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SINGULAR VALUES OF COMPACT OPERATORS VIA OPERATOR MATRICES

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Abstract. This paper finds new upper bounds for the singular values of certain operator forms. Compared with the existing literature, numerous numerical examples will be given to show that the obtained forms add a new set of independent bounds, that are incomparable with some celebrated known results.

1. Introduction

In the sequel, let $\mathbb{K}(\mathbb{H})$ denote the two-sided ideal of compact linear operators from a separable complex Hilbert space $\mathbb{H}$ into itself, treated as a subclass of the C^*-algebra $\mathbb{B}(\mathbb{H})$ of all bounded linear operators on $\mathbb{H}$. The inner product on $\mathbb{H}$ will be denoted by $\langle \cdot,\cdot \rangle$, and the induced norm by $\|\cdot\|$. The zero operator in $\mathbb{B}(\mathbb{H})$ is denoted by $\mathbf{O}_{\mathbb{H}}$, or merely by $\mathbf{O}$ if no confusion arises.

For $A \in \mathbb{K}(\mathbb{H})$, let $s_j(A)$ denote the j-th singular value of A. Thus, $s_1(A) \geqslant s_2(A) \geqslant \cdots$. Singular values form one of the most basic and useful notions in understanding the geometry of $\mathbb{K}(\mathbb{H})$. The singular value decomposition, which allows deforming a compact operator into a simpler form, is an example.

Furthermore, many other key notions are defined via the singular values, such as the Schatten p-norms.

Therefore, it is essential to look into possible relations among the singular values of related operators in a way that extends certain important relations. In this direction, the arithmetic-geometric mean inequality for singular values states that if $A,B \in \mathbb{K}(\mathbb{H})$, then [3]

$$s_j(A^*B) \leqslant \frac{1}{2} s_j\left(|A^*|^2 + |B^*|^2\right), \tag{1}$$

for $j = 1,2,\ldots$, where A^* denote the conjugate transpose of A, and $|A|$ denotes the absolute value of A, defined as $(A^*A)^{\frac{1}{2}}$. This inequality influenced the path of research in operator theory, as one can see in [2, 7, 14, 16, 19], to mention a few.

This paper's sole goal is to complement the existing literature about singular values. We show some new bounds that are related to the existing ones. However, some

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terminologies, definitions, and lemmas are needed as a background. We recall that if $A \in \mathbb{B}(\mathbb{H})$, then A is said to be positive, denoted as $A \geqslant \mathbf{O}$, provided that $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathbb{H}$. If $A \geqslant \mathbf{O}$ and A is invertible, we say A is strictly positive and write $A > \mathbf{O}$.

The theory of operator means was developed in [15, 17]. In particular, if $0 \leqslant t \leqslant 1$ and $A, B \in \mathbb{B}(\mathbb{H})$ are strictly positive, the weighted geometric mean of A and B is defined by

$$A\sharp_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Operator matrices will play a vital role in proving our results. We recall that $A, B, C, D \in \mathbb{B}(\mathbb{H})$, then $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{B}(\mathbb{H} \oplus \mathbb{H})$, where $\oplus$ denotes the direct sum.

The next two lemmas state equivalent conditions for an operator matrix to be positive.

LEMMA 1. [11, Lemma 1] *Let* $A, B, C \in \mathbb{B}(\mathbb{H})$ *be such that* $A, B \geqslant \mathbf{O}$. *Then* $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ *is a positive operator in* $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$ *if and only if* $|\langle Cx, y\rangle|^2 \leqslant \langle Ax, x\rangle \langle By, y\rangle$ *for all* $x, y \in \mathbb{H}$.

LEMMA 2. [1] *Let* $A, B, C \in \mathbb{B}(\mathbb{H})$ *be such that* $A, B \geqslant \mathbf{O}$. *Then* $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ *is a positive operator in* $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$ *if and only if there exists a contraction* K *such that* $C = A^{\frac{1}{2}}KB^{\frac{1}{2}}$.

On the other hand, the following lemma gives an equivalent condition to the fact that a given operator matrix is positive partial transpose. We remark here that the proof given in [10] for this lemma is stated for matrices but is also valid for Hilbert space operators.

LEMMA 3. [10, Theorem 2.1] *Let* $A, B, C \in \mathbb{B}(\mathbb{H})$ *be such that* $A, B \geqslant \mathbf{O}$ *and let* $0 \leqslant t \leqslant 1$. *Then* $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ *and* $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ *are positive operators if and only if* $\begin{bmatrix} A\sharp_t B & C \\ C^* & A\sharp_{1-t} B \end{bmatrix}$ *and* $\begin{bmatrix} A\sharp_t B & C^* \\ C & A\sharp_{1-t} B \end{bmatrix}$ *are positive operators.*

On the other hand, the following result is a refinement of the celebrated Davidson-Power inequality [6].

LEMMA 4. [12] *Let* $A, B \in \mathbb{B}(\mathbb{H})$ *be such that* $A, B \geqslant \mathbf{O}$. *Then*

$$\|A + B\| \leqslant \max\{\|A\|, \|B\|\} + \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|,$$

where $\|\cdot\|$ *denotes the spectral norm (or the usual operator norm).*

As for the singular values, the max-min principle is considered one of the most efficient tools for obtaining singular value bounds. This can be found in [9, Theorem 9.1] or [18, Theorem 1.5], and it is stated as follows.

