2} l_2^p &\leqslant C \sum_{l_1=0}^{N-1} 2^{l_1(p-p')} \sum_{l_2=l_1+1}^{N} \frac{l_2^p}{2^{l_2(1-p')}} 2^{(l_1-l_2)p'} \\ &\leqslant C \sum_{l_1=0}^{N-1} 2^{l_1(p-p')} \sum_{l_2=l_1+1}^{\infty} \frac{l_2^p}{2^{l_2(1-p')}} \leqslant C \sum_{l_1=0}^{\infty} 2^{l_1(p-p')} \leqslant C.\end{aligned}$$

Hence, $\mathfrak{I}_{2,1} \leqslant C$. In a similar way we get,

$$\begin{aligned}\mathfrak{I}_{2,2} \leqslant\ & C \sum_{l_1=0}^{N-1} \sup_{n \geqslant 2^N} \frac{1}{(n+1)^{1-p}} \left(\sum_{l_2=N}^{|n|-1} \sum_{\substack{j \in \{0,\dots,2^{|n|-1}-1\}, \\ j=2^{l_1}+2^{l_2}M, \\ M \ odd}} \left| \sum_{k=2^{|n|-1}}^{2^{|n|}-1} \frac{A_{k,j}}{n-k} \right| \right)^p \\ & + C \sum_{l_1=0}^{N-1} \sup_{n \geqslant 2^N} \frac{1}{(n+1)^{1-p}} \left| \sum_{k=2^{|n|-1}}^{2^{|n|}-1} \frac{A_{k,2^{l_1}}}{n-k} \right|^p \\ \leqslant\ & C \sum_{l_1=0}^{N-1} \sup_{n \geqslant 2^N} \frac{1}{(n+1)^{1-p}} \left(\sum_{l_2=N}^{|n|-1} 2^{|n|-l_2} l_2 2^{l_1-|n|} \right)^p \\ & + C \sum_{l_1=0}^{N-1} \sup_{n \geqslant 2^N} \frac{1}{(n+1)^{1-p}} (|n| 2^{l_1-|n|})^p.\end{aligned}$$

We have

$$\sum_{l_1=0}^{N-1} 2^{l_1 p} \left(\sum_{l_2=N}^{\infty} 2^{-l_2} l_2 \right)^p \leqslant C(N 2^{-N})^p \sum_{l_1=0}^{N-1} 2^{l_1 p} \leqslant C \cdot N^p.$$

Moreover,

$$\begin{aligned}\sum_{l_1=0}^{N-1} \sup_{n \geqslant 2^N} \frac{1}{(n+1)^{1-p}} (|n| 2^{l_1-|n|})^p &\leqslant \sup_{n \geqslant 2^N} \frac{|n|^p}{(n+1)^{1-p}} \sum_{l_1=0}^{N-1} 2^{(l_1-N)p} \\ &\leqslant C \sup_{n \geqslant 2^N} \frac{|n|^p}{(n+1)^{1-p}}.\end{aligned}$$

Therefore, by the definition of $|n|$ we get that

$$\mathfrak{I}_{2,2} \leqslant C \sup_{n \geqslant 2^N} \frac{|n|^p}{(n+1)^{1-p}} \leqslant C \frac{\log^p n}{(n+1)^{1-p}} \leqslant C,$$

which means that $\mathfrak{I}_2 \leqslant C$. It is easily seen that using (7), $\mathfrak{I}_3 \leqslant C$ can be proved in a similar way. We deduce that (15) is verified for every atom a supported on I_N. □

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