Mathematical
Inequalities
& Applications
Volume 27, Number 4 (2024), 775–787 doi:10.7153/mia-2024-27-55

GENERALIZED AMOS–TYPE BOUNDS FOR MODIFIED BESSEL FUNCTION RATIOS

KURT HORNIK AND BETTINA GRÜN*

(*Communicated by S. Varošanec*)

Abstract. Amos-type and generalized Amos-type bounds have been established in the literature as lower and upper bounds for the modified Bessel function ratio $R_\nu(t) = I_{\nu+1}(t)/I_\nu(t)$ for $t > 0$. We complement previous results by providing a family of improved explicit lower bounds of the generalized Amos-type given by $G_{\alpha,\beta,\lambda}(t) = t/(\alpha + \sqrt{\lambda t^2 + \beta^2})$. We show that the difference of two such bounds has a single sign change, and that for every $t > 0$ the optimal bound can easily be determined. We also show that the upper bound for the modified Bessel function ratio established by Amos cannot be improved by considering $\lambda > 0$ instead of fixing $\lambda = 1$.

1. Introduction and overview

The (modified) Bessel function ratios $R_\nu(t) = I_{\nu+1}(t)/I_\nu(t)$, where I_ν is the modified Bessel function of the first kind of order ν, have received attention due to their occurrence in different areas of application such as statistics [7] and finite elasticity [9]. Amos [1] introduced lower and upper bounds for $R_\nu(t)$ on $(0,\infty)$ in terms of $G_{\alpha,\beta,\lambda}(t) = t/A_{\alpha,\beta,\lambda}(t)$ with $\lambda = 1$, where

$$A_{\alpha,\beta,\lambda}(t) = \alpha + \sqrt{\lambda t^2 + \beta^2}. \tag{1}$$

In what follows the parameters are always restricted to $\lambda > 0$ and (without loss of generality) to $\beta \geqslant 0$. Different variants of such "Amos-type" bounds $A_{\alpha,\beta,\lambda}$ with $\lambda = 1$ for R_ν were established in several references (e.g., [2, 4, 5, 6, 8, 10, 11, 13]), and these bounds were also further characterized in detail [3]. The attractiveness of these bounds stems from the fact that they allow both for explicit inversion and integration thus yielding bounds for R_ν^{-1} and the antiderivative of R_ν (equivalently, I_ν and its logarithm).

Yang and Zheng [12] derived new "generalized" Amos-type bounds (with λ not necessarily equal to one) for $W_\nu(t) = t/R_\nu(t)$ in terms of $A_{\alpha,\beta,\lambda}(t)$. (Clearly, when there are no sign changes in $R_\nu(t)$ and $A_{\alpha,\beta,\lambda}(t)$, there is a one-to-one correspondence between such bounds and bounds for $R_\nu(t)$ in terms of $G_{\alpha,\beta,\lambda}(t)$.) Note that as $t \to \infty$,

Mathematics subject classification (2020): 33C10, 39B62.
Keywords and phrases: Bounds, inequalities, modified Bessel function of the first kind, modified Bessel function ratio.

* Corresponding author.

$W_\nu(t)/t = 1/R_\nu(t) \to 1$ and $A_{\alpha,\beta,\lambda}(t)/t \to \sqrt{\lambda}$. Hence, upper bounds of $W_\nu(t)$ in terms of $A_{\alpha,\beta,\lambda}(t)$ need $\lambda \geqslant 1$, and lower bounds need $\lambda \leqslant 1$.

Write

$$\kappa = 2(\nu+1), \qquad \beta_\nu(\alpha) = \kappa - \alpha = 2(\nu+1) - \alpha. \tag{2}$$

Yang and Zheng [12, Proposition 4.4] show that the (uniformly) best upper bounds for $\alpha \leqslant \nu - 1$ have $\alpha = \nu - 1$, $\beta = \beta_\nu(\alpha) = \nu + 3$ and $\lambda = (\nu+3)/(\nu+2)$, and for $\alpha \geqslant \nu + 1/2$ have, $\alpha = \nu + 1/2$, $\beta = \beta_\nu(\alpha) = \nu + 3/2$, and $\lambda = 1$. One can easily show that as $W_\nu(t) \to \kappa$ as $t \to 0+$ (e.g., [3, Lemma 1]); hence, only bounds $A_{\alpha,\beta,\lambda}$ with $\beta = \beta_\nu(\alpha)$ are sharp at zero. For the intermediate range $\nu - 1 < \alpha < \nu + 1/2$, Yang and Zheng [12] give an implicit characterization of the best upper bounds, and ([12, Corollary 4.13]) provide the explicit bounds $W_\nu(t) < U^{YZ}_{\nu,\alpha}(t)$ for all $t > 0$, where

$$U^{YZ}_{\nu,\alpha}(t) = A_{\alpha,\beta_\nu(\alpha),\lambda^{YZ}_\nu}(t), \qquad \lambda^{YZ}_\nu(\alpha) = \frac{4\nu + 5 - 2\alpha}{2\nu + 4}.$$

In this paper we improve these best known explicit bounds. Write

$$\lambda_\nu(\alpha) = \frac{\beta_\nu(\alpha)}{\alpha - 1 + 2\sqrt{\beta_\nu(\alpha) - \alpha}} = \frac{2\nu + 2 - \alpha}{\alpha - 1 + 2\sqrt{2(\nu + 1 - \alpha)}} \tag{3}$$

and

$$U_{\nu,\alpha}(t) = A_{\alpha,\beta_\nu(\alpha),\lambda_\nu(\alpha)}(t). \tag{4}$$

THEOREM 1. *Let* $\nu > -3/2$. *Then for all* $t \geqslant 0$ *and* $\nu - 1 \leqslant \alpha \leqslant \nu + 1/2$, $W_\nu(t) \leqslant U_{\nu,\alpha}(t)$. *For* $\nu - 1 < \alpha < \nu + 1/2$, $\lambda_\nu(\alpha) < \lambda^{YZ}_\nu(\alpha)$ *and hence for all* $t > 0$, $U_{\nu,\alpha}(t) < U^{YZ}_{\nu,\alpha}(t)$.

