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REVERSED HARDY–LITTLEWOOD–SOBOLEV INEQUALITY ON HEISENBERG GROUP $\mathbb{H}^n$ AND CR SPHERE $\mathbb{S}^{2n+1}$

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Abstract. This paper is mainly devoted to the study of the reversed Hardy-Littlewood-Sobolev (HLS) inequality on Heisenberg group $\mathbb{H}^n$ and CR sphere $\mathbb{S}^{2n+1}$. First, we establish the roughly reversed HLS inequality and give an explicitly lower bound for the sharp constant. Then, the existence of the extremal functions with sharp constant is proved by *subcritical approach* and some compactness techniques. Our method is *rearrangement free* and can be applied to study the classical HLS inequality and other similar inequalities.

1. Introduction

Heisenberg group is one of the simplest noncommutative geometries and is the model space of CR manifolds, which arise from the study of real hypersurfaces of complex manifolds. It is well-known that the non-commutativity and the complex structure induced from complex manifolds inspire many interesting geometric properties and bring some new difficulties. In the past few decades, sharp inequalities such as Sobolev inequality [22, 23, 36, 43], Hardy-Littlewood-Sobolev(HLS) inequality [22, 25], Moser-Trudinger inequality [3, 13, 14], Hardy inequality [26, 51], Hardy-Sobolev inequality [34], etc., play important roles in the study of problems defined on Heisenberg group and CR manifolds. In this paper, we mainly concern with *reversed Hardy-Littlewood-Sobolev inequality* on $\mathbb{H}^n$ and CR sphere $\mathbb{S}^{2n+1}$.

1.1. HLS and reversed HLS inequalities on $\mathbb{R}^n$

The classical HLS inequality [40, 41, 53] on $\mathbb{R}^n$ states that

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x-y|^{n-\alpha}}dxdy\right| \leqslant N_{p,\alpha,n}\|f\|_p\|g\|_t \tag{1.1}$$

holds for all $f\in L^p(\mathbb{R}^n)$, $g\in L^t(\mathbb{R}^n)$, where $0<\alpha<n$ and $1<p,t<+\infty$ satisfying

$$\frac{1}{p}+\frac{1}{t}+\frac{n-\alpha}{n}=2. \tag{1.2}$$

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Using rearrangement inequalities, Lieb [46] proved the existence of the extremal functions to the inequality (1.1) with the sharp constant. For the conformal case $p = t = \frac{2n}{n+\alpha}$, he classified the extremal functions and computed the best constant (different discussions can be found in [5,47]). In fact, he proved that the extremal functions with $p = t$ are given by

$$f_\varepsilon(x) = c_1 g_\varepsilon(x) = c\left(\frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2}\right)^{(n+\alpha)/2}, \tag{1.3}$$

where c_1, c and ε are constants, x_0 is some point in $\mathbb{R}^n$. Recently, the solutions of the Euler-Lagrange equation in the conformal case were classified by the method of moving planes [12] and the method of moving spheres [45], respectively.

For $0 < p,t < 1$ and $\alpha > n$ satisfying (1.2), Dou and Zhu [19] (also see [2,49]) established a class of reversed HLS inequality

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}} dxdy \geqslant N_{p,\alpha,n}\|f\|_p\|g\|_t, \tag{1.4}$$

where $f \in L^p(\mathbb{R}^n)$, $g \in L^t(\mathbb{R}^n)$ are nonnegative functions. Employing the rearrangement inequalities and the method of moving spheres, they also classified the extremal functions and computed the best constant in the conformal case. In fact, they found that the extremal functions of (1.4) in the conformal case are given as (1.3), too.

As stated above, it can be found that *rearrangement inequalities, the method of moving planes and the method of moving spheres* are basic and important tools in the study of HLS inequalities. More applications of these techniques can be found in the study of HLS inequalities and reversed HLS inequalities on the upper half space (see [8,11,17,20,27,39,50] and the references therein).

Note that f_ε and g_ε will blow up as $\varepsilon \to 0^+$, and vanish as $\varepsilon \to +\infty$. The phenomenon makes it difficult to study the extremal problems. To overcome the difficulty, we often renormalize the extremal sequence. For example, Lieb [46] renormalized the extremal sequence $\{f_j(x)\}$ so that it satisfies $f_j(x) > \beta > 0$ if $|x| = 1$. The technique can also be found in [19].

Recently, Dou, Guo and Zhu [17] adopted the *subcritical approach* to study sharp HLS type inequalities on the upper half space. By Young inequality, they first established two classes of HLS type inequalities with subcritical power on a ball. Then, using the conformal transformation between ball and upper half space and the method of moving planes, they proved that the extremal functions of HLS type inequalities with subcritical power are constant functions. Passing to the limit from subcritical power to critical power, they obtained two classes of sharp HLS type inequalities on the upper half space. In the process of taking the limit, since these extremal functions of HLS type inequalities with subcritical power are constant functions, we can choose every extremal function to be $f \equiv 1$ and avoid efficiently the blow-up phenomenon. Inspired by the idea of [17], Gluck [27] established a class of sharp HLS type inequalities with more general kernels

$$K_{\alpha,\beta}(x) = \frac{x_n^\beta}{(|x'|^2 + x_n^2)^{(n-\alpha)/2}}, \quad x = (x', x_n) \in \mathbb{R}^{n-1} \times (0,\infty)$$

on the upper half space $\mathbb{R}^n_+$.