LEMMA 5. *Let* $A \in \mathbb{K}(\mathbb{H})$. *Then for* $j = 1,2,\ldots,$

$$s_j(A) = \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\|=1}} \|Ax\|.$$

2. Upper bounds for the singular values

In this section, we present our main findings, where upper bounds for the singular values of certain operators are explicitly found. These results are compared with the literature and among themselves.

PROPOSITION 1. *Let* $A,B,C \in \mathbb{K}(\mathbb{H})$ *be such that* $A,B \geqslant \mathbf{O}$. *If* $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \geqslant \mathbf{O}$ *in* $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$, *then*

$$s_j(C) \leqslant \min\left\{ \|B\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(A), \|A\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(B) \right\},$$

for $j = 1,2,\ldots.$

Proof. If $x,y \in \mathbb{H}$, it follows from Lemma 1 that

$$|\langle Cx,y\rangle| \leqslant \sqrt{\langle Ax,x\rangle \langle By,y\rangle}.$$

If we take the supremum over $y \in \mathbb{H}$ with $\|y\| = 1$, we get

$$\|Cx\| \leqslant \|B\|^{\frac{1}{2}} \langle Ax,x\rangle^{\frac{1}{2}} \leqslant \|B\|^{\frac{1}{2}} \|Ax\|^{\frac{1}{2}} \|x\|^{\frac{1}{2}},$$

where we used Cauchy-Schwarz inequality to obtain the latter inequality. Then Lemma 5 yields

$$\begin{aligned} s_j(C) &= \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\|=1}} \|Cx\| \\ &\leqslant \|B\|^{\frac{1}{2}} \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\|=1}} \|Ax\|^{\frac{1}{2}} \\ &= \|B\|^{\frac{1}{2}} \left(\max_{\dim M = j} \min_{\substack{x \in M \\ \|x\|=1}} \|Ax\| \right)^{\frac{1}{2}} \\ &= \|B\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(A). \end{aligned}$$

That is, for $j = 1,\ldots,n,$

$$s_j(C) \leqslant \|B\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(A). \tag{2}$$

Notice that $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \geqslant \mathbf{O}$ is equivalent to $\begin{bmatrix} B & C \\ C^* & A \end{bmatrix} \geqslant \mathbf{O}$. Utilizing (2) and the fact that $s_j(C) = s_j(C^*)$, we infer that

$$s_j(C) \leqslant \|A\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(B). \tag{3}$$

Combining inequalities (2) and (3) implies the desired result. □

Strengthening Proposition 1, we have the following result.

THEOREM 1. *Let $A, B, C \in \mathbb{K}(\mathbb{H})$ be such that $A, B \geqslant \mathbf{O}$. If $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \geqslant \mathbf{O}$ in $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$, let K be the contraction that satisfies $C = A^{\frac{1}{2}} K B^{\frac{1}{2}}$, as in Lemma 2. Then*

$$s_j(C) \leqslant \min\{\alpha_j, \beta_j\}$$

where

$$\alpha_j = \min\left\{ \|B\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\left(A^{\frac{1}{2}} K K^* A^{\frac{1}{2}}\right), \left\|B^{\frac{1}{2}} K^* K B^{\frac{1}{2}}\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(A) \right\},$$

$$\beta_j = \min\left\{ \|A\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\left(B^{\frac{1}{2}} K^* K B^{\frac{1}{2}}\right), \left\|A^{\frac{1}{2}} K K^* A^{\frac{1}{2}}\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(B) \right\},$$

for $j = 1, 2, \ldots$.

Proof. Let $x, y \in \mathbb{H}$. Then, by Lemma 2 and the Cauchy-Schwarz inequality, we know that a contraction $K \in \mathbb{B}(\mathbb{H})$ exists such that

$$\begin{aligned} |\langle Cx, y \rangle| &= \left| \left\langle A^{\frac{1}{2}} K B^{\frac{1}{2}} x, y \right\rangle \right| \\ &= \left| \left\langle K B^{\frac{1}{2}} x, A^{\frac{1}{2}} y \right\rangle \right| \\ &\leqslant \left\| K B^{\frac{1}{2}} x \right\| \left\| A^{\frac{1}{2}} y \right\| \\ &= \sqrt{\left\langle B^{\frac{1}{2}} K^* K B^{\frac{1}{2}} x, x \right\rangle \langle Ay, y \rangle}. \end{aligned}$$

That is,

$$|\langle Cx, y \rangle| \leqslant \sqrt{\left\langle B^{\frac{1}{2}} K^* K B^{\frac{1}{2}} x, x \right\rangle \langle Ay, y \rangle}.$$

If we take the supremum over $y \in \mathbb{H}$ with $\|y\| = 1$, we get

$$\|Cx\| \leqslant \|A\|^{\frac{1}{2}} \left\langle B^{\frac{1}{2}} K^* K B^{\frac{1}{2}} x, x \right\rangle^{\frac{1}{2}} \leqslant \|A\|^{\frac{1}{2}} \left\| B^{\frac{1}{2}} K^* K B^{\frac{1}{2}} x \right\|^{\frac{1}{2}} \|x\|^{\frac{1}{2}}.$$