The new improved bounds are mutually incomparable over $(0,\infty)$, with their differences having exactly one sign change, and the function $\alpha \mapsto -U_{\nu,\alpha}(t)$ is unimodal for all $t > 0$. More precisely, we have the following. Write

$$\Phi_\nu(u) = \frac{u(2-u)(\kappa + u^2)^2}{4(u-1)^2(u+\kappa)^2}(\kappa - 2 + 4u - u^2)^2, \qquad 1 \leqslant u \leqslant 2 \tag{5}$$

and

$$u_\nu(\alpha) = \sqrt{\kappa - 2\alpha}, \qquad \nu - 1 \leqslant \alpha \leqslant \nu + 1/2. \tag{6}$$

THEOREM 2. *Let* $\nu > -3/2$ *and* $\nu - 1 \leqslant \alpha_1 < \alpha_2 \leqslant \nu + 1/2$, *and write*

$$t^*_{\nu,\alpha_1,\alpha_2} = 2\frac{\sqrt{(\alpha_2 - \alpha_1)(\lambda_\nu(\alpha_2)\beta_\nu(\alpha_1) - \lambda_\nu(\alpha_1)\beta_\nu(\alpha_2))}}{\lambda_\nu(\alpha_1) - \lambda_\nu(\alpha_2)}. \tag{7}$$

(a) $U_{\nu,\alpha_1}(t) < U_{\nu,\alpha_2}(t)$ *for all* $0 < t < t^*_{\nu,\alpha_1,\alpha_2}$, *and* $U_{\nu,\alpha_1}(t) > U_{\nu,\alpha_2}(t)$ *for all* $t > t^*_{\nu,\alpha_1,\alpha_2}$.

(b) The functions $\alpha \mapsto t^*_{\nu,\alpha_1,\alpha}$ *and* $\alpha \mapsto t^*_{\nu,\alpha,\alpha_2}$ *are increasing on* $(\alpha_1, \nu + 1/2]$ *and, respectively,* $[\nu - 1, \alpha_2)$.

(c) We have

$$\lim_{\alpha_1 \to \alpha-} t^*_{\nu,\alpha_1,\alpha} = \lim_{\alpha_2 \to \alpha+} t^*_{\nu,\alpha,\alpha_2} = \sqrt{\Phi_\nu(u_\nu(\alpha))}. \tag{8}$$

with the limit increasing in α over $(\nu-1, \nu+1/2)$.

THEOREM 3. *Let $\nu > -3/2$ and $t > 0$.*

(a) The function $\alpha \mapsto U_{\nu,\alpha}(t)$ attains its minimum over $[\nu-1, \nu+1/2]$ at its only critical point $\alpha^*_\nu(t)$ in $(\nu-1, \nu+1/2)$, and is decreasing for $\nu-1 \leqslant \alpha < \alpha^*_\nu(t)$ and increasing for $\alpha^*_\nu(t) < \alpha \leqslant \nu+1/2$.

(b) $\alpha^*_\nu(t)$ solves the equation $t^2 = \Phi_\nu(u_\nu(\alpha^*_\nu(t)))$.

(c) The function $t \mapsto \alpha^*_\nu(t)$ is increasing over $(0,\infty)$, with $\lim_{t\to 0+} \alpha^*_\nu(t) = \nu - 1$ and $\lim_{t\to\infty} \alpha^*_\nu(t) = \nu + 1/2$.

Therefore, for every $t > 0$ the optimal upper bound $\min_{\nu-1\leqslant\alpha\leqslant\nu+1/2} U_{\nu,\alpha}(t) = U_{\nu,\alpha^*_\nu(t)}(t)$ can conveniently be found by direct minimization via golden search or bisection to find the unique critical point (or less practically, via solving $\Phi_\nu(u) = t^2$, which is a polynomial equation of degree 6 in u).

For $\nu > -1$ and $t > 0$, $R_\nu(t) > 0$ and $A_{\alpha,\beta_\nu(\alpha),\lambda}(t) > A_{\alpha,\beta_\nu(\alpha),\lambda}(0) = \kappa > 0$, giving the following lower generalized Amos-type bounds for R_ν.

COROLLARY 1. *Let $\nu \geqslant -1$. Then for all $t > 0$ and $\nu-1 \leqslant \alpha \leqslant \nu+1/2$, $R_\nu(t) \geqslant G_{\alpha,\beta_\nu(\alpha),\lambda_\nu(\alpha)}(t)$.*

Hornik & Grün [3, Theorem 3] show that $\alpha = \nu+1/2$ and $\beta = \beta_\nu(\alpha) = \nu+3/2$ gives the uniformly best lower Amos-type bound $G_{\alpha,\beta,1}$ for R_ν. Using Theorem 3 (note that $\lambda_\nu(\nu+1/2) = 1$) we can see that $G_{\nu+1/2,\nu+3/2,1}(t) < G_{\alpha,\beta_\nu(\alpha),\lambda_\nu(\alpha)}(t)$ for all $t > 0$ and $\nu-1 \leqslant \alpha < \nu+1/2$, again illustrating the fact that the generalized Amos-type bounds with $\beta = \beta_\nu(\alpha)$ can successfully be employed for obtaining improved lower bounds for R_ν.

Amos [1] also established that for $\nu \geqslant 0$ and all $t \geqslant 0$, $R_\nu(t) \leqslant G_{\nu,\nu+2,1}(t)$, or equivalently, $W_\nu(t) \geqslant A_{\nu,\nu+2,1}(t)$. Noting that $\nu+2 = \beta_\nu(\nu)$, it is of interest whether this can be improved by new generalized lower bounds $A_{\alpha,\beta,\lambda}$ for W_ν with $\beta = \beta_\nu(\alpha)$. However, this is not the case. We have the following:

THEOREM 4. *Let $\nu > -2$ and $\alpha < 2(\nu+1)$. Then $W_\nu(t) \geqslant A_{\alpha,\beta_\nu(\alpha),\lambda}(t)$ for all $t \geqslant 0$ if and only if $\lambda \leqslant \min(\beta_\nu(\alpha)/(\nu+2), 1)$, and for all such λ, $A_{\nu,\nu+2,1}(t) > A_{\alpha,\beta_\nu(\alpha),\lambda}(t)$ for all $t > 0$ unless $\alpha = \nu$ and $\lambda = 1$.*

Thus, $A_{\nu,\nu+2,1}$ is the uniformly best generalized Amos-type lower bound for W_ν of the form $A_{\alpha,\beta_\nu(\alpha),\lambda}$.

2. Lemmas

To prove the results, we first establish several lemmas.