In [54], Stein and Weiss established a class of weighted HLS type inequality, named as Stein-Weiss inequality. Applying the rearrangement inequalities and under the conditions $\alpha \geqslant 0$ and $\beta \geqslant 0$, Lieb [46] proved the existence of the extremal functions of Stein-Weiss inequality. Recently, more interesting results have been presented, such as reverse Stein-Weiss inequality [6], Stein-Weiss inequality and reverse Stein-Weiss inequality on the upper half space [9, 15], Stein-Weiss inequalities with Poisson type kernel [10,55], etc.. Because of technical reason, the conditions $\alpha \geqslant 0$ and $\beta \geqslant 0$ were assumed in the aforementioned articles.

The conditions were removed by Chen, Lu and Tao in [7]. In fact, by *concentration-compactness principles*, they established the existence of extremal functions for two kinds of Stein-Weiss inequalities on the Heisenberg group, which can also be applied to the corresponding problems in [6, 9, 10, 15, 46, 55]. Tao and Wang [56] applied the idea to prove the existence of extremal functions for a class of Stein-Weiss inequalities with an extended Poisson kernel.

1.2. HLS inequlity on the Heisenberg group

We first recall Heisenberg group and some notations.

Heisenberg group $\mathbb{H}^n$ consists of the set

$$\mathbb{C}^n \times \mathbb{R} = \{(z,t) : z = (z_1, \cdots, z_n) \in \mathbb{C}^n, t \in \mathbb{R}\}$$

with the multiplication law

$$(z,t)(z',t') = (z+z', t+t'+2Im(z\cdot\overline{z'})),$$

where $z\cdot\overline{z'} = \sum_{j=1}^n z_j\overline{z'_j}$, $z_j = x_j+\sqrt{-1}y_j$ and $\overline{z_j} = x_j-\sqrt{-1}y_j$.

For any points $u = (z,t)$, $v = (z',t') \in \mathbb{H}^n$, denote the norm function by $|u| = (|z|^4+t^2)^{1/4}$ and the distance between u and v by $|v^{-1}u|$. Moreover, there exists a constant $\gamma \geqslant 1$ such that $|uv| \leqslant \gamma(|u|+|v|)$ holds for all $u, v \in \mathbb{H}^n$. Write $B(u,R) = \{v \in \mathbb{H}^n : |u^{-1}v| < R\}$ as the ball centered at u with radius R. For any $\lambda > 0$, the dilation $\delta_\lambda(u)$ is defined as $\delta_\lambda(u) = (\lambda z, \lambda^2 t)$, and $Q = 2n+2$ is the homogeneous dimension with respect to the dilations. For more details about Heisenberg group, please see [21, 22] and the references therein.

To study the sigular integral operator on CR manifolds, Folland and Stein [22] established the following HLS inequality

$$\left|\int_{\mathbb{H}^n}\int_{\mathbb{H}^n}\overline{f(u)}g(v)|v^{-1}u|^{\alpha-Q}dvdu\right| \leqslant D(n,\alpha,p)\|f\|_{L^q(\mathbb{H}^n)}\|g\|_{L^p(\mathbb{H}^n)}, \tag{1.5}$$

where $f \in L^q$, $g \in L^p$, $0 < \alpha < Q$, $\frac{1}{q}+\frac{1}{p}+\frac{Q-\alpha}{Q} = 2$ and $du = dzdt = dxdydt$ is the Haar measure on $\mathbb{H}^n$. In fact, the inequality (1.5) can be deduced from Proposition 8.7 of [22].

Since rearrangement inequalities do not work efficiently on Heisenberg group, it took a quite long time to study the problems about the sharp constant and extremal functions of (1.5). In 2012, Frank and Lieb [25] studied the conformal case $p = q = \frac{2Q}{Q+\alpha}$ of (1.5). They introduced a class of *rearrangement free method* and classified the extremal functions. Then, sharp constants were computed for the HLS inequality, Sobolev inequality and their limiting cases on Heisenberg group and CR sphere $\mathbb{S}^{2n+1}$. Their results about HLS inequality on $\mathbb{H}^n$ can be stated as follows.

THEOREM A. (Sharp HLS inequality on $\mathbb{H}^n$) *Let* $0 < \alpha < Q$ *and* $p_\alpha = \frac{2Q}{Q+\alpha}$. *Then for any* $f, g \in L^{p_\alpha}(\mathbb{H}^n)$,

$$\left| \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \overline{f(u)} |v^{-1}u|^{-(Q-\alpha)} g(v) dv du \right| \leqslant D_{n,\alpha} ||f||_{L^{p_\alpha}(\mathbb{H}^n)} ||g||_{L^{p_\alpha}(\mathbb{H}^n)}, \tag{1.6}$$

where

$$D_{n,\alpha} := \left(\frac{\pi^{n+1}}{2^{n-1} n!} \right)^{(Q-\alpha)/Q} \frac{n! \Gamma(\alpha/2)}{\Gamma^2((Q+\alpha)/4)}. \tag{1.7}$$

And the equality holds if and only if

$$f(u) = c_1 g(u) = c_2 H(\delta_r(u_0^{-1} u), \tag{1.8}$$

for some $c_1,\ c_2 \in \mathbb{C},\ r > 0$ *and* $u_0 \in \mathbb{H}^n$ *(unless* $f \equiv 0$ *or* $g \equiv 0$*). Here* H *is defined as*

$$H(u) = H(z,t) = ((1+|z|^2)^2 + t^2)^{-(Q+\alpha)/4}. \tag{1.9}$$

REMARK A.1. Using the Green's function of the sub-Laplacian [24] and making a duality argument, we see that HLS inequality (1.6) with $\alpha = 2$ is equivalent to the sharp Sobolev inequality established by Jeison and Lee [43]. Based on the idea introduced by Obata [52], they classified the extremal functions and computed the sharp constant of the sharp Sobolev inequality (see [43]).