Consequently, using Lemma 5,

$$\begin{aligned} s_j(C) &= \max_{\dim M = j} \min_{\substack{x\in M\\ \|x\|=1}} \|Cx\| \\ &\leqslant \|A\|^{\frac{1}{2}} \max_{\dim M = j} \min_{\substack{x\in M\\ \|x\|=1}} \left\|B^{\frac{1}{2}}K^*KB^{\frac{1}{2}}x\right\|^{\frac{1}{2}} \\ &= \|A\|^{\frac{1}{2}} \left(\max_{\dim M = j} \min_{\substack{x\in M\\ \|x\|=1}} \left\|B^{\frac{1}{2}}K^*KB^{\frac{1}{2}}x\right\| \right)^{\frac{1}{2}} \\ &= \|A\|^{\frac{1}{2}}\, s_j^{\frac{1}{2}}\left(B^{\frac{1}{2}}K^*KB^{\frac{1}{2}}\right). \end{aligned}$$

Therefore,

$$s_j(C) \leqslant \|A\|^{\frac{1}{2}}\, s_j^{\frac{1}{2}}\left(B^{\frac{1}{2}}K^*KB^{\frac{1}{2}}\right). \tag{4}$$

Again, if we apply Lemma 2 and the Cauchy-Schwarz inequality, we can write

$$\begin{aligned} |\langle Cx, y\rangle| &= \left|\left\langle A^{\frac{1}{2}}KB^{\frac{1}{2}}x, y\right\rangle\right| \\ &= \left|\left\langle B^{\frac{1}{2}}x, K^*A^{\frac{1}{2}}y\right\rangle\right| \\ &\leqslant \left\|B^{\frac{1}{2}}x\right\| \left\|K^*A^{\frac{1}{2}}y\right\| \\ &= \sqrt{\langle Bx, x\rangle \left\langle A^{\frac{1}{2}}KK^*A^{\frac{1}{2}}y, y\right\rangle}, \end{aligned}$$

where K is a contraction and $x, y \in \mathbb{H}$. Arguing like before, we obtain

$$\|Cx\| \leqslant \left\|A^{\frac{1}{2}}KK^*A^{\frac{1}{2}}\right\|^{\frac{1}{2}} \langle Bx, x\rangle^{\frac{1}{2}} \leqslant \left\|A^{\frac{1}{2}}KK^*A^{\frac{1}{2}}\right\|^{\frac{1}{2}} \|Bx\|^{\frac{1}{2}} \|x\|^{\frac{1}{2}}$$

for $x \in \mathbb{H}$. So,

$$\begin{aligned} s_j(C) &= \max_{\dim M = j} \min_{\substack{x\in M\\ \|x\|=1}} \|Cx\| \\ &\leqslant \left\|A^{\frac{1}{2}}KK^*A^{\frac{1}{2}}\right\|^{\frac{1}{2}} \max_{\dim M = j} \min_{\substack{x\in M\\ \|x\|=1}} \|Bx\|^{\frac{1}{2}} \\ &= \left\|A^{\frac{1}{2}}KK^*A^{\frac{1}{2}}\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(B). \end{aligned}$$

Thus,

$$s_j(C) \leqslant \left\|A^{\frac{1}{2}}KK^*A^{\frac{1}{2}}\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(B). \tag{5}$$

Combining inequalities (4) and (5) together implies that

$$s_j(C) \leqslant \min\left\{ \|A\|^{\frac{1}{2}}\, s_j^{\frac{1}{2}}\left(B^{\frac{1}{2}}K^*KB^{\frac{1}{2}}\right), \left\|A^{\frac{1}{2}}KK^*A^{\frac{1}{2}}\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(B) \right\}. \tag{6}$$

Noting the equivalence of the facts that $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \geqslant \mathbf{O}$ and $\begin{bmatrix} B & C^* \\ C & A \end{bmatrix} \geqslant \mathbf{O}$, and utilizing (6) and the fact $s_j(C) = s_j(C^*)$, we infer that

$$s_j(C) \leqslant \alpha_j = \min\left\{ \|B\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\left(A^{\frac{1}{2}} K K^* A^{\frac{1}{2}}\right), \left\|B^{\frac{1}{2}} K^* K B^{\frac{1}{2}}\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(A) \right\}. \tag{7}$$

The desired result follows from (6) and (7). □

As an application, we present the following new bound for $s_j(A^*B)$.

COROLLARY 1. *Let $A, B \in \mathbb{K}(\mathbb{H})$. Then there exists a contraction K such that*

$$s_j(A^*B) \leqslant \min\{\alpha_j', \beta_j'\}$$

where

$$\alpha_j' = \min\{\|B\| s_j(|A| K), \|K|B|\| s_j(A)\},$$
$$\beta_j' = \min\{\|A\| s_j(K|B|), \||A| K\| s_j(B)\},$$

for $j = 1, 2, \ldots$.