LEMMA 1. *Let* $\nu > -2$, $\beta > 0$, $\alpha + \beta = \kappa$. *Then if* $\lambda > \beta/(\nu+2)$ ($\lambda < \beta/(\nu+2)$), $W_\nu(t) < A_{\alpha,\beta,\lambda}(t)$ ($W_\nu(t) > A_{\alpha,\beta,\lambda}(t)$) *for all* $t > 0$ *sufficiently small.*

Proof. As $t \to 0$, using Lemma 1 and Lemma 2 in [3],

$$W_\nu(t) = \kappa + \frac{t^2}{2(\nu+2)} + O(t^4), \qquad A_{\alpha,\beta,\lambda}(t) = (\alpha+\beta) + \frac{\lambda t^2}{2\beta} + O(t^4)$$

(note that the reference writes v_ν instead of W_ν), from which the lemma immediately follows. □

LEMMA 2. *Let* $\nu > -3/2$ *and* $\alpha < \nu + 1/2$. *Then* $\beta_\nu(\alpha) > \max(\alpha, 0)$.

Proof. For $\alpha < \nu + 1/2$, $\beta_\nu(\alpha) = 2\nu + 2 - \alpha > \nu + 3/2$ which is positive for $\nu > -3/2$, and $\beta_\nu(\alpha) - \alpha = 2(\nu + 1 - \alpha) > 1$. □

LEMMA 3. *For all* ν, *the transformation* $\alpha \mapsto u_\nu(\alpha)$ *is decreasing from* $[\nu-1, \nu+1/2]$ *onto* $[1,2]$, *with inverse* $\alpha_\nu(u) = (\kappa - u^2)/2$.

Proof. Clearly, as α increases from $\nu - 1$ to $\nu + 1/2$, $\kappa - 2\alpha = 2(\nu + 1 - \alpha)$ decreases from 4 to 1, and hence $u_\nu(\alpha)$ decreases from 2 to 1. The expression for the inverse is immediate. □

LEMMA 4. *Let* $\nu > -3/2$. *Write* $D_\nu(\alpha) = \alpha - 1 + 2\sqrt{\beta_\nu(\alpha) - \alpha}$. *Then as* α *increases from* $\nu - 1$ *to* $\nu + 1/2$, $D_\nu(\alpha)$ *decreases from* $\nu + 2$ *to* $\nu + 3/2$, *and* $\lambda_\nu(\alpha)$ *decreases from* $(\nu+3)/(\nu+2)$ *to 1.*

Proof. We have

$$D_\nu(\alpha_\nu(u)) = \frac{\kappa - u^2}{2} - 1 + 2u = \frac{\kappa - 2 + 4u - u^2}{2}$$

and, as $\kappa - \alpha_\nu(u) = \kappa - (\kappa - u^2)/2 = (\kappa + u^2)/2$,

$$\lambda_\nu(\alpha_\nu(u)) = \frac{\kappa - \alpha_\nu(u)}{D_\nu(\alpha_\nu(u))} = \frac{\kappa + u^2}{\kappa - 2 + 4u - u^2}.$$

The function $u \mapsto Q(u) = \kappa - 2 + 4u - u^2$ has derivative $4 - 2u$ which is positive for $1 \leqslant u < 2$. Thus, as u increases from 1 to 2, Q increases from $Q(1) = \kappa + 1 = 2\nu + 3$ (which is positive for $\nu > -3/2$) to $Q(2) = \kappa + 2 = 2\nu + 4$, and using Lemma 3, as α increases from $\nu - 1$ to $\nu + 1/2$, $D_\nu(\alpha)$ decreases from $\nu + 2$ to $\nu + 3/2$. Next,

$$\frac{d}{du} \frac{\kappa + u^2}{Q(u)} = \frac{2uQ(u) - (\kappa + u^2)(4 - 2u)}{Q(u)^2},$$

where the numerator equals $4(u-1)(u+\kappa)$ and hence is positive for $1 < u \leqslant 2$. Thus, $u \mapsto \lambda_\nu(\alpha_\nu(u))$ is increasing for $1 \leqslant u \leqslant 2$, and again using Lemma 3, $\alpha \mapsto \lambda_\nu(\alpha)$ is decreasing for $\nu - 1 \leqslant \alpha \leqslant \nu + 1/2$. As clearly

$$\lambda_\nu(\alpha_\nu(1)) = \frac{\kappa+1}{\kappa+1} = 1, \qquad \lambda_\nu(\alpha_\nu(2)) = \frac{\kappa+4}{\kappa+2} = \frac{\nu+3}{\nu+2},$$

the proof is complete. □

LEMMA 5. *Let $\nu > -3/2$. For $\nu - 1 \leqslant \alpha \leqslant \nu + 1/2$,*

$$\frac{2(\nu+1)-\alpha}{\nu+2} \leqslant \lambda_\nu(\alpha) \leqslant \frac{4\nu+5-2\alpha}{2\nu+4} \tag{9}$$

where the first inequality is strict unless $\alpha = \nu - 1$, and the second inequality is strict unless $\alpha = \nu + 1/2$.

Proof. Again, it helps to substitute $\alpha = \alpha_\nu(u)$. We have

$$\frac{2(\nu+1)-\alpha_\nu(u)}{\nu+2} = \frac{(\kappa+u^2)/2}{\nu+2} = \frac{\kappa+u^2}{\kappa+2}$$

and

$$\frac{4\nu+5-2\alpha_\nu(u)}{2\nu+4} = \frac{2\kappa+1-(\kappa-u^2)}{\kappa+2} = \frac{\kappa+1+u^2}{\kappa+2},$$

so the assertions are equivalent to

$$\frac{\kappa+u^2}{\kappa+2} \leqslant \frac{\kappa+u^2}{\kappa-2+4u-u^2} \leqslant \frac{\kappa+1+u^2}{\kappa+2}$$

for $1 \leqslant u \leqslant 2$, with the first inequality strict unless $u = 2$, and the second strict unless $u = 1$. Note that $\kappa - 2 + 4u - u^2 > 0$ by Lemma 4. The first inequality holds iff $\kappa + 2 \geqslant \kappa - 2 + 4u - u^2$, or equivalently, $0 \leqslant u^2 - 4u + 4 = (u-2)^2$, which indeed holds for all u and strictly so unless $u = 2$. With $C(u) = u^3 - 3u^2 + (\kappa+2)u - (3\kappa+2)$, the second inequality is equivalent to

$$0 \leqslant (\kappa+1+u^2)(\kappa-2+4u-u^2) - (\kappa+2)(\kappa+u^2) = (1-u)C(u).$$