In view of the efficiency of the method of moving planes and the method of moving spheres in the study of Euler-Lagrange equation of (1.1), a natural question is whether one can adapt them on the Heisenberg group. There have been a number of attempts by several mathematicians in the directions (see [4, 35] and the references therein). But, it seems that these methods are not suitable very well with Heisenberg group.

For the case $p \neq q$, Han [30] used the concentration-compactness principles to study the existence of extremal functions of (1.5). Recently, Han, Lu, Zhu [31] and Chen, Lu, Tao [7] established two classes of weighted HLS inequalities on Heisenberg group and proved the existence of extremal functions by the concentration-compactness principles.

1.3. Reversed HLS inequalities on the Heisenberg group

If $\alpha > Q$, we will establish the following reversed HLS inequality.

PROPOSITION 1.1. *Let $\alpha > Q \geqslant 4$ and $p_\alpha = \frac{2Q}{Q+\alpha}$. Then for any nonnegative functions $f,g \in L^{p_\alpha}(\mathbb{H}^n)$, there exists a sharp constant $N_{Q,\alpha,\mathbb{H}}$ such that*

$$\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{F(u)G(v)}{|v^{-1}u|^{Q-\alpha}} dudv \geqslant N_{Q,\alpha,\mathbb{H}} \|F\|_{L^{p_\alpha}(\mathbb{H}^n)} \|G\|_{L^{p_\alpha}(\mathbb{H}^n)}. \tag{1.10}$$

The sharp constant satisfies

$$N_{Q,\alpha,\mathbb{H}} \geqslant \frac{(8|B_1|)^{(Q-\alpha)/Q}}{2p_\alpha^2},$$

where $B_1 := B(0,1)$ and the volume of B_1 is given (see [13,30]*) as*

$$|B_1| = \int_{|u|<1} du = \frac{2\pi^{\frac{Q-2}{2}}\Gamma(\frac{1}{2})\Gamma(\frac{Q+2}{4})}{(Q-2)\Gamma(\frac{Q-2}{2})\Gamma(\frac{Q+4}{4})}.$$

Since $p_\alpha \in (0,1)$, the extremal problem of (1.10) is analytically different from the case $\alpha \in (0,Q)$. This brings some difficulties to study the case $\alpha > Q$ by the method of [25] and [30].

We will discuss the extremal problem by *subcritical approach*. However, because of the non-commutativity and the complex structure of Heisenberg group and CR sphere, which make the method of moving planes and the method of moving spheres ineffective, it is not easy to prove that the extremal functions of HLS inequalities with subcritical power on the CR sphere should be constant functions. We will encounter the blow-up phenomenon and circumvent it by *renormalization method* (see Section 4). Furthermore, our method is rearrangement free and different from the method in [25]. Recently, we have successfully experimented with the method and provided a new proof for the existence of extremal functions of (1.1) and (1.4) (see [58]).

The unit CR sphere is the sphere $\mathbb{S}^{2n+1} = \{\xi = (\xi_1,\cdots,\xi_{n+1} \in \mathbb{C}^{n+1} : \|\xi\| = 1\}$ endowed with standard CR structure. *Cayley transformation* $\mathscr{C} : \mathbb{H}^n \to \mathbb{S}^{2n+1} \setminus \mathfrak{S}$ and its reverse are defined respectively as

$$\mathscr{C}(z,t) = \Big(\frac{2z}{1+|z|^2+it}, \frac{1-|z|^2-it}{1+|z|^2+it}\Big),$$

$$\mathscr{C}^{-1}(\xi) = \Big(\frac{\xi_1}{1+\xi_{n+1}}, \cdots, \frac{\xi_n}{1+\xi_{n+1}}, \mathrm{Im}\frac{1-\xi_{\mathrm{n}+1}}{1+\xi_{\mathrm{n}+1}}\Big),$$

where $\mathfrak{S} = (0,\cdots,0,-1)$ is the south pole. The Jacobian of the Cayley transformation is

$$J_{\mathscr{C}}(z,t) = \frac{2^{2n+1}}{((1+|z|^2)^2+t^2)^{n+1}}$$

which implies that

$$\int_{\mathbb{S}^{2n+1}} \phi(\xi)d\xi = \int_{\mathbb{H}^n} \phi(\mathscr{C}(u))J_{\mathscr{C}}(u)du \tag{1.11}$$

for all integrable function ϕ on $\mathbb{S}^{2n+1}$, where $d\xi$ is the Euclidean volume element of $\mathbb{S}^{2n+1}$. Under the Cayley transformation, we have the following relations between two distance functions

$$|1-\xi\cdot\bar{\eta}| = 2|u^{-1}v|^2\left((1+|z|^2)^2+t^2\right)^{-1/2}\left((1+|z'|^2)^2+t'^2\right)^{-1/2}, \tag{1.12}$$

where $\zeta = \mathscr{C}(u),\ \eta = \mathscr{C}(v),\ u=(z,t)$ and $v=(z',t')$.