Proof. We know that $\begin{bmatrix} |A|^2 & A^*B \\ B^*A & |B|^2 \end{bmatrix} \geqslant \mathbf{O}$ for any $A, B \in \mathbb{B}(\mathbb{H})$. The result follows from Theorem 1 and the facts

$$\left\| |B|^2 \right\|^{\frac{1}{2}} = \| |B| \| = \|B\|,$$

$$s_j^{\frac{1}{2}}(|A| K K^* |A|) = s_j^{\frac{1}{2}}\left(|K^*|A||^2\right) = s_j(|K^*|A||) = s_j(K^*|A|) = s_j(|A| K),$$

and

$$\| |B| K^* K |B| \|^{\frac{1}{2}} = \left\| |K|B||^2 \right\|^{\frac{1}{2}} = \| |K|B| | \| = \|K|B|\|. \quad \square$$

REMARK 1. In this remark, we compare between the two bounds for $s_j(A^*B)$ in (1) and Corollary 1. We use the notation $s(\cdot)$ to denote the list of singular values. For this purpose, let $A = \begin{bmatrix} 1 & 5 \\ -3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ -1 & -4 \end{bmatrix}$. Then numerical calculations show that

$$s(A^*B) \approx \{36.6513, 1.63705\}, \ \frac{1}{2} s\left(|A^*|^2 + |B^*|^2\right) \approx \{40.7195, 1.78047\}.$$

Furthermore, when $K \approx \begin{bmatrix} -0.588172 & 0.808736 \\ 0.808736 & 0.588172 \end{bmatrix}$,

$$\|B\| s(|A| K) \approx \{36.9823, 6.34516\}, \ \| K|B| \| s(A) \approx \{36.9823, 6.34516\}$$
$$\|A\| s(K|B|) \approx \{36.9823, 9.45603\}, \ \| |A| K \| s(B) \approx \{36.9823, 9.45603\}.$$

Consequently, for $j=1$, Corollary 1 is better than (1), while the opposite conclusion holds for $j=2$. This shows that the two bounds are, in general, non-comparable.

We point out that the unitary K is the contraction that results from Lemma 2 when applied to the positive form $\begin{bmatrix} |A|^2 & A^*B \\ B^*A & |B|^2 \end{bmatrix}$, as in Corollary 1.

In the above discussion, we have seen how Proposition 1 was utilized to obtain Corollary 1, which gave a new upper bound for the singular values of the product of two compact operators. In the following result, we present an extended version of Proposition 1 in a way that will allow obtaining another interesting bound for $s_j(A^*B)$. The fact that Theorem 2 implies Proposition 1 can be seen by letting $t=0,1$ in the theorem.

THEOREM 2. *Let $A,B,C \in \mathbb{K}(\mathbb{H})$ be such that $A,B > \mathbf{O}$ and let $C = U|C|$ be the polar decomposition of C. If $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \geqslant \mathbf{O}$ in $\mathbb{B}(\mathbb{H}\oplus\mathbb{H})$, then for any $0 \leqslant t \leqslant 1$,*

$$s_j(C) \leqslant \|A\sharp_{1-t}(U^*BU)\|^{\frac{1}{2}}\, s_j^{\frac{1}{2}}(A\sharp_t(U^*BU)),$$

where $j=1,2,\ldots$.

Proof. For the given operators, one can check that

$$\begin{bmatrix} I & O \\ O & U \end{bmatrix}^* \begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \begin{bmatrix} I & O \\ O & U \end{bmatrix} = \begin{bmatrix} A & C^*U \\ U^*C & U^*BU \end{bmatrix} \geqslant \mathbf{O}.$$

Since $C = U|C|$ is the polar decomposition of C, then $U^*C = U^*U|C| = |C|$ (see, e.g., [8, p. 57]). So,

$$\begin{bmatrix} A & |C| \\ |C| & U^*BU \end{bmatrix} \geqslant \mathbf{O}.$$

Thus, by Lemma 3, we infer that

$$\begin{bmatrix} A\sharp_t(U^*BU) & |C| \\ |C| & A\sharp_{1-t}(U^*BU) \end{bmatrix} \geqslant \mathbf{O}.$$

Now, using Lemma 1, if $x,y \in \mathbb{H}$, we have

$$|\langle |C|x,y\rangle| \leqslant \sqrt{\langle (A\sharp_t(U^*BU))x,x\rangle \langle (A\sharp_{1-t}(U^*BU))y,y\rangle}. \tag{8}$$

By taking the supremum over $y \in \mathbb{C}^n$ with $\|y\|=1$, we deduce that

$$\begin{aligned} \|\,|C|x\,\| &\leqslant \|A\sharp_{1-t}(U^*BU)\|^{\frac{1}{2}} \langle (A\sharp_t(U^*BU))x,x\rangle^{\frac{1}{2}} \\ &\leqslant \|A\sharp_{1-t}(U^*BU)\|^{\frac{1}{2}} \|(A\sharp_t(U^*BU))x\|^{\frac{1}{2}} \|x\|^{\frac{1}{2}}. \end{aligned}$$

Applying Lemma 5 yields

$$\begin{aligned} s_j(C) &= s_j(|C|) \\ &= \max_{\dim M=j} \min_{\substack{x\in M \\ \|x\|=1}} \| \, |C|x \, \| \\ &\leqslant \|A\sharp_{1-t}(U^*BU)\|^{\frac{1}{2}} \max_{\dim M=j} \min_{\substack{x\in M \\ \|x\|=1}} \|(A\sharp_t(U^*BU))x\|^{\frac{1}{2}} \\ &= \|A\sharp_{1-t}(U^*BU)\|^{\frac{1}{2}} \left(\max_{\dim M=j} \min_{\substack{x\in M \\ \|x\|=1}} \|(A\sharp_t(U^*BU))x\| \right)^{\frac{1}{2}} \\ &= \|A\sharp_{1-t}(U^*BU)\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(A\sharp_t(U^*BU)), \end{aligned}$$

as required. □

REMARK 2. Both Proposition 1 and Theorem 2 gave upper bounds for $s_j(C)$, provided that $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \geqslant \mathbf{O}$. In this remark, we compare the found bounds numerically. As we will see. The two bounds are, in general, not comparable. For this purpose, let