As $C''(u) = 6(u-1)$, $u = 1$ is the inflection point of C, and C cannot have a local maximum for $u > 1$. Thus, $\max_{1 \leqslant u \leqslant 2} C(u) = \max(C(1), C(2)) = \max(-2\kappa-2, -\kappa-2)$. Thus if $\nu > -3/2$, $\kappa > -1$, so that for $1 \leqslant u \leqslant 2$, $C(u) < 0$ and $(1-u)C(u) \geqslant 0$ with strict inequality unless $u = 1$. □

Let

$$Q_{\alpha,\beta,\lambda}(s) = (1-\lambda)s^2 + (\beta - (\alpha+1)\lambda)s - \beta\lambda \tag{10}$$

and for $\lambda > 1$, write

$$M_{\alpha,\beta}(\lambda) = \max_{-\infty < s < \infty} Q_{\alpha,\beta,\lambda}(s). \tag{11}$$

LEMMA 6. *Let* $\alpha < \beta$ *satisfy* $\beta > 0$ *and* $0 < \alpha - 1 + 2\sqrt{\beta - \alpha} < \beta$ *so that* $\lambda_{\alpha,\beta} := \beta/(\alpha - 1 + 2\sqrt{\beta - \alpha}) > 1$. *Then* $M_{\alpha,\beta}(\lambda_{\alpha,\beta}) = 0$ *and* $M'_{\alpha,\beta}(\lambda_{\alpha,\beta}) < 0$.

Proof. If $\lambda > 1$, $Q_{\alpha,\beta,\lambda}$ is maximized at the unique critical point s_0 solving $2(1 - \lambda)s + (\beta - (\alpha + 1)\lambda) = 0$, so that $s_0 = (\beta - (\alpha + 1)\lambda)/(2(\lambda - 1))$, and $M_{\alpha,\beta}(\lambda) = Q_{\alpha,\beta,\lambda}(s_0) = N_{\alpha,\beta}(\lambda)/(4(\lambda - 1))$, where

$$N_{\alpha,\beta}(\lambda) = ((\alpha + 1)^2 - 4\beta)\lambda^2 - 2(\alpha - 1)\beta\lambda + \beta^2. \tag{12}$$

As $(\alpha + 1)^2 - 4\beta = (\alpha - 1 - 2\sqrt{\beta - \alpha})(\alpha - 1 + 2\sqrt{\beta - \alpha})$ we have

$$N_{\alpha,\beta}(\lambda) = ((\alpha - 1 - 2\sqrt{\beta - \alpha})\lambda - \beta)((\alpha - 1 + 2\sqrt{\beta - \alpha})\lambda - \beta)$$

so that $N_{\alpha,\beta}(\lambda_{\alpha,\beta}) = 0$, $N'_{\alpha,\beta}(\lambda_{\alpha,\beta}) = -4\beta\sqrt{\beta - \alpha}$ and finally

$$M'_{\alpha,\beta}(\lambda_{\alpha,\beta}) = \frac{N'_{\alpha,\beta}(\lambda_{\alpha,\beta})(\lambda_{\alpha,\beta} - 1) - N_{\alpha,\beta}(\lambda_{\alpha,\beta})}{4(\lambda_{\alpha,\beta} - 1)^2} = -\frac{\beta\sqrt{\beta - \alpha}}{\lambda_{\alpha,\beta} - 1} < 0,$$

as asserted. □

LEMMA 7. *Let* $\alpha + \beta = \kappa$ *and* $\Delta_{\alpha,\beta,\lambda}(t) = W_\nu(t) - A_{\alpha,\beta,\lambda}(t)$. *If* $\Delta_{\alpha,\beta,\lambda}(t) = 0$ *for some* $t > 0$, *then*

$$t\Delta'_{\alpha,\beta,\lambda}(t) = \frac{(s - \beta)Q_{\alpha,\beta,\lambda}(s)}{\lambda s}, \qquad s = \sqrt{\lambda t^2 + \beta^2}. \tag{13}$$

Proof. W_ν satisfies $tW'_\nu(t) = t^2 + \kappa W_\nu(t) - W_\nu(t)^2$ (e.g., [10], Equation (3)), and clearly

$$A'_{\alpha,\beta,\lambda}(t) = \frac{\lambda t}{\sqrt{\lambda t^2 + \beta^2}}.$$

Thus, if $\Delta_{\alpha,\beta,\lambda}(t) = 0$ for some $t > 0$, we have

$$\begin{aligned} t\Delta'_{\alpha,\beta,\lambda}(t) &= t^2 + (\alpha + \beta)W_\nu(t) - W_\nu(t)^2 - \frac{\lambda t^2}{\sqrt{\lambda t^2 + \beta^2}} \\ &= t^2 + (\alpha + \beta)A_{\alpha,\beta,\lambda}(t) - A_{\alpha,\beta,\lambda}(t)^2 - \frac{\lambda t^2}{\sqrt{\lambda t^2 + \beta^2}}. \end{aligned}$$

For $s = \sqrt{\lambda t^2 + \beta^2}$ we have $A_{\alpha,\beta,\lambda}(t) = \alpha + s$ and $t^2 = (s^2 - \beta^2)/\lambda$, and the above can be written as

$$\frac{s^2 - \beta^2}{\lambda} + (\alpha + \beta)(\alpha + s) - (\alpha + s)^2 - \frac{s^2 - \beta^2}{s} = \frac{s - \beta}{\lambda s} Q_{\alpha,\beta,\lambda}(s). \quad \square$$

LEMMA 8. *Let* $\beta_1, \beta_2, \lambda_1, \lambda_2 > 0$, *and write* $\Delta(t) = A_{\alpha_1,\beta_1,\lambda_1}(t) - A_{\alpha_2,\beta_2,\lambda_2}(t)$. *Then for* $t > 0$,

$$\frac{\Delta'(t)}{t} = \frac{\lambda_1\lambda_2(\lambda_1 - \lambda_2)t^2 + (\lambda_1^2\beta_2^2 - \lambda_2^2\beta_1^2)}{s_1 s_2(\lambda_1 s_2 + \lambda_2 s_1)}, \tag{14}$$

where $s_1 = \sqrt{\lambda_1 t^2 + \beta_1^2}$ *and* $s_2 = \sqrt{\lambda_2 t^2 + \beta_2^2}$.

Proof. We have

$$\begin{aligned}\frac{\Delta'(t)}{t} &= \frac{\lambda_1}{s_1} - \frac{\lambda_2}{s_2} \\ &= \frac{\lambda_1 s_2 - \lambda_2 s_1}{s_1 s_2}\frac{\lambda_1 s_2 + \lambda_2 s_1}{\lambda_1 s_2 + \lambda_2 s_1} \\ &= \frac{\lambda_1\lambda_2(\lambda_1 - \lambda_2)t^2 + (\lambda_1^2\beta_2^2 - \lambda_2^2\beta_1^2)}{s_1 s_2(\lambda_1 s_2 + \lambda_2 s_1)}. \quad \square\end{aligned}$$

3. Proofs

We can now prove the theorems.