For any $f\in L^p(\mathbb{S}^{2n+1})$, there is a corresponding function

$$F(u) = |J_{\mathscr{C}}(u)|^{1/p} f(\mathscr{C}(u)) \in L^p(\mathbb{H}^n)$$

such that $\|f\|_{L^p(\mathbb{S}^{2n+1})} = \|F\|_{L^p(\mathbb{H}^n)}$.

Applying the Cayley transformation to (1.10), we have that

$$\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}} \frac{f(\xi)g(\eta)}{|1-\xi\cdot\bar{\eta}|^{\frac{Q-\alpha}{2}}}d\xi d\eta \geqslant N_{Q,\alpha}\|f\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}\|g\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})} \tag{1.13}$$

holds for all nonnegative functions $f,g\in L^{p_\alpha}(\mathbb{S}^{2n+1})$, where $N_{Q,\alpha}$ is the sharp constant and satifies

$$N_{Q,\alpha} \geqslant \frac{(8|B_1|)^{\frac{Q-\alpha}{Q}}}{2^{1+n\frac{\alpha-Q}{Q}}p_\alpha^2}.$$

Define the extremal problem of (1.13) as

$$\begin{aligned} N_{Q,\alpha} &= \inf_{\|f\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}=\|g\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}=1}\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}\frac{f(\xi)g(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\xi d\eta \\ &= \inf_{f,g\in L^{p_\alpha}(\mathbb{S}^{2n+1})\setminus\{0\}} \frac{\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}} f(\xi)g(\eta)|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2}d\xi d\eta}{\|f\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}\|g\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}}. \end{aligned} \tag{1.14}$$

Then, it is easy to get the following estimate.

PROPOSITION 1.2. (Upper and lower bound for the sharp constant)

$$0 < \frac{(8|B_1|)^{\frac{Q-\alpha}{Q}}}{2^{1+n\frac{\alpha-Q}{Q}}p_\alpha^2} \leqslant N_{Q,\alpha} \leqslant \left(\frac{2\pi^{n+1}}{n!}\right)^{(Q-\alpha)/Q}\frac{n!\Gamma(\alpha/2)}{\Gamma^2((Q+\alpha)/2)}, \tag{1.15}$$

where

$$\left(\frac{2\pi^{n+1}}{n!}\right)^{(Q-\alpha)/Q}\frac{n!\Gamma(\alpha/2)}{\Gamma^2((Q+\alpha)/2)} = |\mathbb{S}^{2n+1}|^{1-\frac{2}{p\alpha}}\int_{\mathbb{S}^{2n+1}}|1-\xi\cdot\bar{\eta}|^{\frac{\alpha-Q}{2}}d\eta. \tag{1.16}$$

Combining the subcritical approach and renormalization method, we prove the following attainability of the sharp constant $N_{Q,\alpha}$.

THEOREM 1.3. (Attainability) *$N_{Q,\alpha}$ can be attained by a pair of positive functions $f,g \in C^1(\mathbb{S}^{2n+1})$. Applying the Cayley transformation, we also have that $N_{Q,\alpha,\mathbb{H}}$ is attained by a pair of positive functions $F,G \in L^{p_\alpha}(\mathbb{H}^n) \cap C^1(\mathbb{H}^n)$.*

In the following, we outline the ideas of the proof of Theorem 1.3. First, consider the extremal problems with subcritical power $p \in (0, p_\alpha)$ and get the existence of extremal function pairs $\{f_p, g_p\}$, see Section 3. Then, prove that the sequence $\{f_p, g_p\}$ form a minimizing sequence of (1.14) as $p \to p_\alpha$. Lastly, we circumvent the blow-up phenomenon by renormalization method and show the attainability of the sharp constant $N_{Q,\alpha}$.

Moreover, since nonlinear terms with negative power appear in the Euler-Lagrange equations (see Section 3 and Section 4), we need not only a upper bound to control the blow up of the sequence, but also a lower bound to avoid the blow up of terms with negative power. So, different techniques are needed for the extremal problem (1.14). More details can be seen in Section 3 and Section 4.

The paper is organized as follows. Section 2 is devoted to establishing the roughly reversed HLS inequalities (1.10). In Section 3, we study the extremal problems related to subcritical case and get the existence of the corresponding extremal functions. These functions will provide a minimizing sequence of (1.14). Then, we prove the attainability of $N_{Q,\alpha}$ in Section 4.

We always use $C, C_1, C_2, \cdots$, etc. to denote positive universal constants though their actual values may differ from line to line or within the same line itself.

2. Roughly reversed HLS inequalities on $\mathbb{H}^n$

This section is mainly devoted to establishing the roughly reversed HLS inequality (1.10). In fact, we present a more general reversed HLS inequalities as follows.