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}.$$

Then it can be seen that $A = |T_1|^2$, $B = |T_2|^2$, $C = T_2^* T_1$, where

$$T_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

Consequently, $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \geqslant \mathbf{O}$. Then numerical calculations show that

$$\|B\|s(A) \approx \{27.4164, 0.583592\}, \quad \|A\|s(B) \approx \{27.4164, 6.8541\}$$

and, for $t = 0.4$,

$$\|A\sharp_{1-t}(U^*BU)\| \, s(A\sharp_t(U^*BU)) \approx \{12.9512, 1.411\},$$

where $U \approx \begin{bmatrix} -0.242536 & 0.970143 \\ 0.970143 & 0.242536 \end{bmatrix}$ is the unitary part in the polar decomposition of C.

It is evident from the above calculations that when $j = 1$, the bound found in Theorem 2 is much sharper than that in Proposition 1. In fact, we have $s^2(C) \approx \{12.6847, 0.315342\}$, which shows how close the bound in Theorem 2 is close to the exact value when $j = 1$. However, for $j = 2$, Proposition 1 provides a better bound.

REMARK 3. In this remark, we give a numerical example that shows that neither the bound in Theorem 1 nor that in Theorem 2 is uniformly better than the other. If we let

$$A=\begin{bmatrix}5&2\\2&4\end{bmatrix},\ B=\begin{bmatrix}5&6\\6&8\end{bmatrix},\ C=\begin{bmatrix}5&2\\6&4\end{bmatrix},$$

then numerical calculations show that, for $t=0.6$,

$$s^2(C)=\{80.202,0.797985\},\ \|A\sharp_{1-t}(U^*BU)\|\,s(A\sharp_t(U^*BU))\approx\{80.2794,6.07693\}.$$

On the other hand,

$$\begin{aligned}\|B\|\,s\left(A^{\frac12}KK^*A^{\frac12}\right)&\approx\{83.2311,30.9309\}\\ \left\|BK^*KB^{\frac12}\right\|^{\frac12}s(A)&\approx\{83.2311,30.9309\}\\ \|A\|\,s\left(B^{\frac12}K^*KB^{\frac12}\right)&\approx\{83.2311,2.06913\}\\ \left\|A^{\frac12}KK^*A^{\frac12}\right\|\,s(B)&\approx\{83.2311,2.06913\}.\end{aligned}$$

These computations indicate that for $j=1$, Theorem 2 is better than Theorem 1, while the opposite conclusion can be deduced for $j=2$. Thus, the two bounds are, in general, not comparable.

We point out that, in these calculations,

$$K\approx\begin{bmatrix}0.882353&0.470588\\-0.470588&0.882353\end{bmatrix}\ \text{and}\ U\approx\begin{bmatrix}0.913812&-0.406138\\0.406138&0.913812\end{bmatrix}.$$

So far, we have found some upper bounds for the singular values of the off-diagonal operator C (or C^*) under the assumption that a given operator matrix is positive. We continue with this theme by presenting the following more straightforward form. An interesting application of this result is stated next, where a new form of (1) is found.

THEOREM 3. *Let $A,B,C\in\mathbb{K}(\mathbb{H})$ be such that $A,B>\mathbf{O}$ and let $C=U|C|$ be the polar decomposition of C. If $\begin{bmatrix}A&C^*\\C&B\end{bmatrix}$ is a positive operator, then for any $0<\nu<1$,*

$$s_j^2(C)\leqslant s_j\left(\nu A^{\frac{1}{\nu}}+(1-\nu)U^*B^{\frac{1}{1-\nu}}U\right),$$

where $j=1,2,\ldots$.

Proof. It follows from (8) that, for $x\in\mathbb{H}$,

$$\begin{aligned}\left\||C|^{\frac12}x\right\|^4&\leqslant\langle Ax,x\rangle\langle U^*BUx,x\rangle\\&=\left\langle A^{\frac{\nu}{\nu}}x,x\right\rangle\left\langle U^*B^{\frac{1-\nu}{1-\nu}}Ux,x\right\rangle\\&=\left\langle A^{\frac{\nu}{\nu}}x,x\right\rangle\left\langle\left(U^*B^{\frac{1}{1-\nu}}U\right)^{1-\nu}x,x\right\rangle\end{aligned}$$

$$\leqslant \left\langle A^{\frac{1}{\nu}}x,x\right\rangle^{\nu}\left\langle U^*B^{\frac{1}{1-\nu}}Ux,x\right\rangle^{1-\nu}$$

(by the Hölder-McCarthy inequality)

$$\leqslant \nu\left\langle A^{\frac{1}{\nu}}x,x\right\rangle+(1-\nu)\left\langle U^*B^{\frac{1}{1-\nu}}Ux,x\right\rangle$$