Proof of Theorem 1. From the results of [12], we know that the theorem is correct for $\alpha = \nu - 1$ (where $\beta_\nu(\alpha) = \nu + 3$ and $\lambda_\nu(\alpha) = (\nu+3)/(\nu+2)$) and for $\alpha = \nu + 1/2$ (where $\beta_\nu(\alpha) = \nu + 3/2$ and $\lambda_\nu(\alpha) = 1$), so we may restrict our attention to $\nu - 1 < \alpha < \nu + 1/2$.

Let $\beta = \beta_\nu(\alpha) = \kappa - \alpha$. By Lemma 2, $\beta > \max(\alpha, 0)$, and by Lemma 4, $D_\nu(\alpha) = \alpha - 1 + 2\sqrt{\beta_\nu(\alpha) - \alpha} > \nu + 3/2 > 0$ and $\lambda_\nu(\alpha) = \beta_\nu(\alpha)/D_\nu(\alpha) > 1$ so that $\beta_\nu(\alpha) > D_\nu(\alpha) > 0$. Using Lemma 6, we find that for all $\lambda > \lambda_\nu(\alpha)$ sufficiently small, $M_{\alpha,\beta}(\lambda) < 0$. For such λ, $\Delta_{\alpha,\beta,\lambda}(t) = W_\nu(t) - A_{\alpha,\beta,\lambda}(t) < 0$ for all $t > 0$ sufficiently small by Lemma 1. If $\Delta_{\alpha,\beta,\lambda}(t) = 0$ for some $t > 0$, Lemma 7 yields that

$$\begin{aligned} t\Delta'_{\alpha,\beta,\lambda}(t) &= \frac{\sqrt{\lambda t^2 + \beta^2} - \beta}{\lambda\sqrt{\lambda t^2 + \beta^2}} Q_{\alpha,\beta,\lambda}(\sqrt{\lambda t^2 + \beta^2}) \\ &\leqslant \frac{\sqrt{\lambda t^2 + \beta^2} - \beta}{\lambda\sqrt{\lambda t^2 + \beta^2}} M_{\alpha,\beta}(\lambda) \\ &< 0, \end{aligned}$$

which is impossible for the first such root. Thus we must have $\Delta_{\alpha,\beta,\lambda}(t) < 0$ for all $t > 0$, and the first assertion of the theorem follows by taking the infimum over all sufficiently small $\lambda > \lambda_\nu(\alpha)$. The second assertion is immediate from Lemma 5. $\square$

Proof of Theorem 3 part one. Again, it will be convenient to substitute $\alpha = \alpha_\nu(u)$, and show that

$$u \mapsto H(u) = U_{\nu,\alpha_\nu(u)}(t)$$

attains its minimum over $[1,2]$ at its only critical point u^* in $(1,2)$, and is decreasing for $1 \leqslant u < u^*$ and increasing for $u^* < u \leqslant 2$.

Write $s = t^2$, $Q(u) = \kappa - 2 + 4u - u^2$, and

$$\lambda(u) = \frac{\kappa + u^2}{Q(u)}, \qquad \tau(u) = \left(\frac{\kappa + u^2}{2}\right)^2, \qquad S(u) = \sqrt{\lambda(u)s + \tau(u)}.$$

Then

$$H(u) = \frac{\kappa - u^2}{2} + S(u), \qquad H'(u) = -u + \frac{\lambda'(u)s + \tau'(u)}{2S(u)}.$$

Building on the derivations from the proof of Lemma 4 we obtain

$$\lambda'(u) = \frac{4(u-1)(u+\kappa)}{Q(u)^2}, \qquad Q(1) = \kappa + 1, \qquad Q(2) = \kappa + 2,$$

from which

$$S(1) = \sqrt{s + (\kappa+1)^2/4}, \qquad S(2) = \sqrt{\frac{\kappa+4}{\kappa+2}s + \left(\frac{\kappa+4}{2}\right)^2},$$

and clearly $\tau'(u) = u(\kappa + u^2)$. Hence,

$$H'(1) = -1 + \frac{\kappa+1}{2S(1)} = -1 + \frac{\kappa+1}{\sqrt{(\kappa+1)^2 + 4s}} < 0$$

and

$$\begin{aligned}
H'(2) &= -2 + \frac{4s/(\kappa+2) + 2(\kappa+4)}{2S(2)} \\
&= \frac{1}{S(2)}\left(\frac{2s}{\kappa+2} + (\kappa+4) - 2S(2)\right)\frac{\frac{2s}{\kappa+2} + (\kappa+4) + 2S(2)}{\frac{2s}{\kappa+2} + (\kappa+4) + 2S(2)} \\
&= \frac{1}{S(2)}\frac{\left(\frac{2s}{\kappa+2} + (\kappa+4)\right)^2 - 4\left(\frac{\kappa+4}{\kappa+2}s + \left(\frac{\kappa+4}{2}\right)^2\right)}{\frac{2s}{\kappa+2} + (\kappa+4) + 2S(2)} \\
&= \frac{1}{S(2)}\frac{\frac{4s^2}{(\kappa+2)^2}}{\frac{2s}{\kappa+2} + (\kappa+4) + 2S(2)} \\
&> 0
\end{aligned}$$

and the proof can be completed by showing that $H'(u) = 0$ implies $H''(u) > 0$.