PROPOSITION 2.1. *Assume $\lambda > 0$, $0 < p,t < 1$ with $\frac{1}{p} + \frac{1}{t} - \frac{\lambda}{Q} = 2$. Then, for any nonnegative functions $F \in L^p(\mathbb{H}^n)$ and $G \in L^t(\mathbb{H}^n)$, there exists some positive constant $C(Q,\lambda,p,\mathbb{H})$ such that*

$$\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} F(u)|v^{-1}u|^\lambda G(v)\,du\,dv \geqslant C(Q,\lambda,p,\mathbb{H})\|F\|_{L^p(\mathbb{H}^n)}\|G\|_{L^t(\mathbb{H}^n)}. \tag{2.1}$$

Moreover, the constant satisfies

$$C(Q,\lambda,p,\mathbb{H}) \geqslant \frac{(4|B_1|)^{-\lambda/Q}}{2pt}\left(\frac{\lambda}{Q}\max\left\{\frac{p}{1-p}, \frac{t}{1-t}\right\}\right)^{-\lambda/Q}. \tag{2.2}$$

Proof. Our proof is similar to the argument given by Ngô and Nguyen [49, Section 2], where authors adopted the idea of Lieb and Loss [47]. For completeness, we will give the detailed proof. Since the homogeneity of (2.1), without loss of generality, we assume that $\|F\|_{L^p(\mathbb{H}^n)} = \|G\|_{L^t(\mathbb{H}^n)} = 1$. So, it is sufficient to show that the right side of (2.2) is a lower bound of the left side of (2.1).

By the layer cake representation [47, Theorem 1.13],

$$
\begin{aligned}
I :=& \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} F(u)|v^{-1}u|^{\lambda} G(v)\,du\,dv \\
=&\lambda \int_0^\infty \int_0^\infty \int_0^\infty c^{\lambda-1} J(a,b,c)\,da\,db\,dc, \qquad (2.3)
\end{aligned}
$$

where

$$
J(a,b,c) := \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \chi_{\{F>a\}}(u)\chi_{\mathbb{H}^n\setminus B_c}(u^{-1}v)\chi_{\{G>b\}}(v)\,du\,dv
$$

and $\chi_\Omega(u)$ is the characteristic function of set Ω, $B_c := B(0,c)$. Noting the basic fact $|u^{-1}v| = |v^{-1}u|$, we have $\chi_{\mathbb{H}^n\setminus B_c}(u^{-1}v) = \chi_{\mathbb{H}^n\setminus B_c}(v^{-1}u)$. Write $\phi(a) = \int_{\mathbb{H}^n} \chi_{\{F>a\}}(u)du = |\{F>a\}|$ and $\psi(b) = \int_{\mathbb{H}^n} \chi_{\{G>b\}}(v)dv = |\{G>b\}|$.

If $\phi(a) \geqslant \psi(b)$ and $2|B_c| = 2C^Q|B_1| \leqslant \phi(a)$, then

$$
\begin{aligned}
J(a,b,c) =& \int_{\mathbb{H}^n} \chi_{\{G>b\}}(v)|\{F>a\}\cap(\mathbb{H}^n\setminus B_c(v))|dv \\
\geqslant& \int_{\mathbb{H}^n} \chi_{\{G>b\}}(v)\big(|\{F>a\}| - |B_c(v)|\big)dv \\
\geqslant& \int_{\mathbb{H}^n} \chi_{\{G>b\}}(v)\frac{\phi(a)}{2}dv = \frac{\phi(a)\psi(b)}{2}.
\end{aligned}
$$

Similarly, if $\phi(a) \leqslant \psi(b)$ and $2|B_c| = 2C^Q|B_1| \leqslant \psi(b)$, the above formula also holds. Therefore, if $2|B_c| = 2C^Q|B_1| \leqslant \max\{\phi(a),\psi(b)\}$, it follows

$$
J(a,b,c) \geqslant \frac{\phi(a)\psi(b)}{2}. \qquad (2.4)
$$

Substituting (2.4) into (2.3), we have

$$
\begin{aligned}
I \geqslant& \lambda \int_0^\infty \int_0^\infty \Big(\int_0^{\left(\frac{\max\{\phi(a),\psi(b)\}}{2|B_1|}\right)^{1/Q}} c^{\lambda-1}\frac{\phi(a)\psi(b)}{2}dc\Big)da\,db \\
=& \int_0^\infty \int_0^\infty \frac{\phi(a)\psi(b)}{2}\Big(\frac{\max\{\phi(a),\psi(b)\}}{2|B_1|}\Big)^{\frac{\lambda}{Q}}da\,db \\
\geqslant& \frac{(2|B_1|)^{-\lambda/Q}}{2}\int_0^\infty \int_0^{a^{p/t}} \phi(a)\psi(b)^{1+\frac{\lambda}{Q}}db\,da \\
&+ \frac{(2|B_1|)^{-\lambda/Q}}{2}\int_0^\infty \int_{a^{p/t}}^\infty \phi(a)^{1+\frac{\lambda}{Q}}\psi(b)db\,da \\
=:& \frac{(2|B_1|)^{-\lambda/Q}}{2}(I_1+I_2). \qquad (2.5)
\end{aligned}
$$