(by the weighted arithmetic mean-geometric mean inequality)

$$=\left\langle \left(\nu A^{\frac{1}{\nu}}+(1-\nu)U^*B^{\frac{1}{1-\nu}}U\right)x,x\right\rangle$$

$$\leqslant \left\|\left(\nu A^{\frac{1}{\nu}}+(1-\nu)U^*B^{\frac{1}{1-\nu}}U\right)x\right\|\|x\|.$$

Indeed, we have shown that if $x\in\mathbb{H}$, then

$$\left\|\,|C|^{\frac{1}{2}}x\,\right\|^4\leqslant \left\|\left(\nu A^{\frac{1}{\nu}}+(1-\nu)U^*B^{\frac{1}{1-\nu}}U\right)x\right\|\|x\|.$$

Thus, implementing Lemma 5,

$$\begin{aligned} s_j^2(C) &= \max_{\dim M=j}\ \min_{\substack{x\in M\\ \|x\|=1}} \left\|\,|C|^{\frac{1}{2}}x\,\right\|^4 \\ &\leqslant \max_{\dim M=j}\ \min_{\substack{x\in M\\ \|x\|=1}} \left\|\left(\nu A^{\frac{1}{\nu}}+(1-\nu)U^*B^{\frac{1}{1-\nu}}U\right)x\right\| \\ &= s_j\left(\nu A^{\frac{1}{\nu}}+(1-\nu)U^*B^{\frac{1}{1-\nu}}U\right), \end{aligned}$$

as required. □

The following result is obtained from Theorem 3 and the fact that $\begin{bmatrix}|A|^2 & A^*B\\ B^*A & |B|^2\end{bmatrix}\geqslant \mathbf{O}$ for any $A,B\in\mathbb{B}(\mathbb{H})$.

COROLLARY 2. *Let $A,B\in\mathbb{K}(\mathbb{H})$. If $B^*A=U|B^*A|$ is the polar decomposition of B^*A, then for any $0<\nu<1$ and $j=1,2,\ldots$*

$$s_j^2(A^*B)\leqslant s_j\left(\nu|A|^{\frac{2}{\nu}}+(1-\nu)U^*|B|^{\frac{2}{1-\nu}}U\right).$$

In particular, when $\nu=\frac{1}{2}$,

$$s_j^2(A^*B)\leqslant \frac{1}{2}s_j\left(|A|^4+U^*|B|^4U\right).$$

REMARK 4. Notice that Corollary 2 provides a new upper bound for $s(A^*B)$. In this remark, we give a numerical example that shows that neither this new bound nor that in (1) is uniformly better than the other. For this, let

$$A=\begin{bmatrix}4&3\\4&4\end{bmatrix},\quad B=\begin{bmatrix}3&5\\2&2\end{bmatrix}.$$

Then it can be checked that $U \approx \begin{bmatrix} -0.066519 & 0.997785 \\ 0.997785 & 0.066519 \end{bmatrix}$ is the unitary part in the polar decomposition of B^*A. Furthermore,

$$s^2(A^*B) \approx \{2001.87, 0.12788\}, \quad \left(\frac{1}{2}s\left(|A^*|^2 + |B^*|^2\right)\right)^2 \approx \{2233.22, 5.0313\}$$

and

$$\frac{1}{2}s\left(|A|^4 + U^*|B|^4U\right) \approx \{2471.59, 2.91099\}.$$

From this, it is evident that neither bound is uniformly better.

REMARK 5. In [7], it is shown that if A and B are two $n \times n$ complex matrices, then for each $j = 1, \dots, n$, one has

$$s_j(AB) \leqslant \frac{1}{4}s_j(A+B)^2. \tag{9}$$

This inequality was posed as an open question in [4]. In this remark, we show that the bound found in Corollary 2 can provide better estimates in some examples than (9). For this, let

$$A = \begin{bmatrix} 14 & 2 & 0 \\ 2 & 17 & -7 \\ 0 & -7 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & -2 & -12 \\ -2 & 25 & -10 \\ -12 & -10 & 24 \end{bmatrix}.$$

Then it can be seen that $A, B \geqslant \mathbf{O}$, and that

$$\begin{aligned} s^2(AB) &\approx \{460672, 29462.3, 0.00198003\} \\ \frac{1}{2}s\left(|A|^4 + U^*|B|^4U\right) &\approx \{976015, 117147, 19.2013\} \\ \left(\frac{1}{4}s(A+B)^2\right)^2 &\approx \{553999, 43996.3, 321.698\}. \end{aligned}$$

From this example, we see that (9) is indeed better than Corollary 2 for $j = 1, 2$, while Corollary 2 provides a better estimate than (9) when $j = 3$, showing that neither bound is uniformly better than the other.

We point out here that in the above calculations,

$$U \approx \begin{bmatrix} 0.578583 & 0.3203 & 0.7501 \\ -0.393009 & 0.915341 & -0.0877155 \\ -0.714692 & -0.244045 & 0.655482 \end{bmatrix}.$$

REMARK 6. In [5, Lemma 2.5], it has been shown that if $A, B \in \mathbb{B}(\mathbb{H})$ are positive, then

$$|BA| \leqslant \frac{1}{2}\left(A^2 + VB^2V^*\right), \tag{10}$$

where V is the partial isometry in the polar decomposition $BA = V^*|BA|$. In fact, the proof of [5, Lemma 2.5] can be modified a little to prove that

$$|B^*A| \leqslant \frac{1}{2}\left(A^*A + VB^*BV^*\right) = \frac{1}{2}\left(|A|^2 + V|B|^2V^*\right), \tag{11}$$

where V is the partial isometry in the polar decomposition $B^*A = V^*|B^*A|$. This follows noting the positivity of the form (see the proof of [5, Lemma 2.5])

$$\begin{bmatrix} I & -V \end{bmatrix} \begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} \begin{bmatrix} I \\ -V^* \end{bmatrix}.$$

It is clear that (11) is stronger than the second inequality in Corollary 2.