Clearly,

$$\begin{aligned}
H''(u) &= -1 - \frac{(\lambda'(u)s + \tau'(u))^2}{4S(u)^3} + \frac{\lambda''(u)s + \tau''(u)}{2S(u)} \\
&= \frac{1}{S(u)}\left(-S(u) - \left(\frac{\lambda'(u)s + \tau'(u)}{2S(u)}\right)^2 + \frac{\lambda''(u)s + \tau''(u)}{2}\right)
\end{aligned}$$

and it suffices to show that the parenthesized expression is positive provided that $H'(u) = 0$. Now in that case,

$$\frac{\lambda'(u)s+\tau'(u)}{2S(u)} = u, \qquad S(u) = \frac{\lambda'(u)s+\tau'(u)}{2u}$$

and the parenthesized expression becomes

$$-\frac{\lambda'(u)s+\tau'(u)}{2u} - u^2 + \frac{\lambda''(u)s+\tau''(u)}{2}$$
$$= \frac{s}{2u}(u\lambda''(u)-\lambda'(u)) + \frac{1}{2u}(u\tau''(u)-2u^3-\tau'(u)). \quad (15)$$

For the second term, we have $\tau''(u) = \kappa + 3u^2$ and thus

$$u\tau''(u) - 2u^3 - \tau'(u) = u(\kappa+3u^2) - 2u^3 - u(\kappa+u^2) = 0.$$

Next,

$$\lambda''(u) = 4\frac{(2u+\kappa-1)Q(u)^2 - 2Q(u)Q'(u)(u-1)(u+\kappa)}{Q(u)^4}$$
$$= \frac{4}{Q(u)^3}\left(2u^3+3(\kappa-1)u^2-6\kappa u+\kappa^2+5\kappa+2\right)$$

from which, with $C(u)$ the parenthesized expression,

$$u\lambda''(u) - \lambda'(u) = \frac{4}{Q(u)^3}\left(uC(u) - (u-1)(u+\kappa)Q(u)\right) = \frac{4}{Q(u)^3}F_\kappa(u),$$

and it can easily be verified that

$$F_\kappa(u) = 3u^4 + (4\kappa-8)u^3 - (12\kappa-6)u^2 + 12\kappa u + \kappa^2 - 2\kappa.$$

As a function of κ, this has derivative

$$2\kappa + (4u^3 - 12u^2 + 12u - 2) = 2\kappa + 4(u-1)^3 + 2,$$

which, as by $\nu > -3/2$ we have $\kappa = 2(\nu+1) > -1$, is thus positive for $u \geqslant 1$. Hence, for all $1 \leqslant u \leqslant 2$,

$$F_\kappa(u) > F_{-1}(u) = 3u^4 - 12u^3 + 18u^2 - 12u + 3 = 3(u-1)^4 \geqslant 0.$$

Thus, for all $1 \leqslant u \leqslant 2$, $u\lambda''(u) - \lambda'(u) > 0$, establishing that $H'(u) = 0$ implies $H''(u) > 0$, and completing the proof of (a).

For (b), note that the equation for the critical point is

$$2u\sqrt{\lambda(u)s+\tau(u)} = \lambda'(u)s + \tau'(u).$$

Taking squares and rearranging,

$$\lambda'(u)^2 s^2 + (2\lambda'(u)\tau'(u) - 4u^2\lambda(u))s + (\tau'(u)^2 - 4u^2\tau(u)) = 0.$$

Writing this as $As^2 + Bs + C = 0$, we find that the constant term is

$$C = \tau'(u)^2 - 4u^2\tau(u) = u^2(\kappa + u^2)^2 - 4u^2\left(\frac{\kappa + u^2}{2}\right)^2 = 0,$$

so that at the critical point we must have $s = -B/A$. Now

$$B = 2\frac{4(u-1)(u+\kappa)}{Q(u)^2}u(\kappa + u^2) - 4u^2\frac{\kappa + u^2}{Q(u)} = \frac{4u(\kappa + u^2)}{Q(u)^2}(u-2)(\kappa + u^2)$$

so that

$$s = \frac{4u(2-u)(\kappa + u^2)^2}{Q(u)^2}\frac{Q(u)^4}{16(u-1)^2(u+\kappa)^2} = \Phi_\nu(u),$$

establishing (b).

The limits from (c) follow readily from the fact that $0 < \Phi_\nu(u) < \infty$ for $1 < u < 2$ with limits ∞ and 0 for $u \to 1+$ and $u \to 2-$. Monotonicity will be established following the proof of Theorem 2. □

Proof of Theorem 2. For $i \in \{1,2\}$, let $\beta_i = \beta_\nu(\alpha_i)$ and $\lambda_i = \lambda_\nu(\alpha_i)$. Clearly, $\beta_1 > \beta_2$. Using Lemma 4, $\lambda_1 > \lambda_2$ and $\beta_1/\lambda_1 = D_\nu(\alpha_1) > D_\nu(\alpha_2) = \beta_2/\lambda_2 > 0$. Consider the difference

$$\Delta(t) = U_{\nu,\alpha_1}(t) - U_{\nu,\alpha_2}(t) = A_{\alpha_1,\beta_1,\lambda_1}(t) - A_{\alpha_2,\beta_2,\lambda_2}(t).$$

Write $s = t^2$ and $\delta = \alpha_2 - \alpha_1$. Then

$$\Delta(t) = 0 \Leftrightarrow \sqrt{\lambda_1 s + \beta_1^2} - \sqrt{\lambda_2 s + \beta_2^2} = \delta.$$

Taking squares, it follows that

$$\delta^2 = \lambda_1 s + \beta_1^2 - 2\sqrt{(\lambda_1 s + \beta_1^2)(\lambda_2 s + \beta_2^2)} + \lambda_2 s + \beta_2^2$$

or equivalently,

$$2\sqrt{(\lambda_1 s + \beta_1^2)(\lambda_2 s + \beta_2^2)} = (\lambda_1 + \lambda_2)s + (\beta_1^2 + \beta_2^2 - \delta^2).$$

Taking squares again, we obtain a quadratic equation for s of the form $As^2 + Bs + C = 0$, where

$$A = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2$$

and, as $\beta_1 - \beta_2 = (\kappa - \alpha_1) - (\kappa - \alpha_2) = \delta$,

$$\begin{aligned} B &= 2(\lambda_1 + \lambda_2)(\beta_1^2 + \beta_2^2 - \delta^2) - 4(\lambda_1 \beta_2^2 + \lambda_2 \beta_1^2) \\ &= 4(\beta_1 - \beta_2)(\lambda_1 \beta_2 - \lambda_2 \beta_1) \\ &= 4(\beta_1 - \beta_2)\lambda_1 \lambda_2 \left(\frac{\beta_2}{\lambda_2} - \frac{\beta_1}{\lambda_1} \right) \\ &< 0. \end{aligned}$$

As $\Delta(0) = (\alpha_1 + \beta_1) - (\alpha_2 + \beta_2) = \kappa - \kappa = 0$, $s = 0$ must be one root of the quadratic equation. Hence, $C = 0$ and the other root is obtained as

$$s^* = -\frac{B}{A} = -\frac{4(\beta_1 - \beta_2)(\lambda_1 \beta_2 - \lambda_2 \beta_1)}{(\lambda_1 - \lambda_2)^2} = -4\frac{(\alpha_2 - \alpha_1)(\lambda_1 \beta_2 - \lambda_2 \beta_1)}{(\lambda_1 - \lambda_2)^2} > 0.$$

Using Lemma 8,

$$\frac{\Delta'(t)}{t} = \frac{\lambda_1}{\sqrt{\lambda_1 t^2 + \beta_1^2}} - \frac{\lambda_2}{\sqrt{\lambda_2 t^2 + \beta_2^2}} \to \frac{\lambda_1}{\beta_1} - \frac{\lambda_2}{\beta_2} < 0$$

as $t \to 0+$, so $\Delta(t) < 0$ for all $t > 0$ sufficiently small. On the other hand, as $t \to \infty$, $\Delta(t)/t \to \sqrt{\lambda_1} - \sqrt{\lambda_2} > 0$, so $\Delta(t) > 0$ for all $t > 0$ sufficiently large. Hence, at $t^* = \sqrt{s^*}$ Δ changes from negative to positive, establishing (a).