By reversed Hölder inequality, it yields

$$
\begin{aligned}
I_1 =& \int_0^\infty \int_0^{a^{p/t}} \phi(a)\psi(b)^{1+\frac{\lambda}{Q}}db\,da \\
\geqslant& \int_0^\infty \phi(a)\Big(\int_0^{a^{p/t}} \psi(b)b^{t-1}db\Big)^{\frac{Q+\lambda}{Q}}\Big(\int_0^{a^{p/t}} b^{(t-1)\frac{Q+\lambda}{\lambda}}db\Big)^{-\frac{\lambda}{Q}}da
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty \phi(a)\Big(\int_0^{a^{p/t}} \psi(b) b^{t-1} db\Big)^{\frac{Q+\lambda}{Q}} \Big(\frac{\lambda p}{Qt(1-p)}\Big)^{-\frac{\lambda}{Q}} a^{p-1} da \\
&= \frac{1}{pt}\Big(\frac{\lambda}{Q}\frac{p}{1-p}\Big)^{-\frac{\lambda}{Q}} \int_0^\infty pa^{p-1}\phi(a)\Big(\int_0^{a^{p/t}} tb^{t-1}\psi(b)db\Big)^{\frac{Q+\lambda}{Q}} da \qquad (2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_0^\infty \int_{a^{p/t}}^\infty \phi(a)^{1+\frac{\lambda}{Q}} \psi(b) db\, da = \int_0^\infty \int_0^{b^{t/p}} \phi(a)^{1+\frac{\lambda}{Q}} \psi(b) da\, db \\
&\geqslant \frac{1}{pt}\Big(\frac{\lambda}{Q}\frac{t}{1-t}\Big)^{-\frac{\lambda}{Q}} \int_0^\infty tb^{t-1}\psi(b)\Big(\int_0^{b^{t/p}} pa^{p-1}\phi(a)da\Big)^{\frac{Q+\lambda}{Q}} db. \qquad (2.7)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
1 &= \|F\|^p_{L^p(\mathbb{H}^n)} = p\int_0^\infty a^{p-1}\phi(a)da, \\
1 &= \|G\|^t_{L^t(\mathbb{H}^n)} = t\int_0^\infty b^{t-1}\psi(b)db
\end{aligned}
$$

and $\frac{Q+\lambda}{Q} \geqslant 1$, it follows from Jensen inequality that

$$
I_1 \geqslant \frac{1}{pt}\Big(\frac{\lambda}{Q}\frac{p}{1-p}\Big)^{-\frac{\lambda}{Q}} \Big(\int_0^\infty pa^{p-1}\phi(a)\int_0^{a^{p/t}} tb^{t-1}\psi(b)db\, da\Big)^{\frac{Q+\lambda}{Q}}, \qquad (2.8)
$$

$$
\begin{aligned}
I_2 &\geqslant \frac{1}{pt}\Big(\frac{\lambda}{Q}\frac{t}{1-t}\Big)^{-\frac{\lambda}{Q}} \Big(\int_0^\infty tb^{t-1}\psi(b)\int_0^{b^{t/p}} pa^{p-1}\phi(a)da\, db\Big)^{\frac{Q+\lambda}{Q}} \\
&= \frac{1}{pt}\Big(\frac{\lambda}{Q}\frac{t}{1-t}\Big)^{-\frac{\lambda}{Q}} \Big(\int_0^\infty pa^{p-1}\phi(a)\int_{a^{p/t}}^\infty tb^{t-1}\psi(b)db\, da\Big)^{\frac{Q+\lambda}{Q}}. \qquad (2.9)
\end{aligned}
$$

Write

$$
C_1(Q,\lambda,p) := \frac{(2|B_1|)^{-\lambda/Q}}{2pt}\Big(\frac{\lambda}{Q}\max\Big\{\frac{p}{1-p},\frac{t}{1-t}\Big\}\Big)^{-\frac{\lambda}{Q}}.
$$

Substituting (2.8) and (2.9) into (2.5) and using the convexity of function $x^{\frac{Q+\lambda}{Q}}$, we arrive at

$$
\begin{aligned}
I \geqslant\, & C_1(Q,\lambda,p)\Big(\int_0^\infty pa^{p-1}\phi(a)\int_0^{a^{p/t}} tb^{t-1}\psi(b)db\, da\Big)^{\frac{Q+\lambda}{Q}} \\
& + C_1(Q,\lambda,p)\Big(\int_0^\infty pa^{p-1}\phi(a)\int_{a^{p/t}}^\infty tb^{t-1}\psi(b)db\, da\Big)^{\frac{Q+\lambda}{Q}} \\
\geqslant\, & C_1(Q,\lambda,p)2^{-\frac{\lambda}{Q}}.
\end{aligned}
$$

The inequality (2.1) is established and the proof is completed. □

REMARK 2.2. The inequality (2.1) includes the inequalty (1.10). In fact, suppose that $\lambda = \alpha - Q$ with $\alpha > Q \geqslant 4$ and $p = t = p_\alpha$, (2.1) is reduced to (1.10).

3. Subcritical HLS inequalities on $\mathbb{S}^{2n+1}$

LEMMA 3.1. *Let $p \in (0, p_\alpha)$. There exists some positive constant $\tilde{C} = C(Q, \alpha, p)$ such that*

$$\int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{f(\xi)g(\eta)}{|1 - \xi \cdot \bar{\eta}|^{(Q-\alpha)/2}} d\xi d\eta \geqslant \tilde{C} \|f\|_{L^p(\mathbb{S}^{2n+1})} \|g\|_{L^p(\mathbb{S}^{2n+1})} \tag{3.1}$$

holds for any nonnegative $f, g \in L^p(\mathbb{S}^{2n+1})$.