An extension of Corollary 2 can be stated as follows.

PROPOSITION 2. *Let $A,B,X \in \mathbb{K}(\mathbb{H})$. If $B^*X^*A = U|B^*X^*A|$ is the polar decomposition of B^*X^*A, then for any $0 < t, \nu < 1$ and $j = 1,2,\ldots,$*

$$s_j^2(A^*XB) \leqslant s_j\left(\nu\left||X^*|^{1-t}A\right|^{\frac{2}{\nu}} + (1-\nu)U^*\left||X|^tB\right|^{\frac{2}{1-\nu}}U\right).$$

In particular,

$$s_j^2(A^*XB) \leqslant \frac{1}{4}s_j\left(\left||X^*|^{\frac{1}{2}}A\right|^4 + U^*\left||X|^{\frac{1}{2}}B\right|^4U\right).$$

Proof. Let $X = V|X|$ be the polar decomposition of X. Utilizing Corollary 2, one can write

$$\begin{aligned}
&s_j^2(A^*XB)\\
&= s_j^2(A^*V|X|B)\\
&= s_j^2\left(A^*V|X|^{1-t}|X|^tB\right)\\
&= s_j^2\left(\left(|X|^{1-t}V^*A\right)^*\left(|X|^tB\right)\right)\\
&\leqslant s_j\left(\nu\left(A^*V|X|^{2(1-t)}V^*A\right)^{\frac{1}{\nu}} + (1-\nu)\left(U^*B^*|X|^{2t}BU\right)^{\frac{1}{1-\nu}}\right)\\
&= s_j\left(\nu\left(A^*|X^*|^{2(1-t)}A\right)^{\frac{1}{\nu}} + (1-\nu)U^*\left(B^*|X|^{2t}B\right)^{\frac{1}{1-\nu}}U\right) \quad \text{(by [8, p. 58])}\\
&= s_j\left(\nu\left||X^*|^{1-t}A\right|^{\frac{2}{\nu}} + (1-\nu)U^*\left||X|^tB\right|^{\frac{2}{1-\nu}}U\right),
\end{aligned}$$

as required. □

REMARK 7. In [19], it is shown that for $j = 1,\ldots,n$,

$$s_j(A^*XB) \leqslant \frac{1}{2}s_j\left((AA^* + BB^*)^{\frac{1}{2}}X(AA^* + BB^*)^{\frac{1}{2}}\right), \tag{12}$$

where A,B,X are $n\times n$ complex matrices such that $X\geqslant \mathbf{O}$. Notice that Proposition 2 provides a new upper bound for $s_j(A^*XB)$. In this remark, we give a numerical example that shows that neither (12) nor Proposition 2 is uniformly better than the other. Indeed, if we let

$$A=\begin{bmatrix}1 & 0 & -1\\ 2 & 0 & -1\\ 2 & 2 & 1\end{bmatrix},\quad B=\begin{bmatrix}-2 & 2. & -1\\ 2 & 2 & 2\\ -1 & 2 & -2\end{bmatrix},\quad X=\begin{bmatrix}1 & 0 & 1\\ 0 & 5 & -2\\ 1 & -2 & 9\end{bmatrix}$$

then numerical calculations show that

$$\begin{aligned} s^2(A^*XB) &\approx \{7538.58, 1095.3, 0.123055\}\\ \left(\frac{1}{2}s\left((AA^*+BB^*)^{\frac{1}{2}}X(AA^*+BB^*)^{\frac{1}{2}}\right)\right)^2 &\approx \{9055.84, 1474.72, 5.9325\}\\ \frac{1}{4}s\left(\left||X^*|^{\frac{1}{2}}A\right|^4+U^*\left||X|^{\frac{1}{2}}B\right|^4U\right) &\approx \{10730, 1706.09, 2.87901\}, \end{aligned}$$

showing that the bound found in Proposition 2 can be better than that in (12). However, neither bound is uniformly better than the other.

3. Some norm inequalities

In this section, we present some norm bounds as applications of the results we found earlier.