For (b), note that $U_{\nu,\alpha_1}(t^*_{\nu,\alpha_1,\alpha_2}) = U_{\nu,\alpha_2}(t^*_{\nu,\alpha_1,\alpha_2})$. Now take $\nu - 1 \leqslant \alpha_1 < \alpha < \alpha_2 \leqslant \nu + 1/2$. Theorem 3 (a) implies that $U_{\nu,\alpha_1}(t^*_{\nu,\alpha_1,\alpha_2}) > U_{\nu,\alpha}(t^*_{\nu,\alpha_1,\alpha_2})$. Using Theorem 2 (a) this gives $t^*_{\nu,\alpha_1,\alpha_2} > t^*_{\nu,\alpha_1,\alpha}$. Similarly, we must have $U_{\nu,\alpha_2}(t^*_{\nu,\alpha_1,\alpha_2}) > U_{\nu,\alpha}(t^*_{\nu,\alpha_1,\alpha_2})$ and thus $t^*_{\nu,\alpha_1,\alpha_2} < t^*_{\nu,\alpha,\alpha_2}$. Hence indeed, the functions $\alpha \mapsto t^*_{\nu,\alpha_1,\alpha}$ and $\alpha \mapsto t^*_{\nu,\alpha,\alpha_2}$ are increasing on $(\alpha_1, \nu + 1/2]$ and, respectively, $[\nu - 1, \alpha_2)$.

For (c), parametrize $\alpha_i = \alpha_\nu(u_i)$. Then, writing $\beta(u) = \beta_\nu(\alpha_\nu(u))$ and $\lambda(u) = \lambda_\nu(\alpha_\nu(u))$,

$$\begin{aligned} (t^*_{\nu,\alpha_1,\alpha_2})^2 &= 4\frac{(u_1^2 - u_2^2)/2}{(\lambda(u_1) - \lambda(u_2))^2}\lambda(u_1)\lambda(u_2)\left(\frac{\beta(u_1)}{\lambda(u_1)} - \frac{\beta(u_2)}{\lambda(u_2)}\right) \\ &= 2\lambda(u_1)\lambda(u_2)\frac{\frac{u_1^2 - u_2^2}{u_1 - u_2}}{\left(\frac{\lambda(u_1) - \lambda(u_2)}{u_1 - u_2}\right)^2}\left(\frac{Q(u_1) - Q(u_2)}{2(u_1 - u_2)}\right) \end{aligned}$$

so that as $u_2 \to u_1$,

$$\begin{aligned} (t^*_{\nu,\alpha_1,\alpha_2})^2 &\to 2\lambda(u_1)^2 \frac{2u_1}{\lambda'(u_1)^2}\frac{Q'(u_1)}{2} \\ &= 2\frac{(\kappa + u_1^2)^2}{Q(u_1)^2} u_1 \frac{Q(u_1)^4}{16(u_1 - 1)^2(u_1 + \kappa)^2}(4 - 2u_1) \\ &= \Phi_\nu(u_1). \end{aligned}$$

From (b), we have that if $\alpha_1 < \alpha_2 < \alpha_3$, $t^*_{\nu,\alpha_1,\alpha_2} < t^*_{\nu,\alpha_1,\alpha_3} < t^*_{\nu,\alpha_2,\alpha_3}$. Thus if $\alpha_0 < \alpha_1 < \alpha < \alpha_2 < \alpha_3$,

$$t^*_{\nu,\alpha_0,\alpha_1} < t^*_{\nu,\alpha_1,\alpha} < t^*_{\nu,\alpha,\alpha_2} < t^*_{\nu,\alpha_2,\alpha_3}$$

and by letting $\alpha_0 \to \alpha_1-$ and $\alpha_3 \to \alpha_2+$ we obtain

$$\sqrt{\Phi_\nu(u_\nu(\alpha_1))} \leqslant t^*_{\nu,\alpha_1,\alpha} < t^*_{\nu,\alpha,\alpha_2} \leqslant \sqrt{\Phi_\nu(u_\nu((\alpha_2))},$$

completing the proof. □

Proof of Theorem 3 part two. The monotonicity of Theorem 3 (c) now follows by combining Theorem 3 (b) and Theorem 2 (c). □

Proof of Theorem 4. Let us first verify that $W_\nu(t) \geqslant A_{\nu,\nu+2,1}(t)$ for all $t > 0$. For all $\lambda < 1$, Lemma 1 shows that $\Delta_{\nu,\nu+2,\lambda}(t) = W_\nu(t) - A_{\nu,\nu+2,\lambda}(t) > 0$ for all $t > 0$ sufficiently small. In general,

$$Q_{\alpha,\beta,\lambda}(\beta) = \beta(2\beta - (\alpha+\beta+2)\lambda), \qquad Q'_{\alpha,\beta,\lambda}(\beta) = 3\beta - (\alpha+2\beta+1)\lambda.$$

Thus if $\lambda < 1$,

$$Q_{\nu,\nu+2,\lambda}(\nu+2) > (\nu+2)(2(\nu+2) - (2(\nu+1)+2)) = 0$$

and

$$Q'_{\nu,\nu+2,\lambda}(\nu+2) > 3(\nu+2) - (\nu+2(\nu+2)+1) = 1,$$

so that $Q_{\nu,\nu+2,\lambda}(s) > 0$ for all $s \geqslant \nu+2$. Thus, if $\Delta_{\nu,\nu+2,\lambda}(t) = 0$ for some $t > 0$, Lemma 7 yields that $t\Delta'_{\nu,\nu+2,\lambda}(t) > 0$, which is impossible for the first such root. Hence, we must have $\Delta_{\nu,\nu+2,\lambda}(t) > 0$, or equivalently $W_\nu(t) > A_{\nu,\nu+2,\lambda}(t)$ for all $t > 0$ and $\lambda < 1$, and thus by taking the sup over all $\lambda < 1$, also $W_\nu(t) \geqslant A_{\nu,\nu+2,1}(t)$ for all $t > 0$.