Proof. It is easy to verify that (3.1) holds for any nonnegative $f, g \in L^p(\mathbb{S}^{2n+1}) \cap L^{p_\alpha}(\mathbb{S}^{2n+1})$ by (1.13) and Hölder inequality. Then we complete the proof by a density argument. $\square$

Define the extremal problem of (3.1) as

$$N_{Q,\alpha,p} = \inf_{\|f\|_{L^p(\mathbb{S}^{2n+1})} = \|g\|_{L^p(\mathbb{S}^{2n+1})} = 1} \int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{f(\xi)g(\eta)}{|1 - \xi \cdot \bar{\eta}|^{(Q-\alpha)/2}} d\xi d\eta. \tag{3.2}$$

From (3.1), it is easy to see that

$$\begin{aligned} 0 < \tilde{C} \leqslant N_{Q,\alpha,p} &\leqslant |\mathbb{S}^{2n+1}|^{1-\frac{2}{p}} \int_{\mathbb{S}^{2n+1}} |1 - \xi \cdot \bar{\eta}|^{\frac{\alpha - Q}{2}} d\eta \\ &= \left(\frac{2\pi^{n+1}}{n!} \right)^{2-\frac{2}{p}} \frac{n! \Gamma(\alpha/2)}{\Gamma^2((Q+\alpha)/2)}. \end{aligned} \tag{3.3}$$

Furthermore, inspired by the argument of Lemma 3.2 of [16] and Proposition 2.5 of [17], we will prove the following attainability of sharp constant $N_{Q,\alpha,p}$.

PROPOSITION 3.2. *(1) There exist a pair of nonnegative functions $(f, g) \in C^1(\mathbb{S}^{2n+1}) \times C^1(\mathbb{S}^{2n+1})$ such that $\|f\|_{L^p(\mathbb{S}^{2n+1})} = \|g\|_{L^p(\mathbb{S}^{2n+1})} = 1$ and*

$$N_{Q,\alpha,p} = \int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{f(\xi)g(\eta)}{|1 - \xi \cdot \bar{\eta}|^{(Q-\alpha)/2}} d\xi d\eta.$$

(2) Minimizer pair (f, g) satisfies the following Euler-Lagrange equations

$$\begin{cases} N_{Q,\alpha,p} f^{p-1}(\xi) = \int_{\mathbb{S}^{2n+1}} |1 - \xi \cdot \bar{\eta}|^{(\alpha - Q)/2} g(\eta) d\eta, \\ N_{Q,\alpha,p} g^{p-1}(\xi) = \int_{\mathbb{S}^{2n+1}} |1 - \xi \cdot \bar{\eta}|^{(\alpha - Q)/2} f(\eta) d\eta. \end{cases} \tag{3.4}$$

(3) There exists some positive constant $C = C(Q, \alpha, p)$ such that

$$0 < \frac{1}{C} < f, g < C < +\infty,$$

and

$$\|f\|_{C^1(\mathbb{S}^{2n+1})}, \|g\|_{C^1(\mathbb{S}^{2n+1})} \leqslant C.$$

Proof. We will divide the proof into three parts:

1. We show that $N_{Q,\alpha,p}$ can be attained by a pair of nonnegative functions $(f,g) \in L^1(\mathbb{S}^{2n+1}) \times L^1(\mathbb{S}^{2n+1})$.

By density argument, we can choose a pair of nonnegative minimizing sequence $\{f_j,g_j\}_{j=1}^{+\infty} \subset C^\infty(\mathbb{S}^{2n+1}) \times C^\infty(\mathbb{S}^{2n+1})$ such that

$$\|f_j\|_{L^p(\mathbb{S}^{2n+1})} = \|g_j\|_{L^p(\mathbb{S}^{2n+1})} = 1,\ j = 1,2,\cdots$$

and

$$N_{Q,\alpha,p} = \lim_{j\to+\infty} \int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} f_j(\xi) g_j(\eta) |1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2} d\xi d\eta.$$

Step 1. We prove that

$$\|f_j\|_{L^1(\mathbb{S}^{2n+1})} \leqslant C, \quad \|g_j\|_{L^1(\mathbb{S}^{2n+1})} \leqslant C, \quad \text{uniformly.} \tag{3.5}$$

Indeed, from (3.3) we know that there exist two constant C_1 and C_2 such that

$$0 < C_1 \leqslant \int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{f_j(\xi) g_j(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}} d\xi d\eta \leqslant C_2 < \infty.$$

By reversed Hölder's inequality, it holds that

$$\|I_\alpha f_j\|_{L^{p'}(\mathbb{S}^{2n+1})} = \|g_j\|_{L^p(\mathbb{S}^{2n+1})} \|I_\alpha f_j\|_{L^{p'}(\mathbb{S}^{2n+1})} \leqslant C_2,$$
$$\|I_\alpha g_j\|_{L^{p'}(\mathbb{S}^{2n+1})} = \|f_j\|_{L^p(\mathbb{S}^{2n+1})} \|I_\alpha g_j\|_{L^{p'}(\mathbb{S}^{2n+1})} \leqslant C_2,$$

where $\frac{1}{p}+\frac{1}{p'} = 1$ and $I_\alpha f(\xi) = \int_{\mathbb{S}^{2n+1}} |1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2} f(\eta) d\eta$. Noting that $0 > p' > q_\alpha = \frac{2Q}{Q-\alpha}$, for some constant $M > 0$ determined later, we have