THEOREM 4. *Let $A,B,C\in\mathbb{K}(\mathbb{H})$ be such that $A,B\geqslant O$, and let $C=U|C|$ be the polar decomposition of C. If $\begin{bmatrix}A & C^*\\ C & B\end{bmatrix}$ is a positive operator, then*

$$\|C\|\leqslant\frac{1}{2}\sqrt{\|A^2+U^*B^2U\|+2\|\Re(AU^*BU)\|}.$$

Proof. It follows from (8) that for all unit vectors $x\in\mathbb{H}$,

$$\begin{aligned} \left\| |C|^{\frac{1}{2}}x\right\|^4 &\leqslant \langle Ax,x\rangle\langle U^*BUx,x\rangle\\ &\leqslant\left(\frac{\langle Ax,x\rangle+\langle U^*BUx,x\rangle}{2}\right)^2\\ &\quad\text{(by the arithmetic-geometric mean inequality)}\\ &=\frac{1}{4}\langle (A+U^*BU)x,x\rangle^2\\ &\leqslant\frac{1}{4}\left\langle (A+U^*BU)^2x,x\right\rangle\\ &\quad\text{(by the Hölder-McCarthy inequality)}\\ &=\frac{1}{4}\left\langle\left(A^2+(U^*BU)^2+2\Re(AU^*BU)\right)x,x\right\rangle \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\left\langle \left(A^2 + U^*B^2U + 2\Re\left(AU^*BU\right)\right)x, x\right\rangle \\
&\leqslant \frac{1}{4}\left\|A^2 + U^*B^2U + 2\Re\left(AU^*BU\right)\right\| \\
&\leqslant \frac{1}{4}\left\|A^2 + U^*B^2U\right\| + \frac{1}{2}\left\|\Re\left(AU^*BU\right)\right\| \\
&\quad \text{(by the triangle inequality for the spectral norm).}
\end{aligned}$$

That is, for any unit vector $x \in \mathbb{H}$,

$$\left\| \, |C|^{\frac{1}{2}}x \right\|^4 \leqslant \frac{1}{4}\left\|A^2 + U^*B^2U\right\| + \frac{1}{2}\left\|\Re\left(AU^*BU\right)\right\|.$$

We get the desired result by taking the supremum over all unit vectors $x \in \mathbb{H}$. □

PROPOSITION 3. *Let $A, B, C \in \mathbb{K}(\mathbb{H})$ be such that $A, B \geqslant O$, and let $C = U|C|$ be the polar decomposition of C. If $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator, then*

$$\|C\| \leqslant \frac{1}{2}\sqrt{\|A\|\,\|B\| + \|AU^*BU\| + 2\left\|\Re\left(AU^*BU\right)\right\|}.$$

Proof. We know that [13, Lemma 1.5] if $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \geqslant \mathbf{O}$, then $\begin{bmatrix} tA & C^* \\ C & \frac{1}{t}B \end{bmatrix} \geqslant \mathbf{O}$ for all $t > 0$. So $\begin{bmatrix} \sqrt{\frac{\|B\|}{\|A\|}}A & C^* \\ C & \sqrt{\frac{\|A\|}{\|B\|}}B \end{bmatrix} \geqslant \mathbf{O}$, whenever $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \geqslant \mathbf{O}$. Theorem 4 implies

$$\begin{aligned}
\|C\| &\leqslant \frac{1}{2}\sqrt{\left\|\frac{\|B\|}{\|A\|}A^2 + \frac{\|A\|}{\|B\|}U^*B^2U\right\| + 2\left\|\Re\left(AU^*BU\right)\right\|} \\
&\leqslant \frac{1}{2}\sqrt{\max\left\{\frac{\|B\|}{\|A\|}\left\|A^2\right\|, \frac{\|A\|}{\|B\|}\left\|U^*B^2U\right\|\right\} + \left\|A\left(U^*B^2U\right)^{\frac{1}{2}}\right\| + 2\left\|\Re\left(AU^*BU\right)\right\|} \\
&= \frac{1}{2}\sqrt{\max\left\{\frac{\|B\|}{\|A\|}\left\|A^2\right\|, \frac{\|A\|}{\|B\|}\left\|\left(U^*BU\right)^2\right\|\right\} + \|AU^*BU\| + 2\left\|\Re\left(AU^*BU\right)\right\|} \\
&= \frac{1}{2}\sqrt{\max\left\{\frac{\|B\|}{\|A\|}\|A\|^2, \frac{\|A\|}{\|B\|}\|U^*BU\|^2\right\} + \|AU^*BU\| + 2\left\|\Re\left(AU^*BU\right)\right\|} \\
&= \frac{1}{2}\sqrt{\|A\|\,\|B\| + \|AU^*BU\| + 2\left\|\Re\left(AU^*BU\right)\right\|},
\end{aligned}$$

as required. □

We derive the following result from Proposition 3.

COROLLARY 3. *Let $A,B \in \mathbb{K}(\mathbb{H})$. If $B^*A = U|B^*A|$ is the polar decomposition of A^*B, then*

$$\|A^*B\| \leqslant \frac{1}{2}\sqrt{\|A\|^2\|B\|^2 + \left\||A|^2U^*|B|^2U\right\| + 2\left\|\Re\left(|A|^2U^*|B|^2U\right)\right\|}.$$

REMARK 8. The advantage of Corollary 3 is that it refines the well-known sub-multiplicative property of the spectral norm. More precisely, we have

$$\begin{aligned}\|A^*B\| &\leqslant \frac{1}{2}\sqrt{\|A\|^2\|B\|^2 + \left\||A|^2U^*|B|^2U\right\| + 2\left\|\Re\left(|A|^2U^*|B|^2U\right)\right\|}\\ &\leqslant \frac{1}{2}\sqrt{\|A\|^2\|B\|^2 + 3\left\||A|^2U^*|B|^2U\right\|}\\ &\quad (\text{since } \|\Re X\| \leqslant \|X\| \text{ for any } X \in \mathbb{B}(\mathbb{H}))\\ &\leqslant \|A\|\,\|B\|.\end{aligned}$$

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