Using Lemma 1 and the asymptotics for $t \to \infty$, $W_\nu(t) \geqslant A_{\alpha,\beta_\nu(\alpha),\lambda}(t)$ for all $t > 0$ is only possible if $\lambda \leqslant \min(\beta_\nu(\alpha)/(\nu+2), 1)$, and the proof can be completed by establishing the last assertion of the theorem. To this end, first take $\alpha < \nu$. Then $\beta_\nu(\alpha) > \nu+2$ and $\lambda = 1$. Using Lemma 8, $\Delta = A_{\nu,\nu+2,1} - A_{\alpha,\beta_\nu(\alpha),1}$ has derivative $\Delta'(t) > 0$ for all $t > 0$, so that $A_{\nu,\nu+2,1}(t) > A_{\alpha,\beta_\nu(\alpha),1}(t)$ for all $t > 0$. Second, take $\nu < \alpha < 2(\nu+1)$. Then $\beta_\nu(\alpha) < \nu+2$, so that $\lambda \leqslant \beta_\nu(\alpha)/(\nu+2) < 1$ and $1 \cdot \beta_\nu(\alpha)^2 - \lambda^2(\nu+2)^2 \geqslant 0$. Again using Lemma 8, $\Delta = A_{\nu,\nu+2,1} - A_{\alpha,\beta_\nu(\alpha),\lambda}$ has derivative $\Delta'(t) > 0$ for all $t > 0$, so that again, $A_{\nu,\nu+2,1}(t) > A_{\alpha,\beta_\nu(\alpha),\lambda}(t)$ for all $t > 0$, completing the proof. □

4. Concluding remarks

Hornik and Grün [3] summarize the "best" (in the sense of not being uniformly weaker than other) known Amos-type bounds for $R_\nu(t)$ by:

$$\begin{aligned}
G_{\nu+1/2,\nu+3/2}(t) <\ & R_\nu(t), && \nu \geqslant -1,\\
& R_\nu(t) < G_{\nu,\nu+2}(t), && \nu \geqslant -1,\\
& R_\nu(t) < G_{\nu+1/2,\sqrt{(\nu+1/2)(\nu+3/2)}}(t), && \nu > 0,\\
& R_\nu(t) < G_{\nu+1/2,\nu+1/2}(t), && -1/2 \leqslant \nu \leqslant 0.
\end{aligned}$$

Yang and Zheng [12] generalize the Amos-type bounds by adding the parameter $\lambda > 0$ and their results imply the additional bound

$$G_{\nu-1,\nu+3,(\nu+3)/(\nu+2)}(t) < R_\nu(t), \quad \nu \geqslant -1.$$

In this paper we add the bounds

$$G_{\alpha,\beta_\nu(\alpha),\lambda_\nu(\alpha)}(t) \leqslant R_\nu(t), \quad \nu \geqslant -1, \tag{16}$$

for $\nu - 1 \leqslant \alpha \leqslant \nu + 1/2$. We also show that the upper bound $G_{\nu,\nu+2}(t)$ cannot be improved by generalizing the Amos-type bound. Further research could investigate if the other upper Amos-type bounds might be improved by generalized Amos-type bounds with the additional parameter λ.

REFERENCES

[1] D. E. AMOS, *Computation of Modified Bessel Functions and Their Ratios*, Mathematics of Computation, **28** (1974), pp. 239–251.

[2] Á. BARICZ AND E. NEUMAN, *Inequalities Involving Modified Bessel Functions of the First Kind II*, Journal of Mathematical Analysis and Applications, **332** (2007), pp. 265–271.

[3] K. HORNIK AND B. GRÜN, *Amos-Type Bounds for Modified Bessel Function Ratios*, Journal of Mathematical Analysis and Applications, **408** (2013), pp. 91–101.

[4] A. LAFORGIA AND P. NATALINI, *Some Inequalities for Modified Bessel Functions*, Journal of Inequalities and Applications, **2010** (2010).

[5] I. NÅSELL, *Rational Bounds for Ratios of Modified Bessel Functions*, SIAM Journal on Mathematical Analysis, **9** (1978), pp. 1–11.

[6] E. NEUMAN, *Inequalities Involving Modified Bessel Functions of the First Kind*, Journal of Mathematical Analysis and Applications, **171** (1992), pp. 532–536.

[7] G. SCHOU, *Estimation of the Concentration Parameter in von Mises-Fisher Distributions*, Biometrika, **65** (1978), pp. 369–377.

[8] J. SEGURA, *Bounds for Ratios of Modified Bessel Functions and Associated Turán-type Inequalities*, Journal of Mathematical Analysis and Applications, **374** (2011), pp. 516–528.

[9] H. C. SIMPSON AND S. J. SPECTOR, *On Barrelling for a Special Material in Finite Elasticity*, Quarterly of Applied Mathematics, **42** (1984), pp. 99–111.

[10] H. C. SIMPSON AND S. J. SPECTOR, *Some Monotonicity Results for Ratios of Modified Bessel Functions*, Quarterly of Applied Mathematics, **42** (1984), pp. 95–98.

[11] Z.-H. YANG AND S.-Z. ZHENG, *Monotonicity and Convexity of the Ratios of the First Kind Modified Bessel Functions and Applications*, Mathematical Inequalities and Applications, **21** (2018), pp. 107–125.

[12] Z.-H. YANG AND S.-Z. ZHENG, *Monotonicity of the Ratio of Modified Bessel Functions of the First Kind with Applications*, Journal of Inequalities and Applications, **2018** (2018), p. 57.

[13] L. YUAN AND J. D. KALBFLEISCH, *On the Bessel Distribution and Related Problems*, The Annals of the Institute of Statistical Mathematics, **52** (2000), pp. 438–447.

(Received December 6, 2019)

Kurt Hornik
Wirtschaftsuniversität Wien
Welthandelsplatz 1, 1020 Vienna, Austria
e-mail: Kurt.Hornik@wu.ac.at

Bettina Grün
Wirtschaftsuniversität Wien
Welthandelsplatz 1, 1020 Vienna, Austria
e-mail: Bettina.Gruen@wu.ac.at