$$\begin{aligned} C_2^{p'} &\leqslant \int_{\mathbb{S}^{2n+1}} |I_\alpha f_j|^{p'} d\xi = \int_{I_\alpha f_j \geqslant M} |I_\alpha f_j|^{p'} d\xi + \int_{I_\alpha f_j < M} |I_\alpha f_j|^{p'} d\xi \\ &\leqslant M^{p'} |\mathbb{S}^{2n+1}| + \big|\{I_\alpha f_j < M\}\big|^{1-\frac{p'}{q_\alpha}} \Big(\int_{I_\alpha f_j < M} |I_\alpha f_j|^{q_\alpha} d\xi \Big)^{\frac{p'}{q_\alpha}}. \end{aligned} \tag{3.6}$$

By reversed HLS inequality (1.13) and reversed Hölder inequality, we have

$$\begin{aligned} \|I_\alpha f_j\|_{L^{q_\alpha}(\mathbb{S}^{2n+1})} &\geqslant C_3 \|f_j\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})} \\ &\geqslant C_3 |\mathbb{S}^{2n+1}|^{\frac{1}{p_\alpha}-\frac{1}{p}} \|f_j\|_{L^p(\mathbb{S}^{2n+1})} = C_3 |\mathbb{S}^{2n+1}|^{\frac{1}{p_\alpha}-\frac{1}{p}}. \end{aligned} \tag{3.7}$$

We choose M satisfying $M^{p'}|\mathbb{S}^{2n+1}| = \frac{1}{2} C_2^{p'}$ and follow from (3.6) and (3.7) that

$$\begin{aligned} \frac{1}{2} C_2^{p'} &\leqslant \big|\{I_\alpha f_j < M\}\big|^{1-\frac{p'}{q_\alpha}} \Big(\int_{|I_\alpha f_j| < M} |I_\alpha f_j|^{q_\alpha} d\xi \Big)^{\frac{p'}{q_\alpha}} \\ &\leqslant \big|\{I_\alpha f_j < M\}\big|^{1-\frac{p'}{q_\alpha}} \Big(\int_{\mathbb{S}^{2n+1}} |I_\alpha f_j|^{q_\alpha} d\xi \Big)^{\frac{p'}{q_\alpha}} \\ &\leqslant \big|\{I_\alpha f_j < M\}\big|^{1-\frac{p'}{q_\alpha}} \big(C_3 |\mathbb{S}^{2n+1}|^{\frac{1}{p_\alpha}-\frac{1}{p}} \big)^{p'}, \end{aligned}$$

which leads to

$$\left|\{I_\alpha f_j < M\}\right| \geqslant \left(\frac{C_2'}{C_3|\mathbb{S}^{2n+1}|^{\frac{1}{p\alpha}-\frac{1}{p}}}\right)^{\frac{q\alpha p'}{q\alpha - p'}} > 0,$$

where $(c_2')^{p'} = \frac{1}{2}C_2^{p'}$. So, there exists an $\varepsilon_0 > 0$, such that for any $j \in \mathbb{N}^+$, we can find two points $\xi_j^1, \xi_j^2 \in \{I_\alpha f_j < M\}$ satisfying $|\xi_j^1 - \xi_j^2| \geqslant \varepsilon_0$. Then

$$\begin{aligned}
\int_{\mathbb{S}^{2n+1}} f_j(\xi)d\xi &\leqslant \int_{\mathbb{S}^{2n+1}\setminus\{B(\xi_j^1,\frac{\varepsilon_0}{4})\}} f_j(\xi)d\xi + \int_{\mathbb{S}^{2n+1}\setminus\{B(\xi_j^2,\frac{\varepsilon_0}{4})\}} f_j(\xi)d\xi \\
&\leqslant C_4 \int_{\mathbb{S}^{2n+1}\setminus\{B(\xi_j^1,\frac{\varepsilon_0}{4})\}} |1-\xi_j^1\cdot\bar{\eta}|^{(Q-\alpha)/2} f_j(\xi)d\xi \\
&\quad + C_4 \int_{\mathbb{S}^{2n+1}\setminus\{B(\xi_j^2,\frac{\varepsilon_0}{4})\}} |1-\xi_j^2\cdot\bar{\eta}|^{(Q-\alpha)/2} f_j(\xi)d\xi \\
&\leqslant 2C_4 M.
\end{aligned}$$

Hence, we obtain $\|f_j\|_{L^1(\mathbb{S}^{2n+1})} \leqslant C$. In the same way, we have $\|g_j\|_{L^1(\mathbb{S}^{2n+1})} \leqslant C$.

Step 2. There exist two subsequences of $\{f_j^p\}$ and $\{g_j^p\}$ (still denoted by $\{f_j^p\}$ and $\{g_j^p\}$) and two nonnegative functions $f,g \in L^1(\mathbb{S}^{2n+1})$ such that

$$\int_{\mathbb{S}^{2n+1}} f_j^p d\xi \to \int_{\mathbb{S}^{2n+1}} f^p d\xi, \quad \int_{\mathbb{S}^{2n+1}} g_j^p d\xi \to \int_{\mathbb{S}^{2n+1}} g^p d\xi, \quad \text{as} \quad j \to +\infty. \tag{3.8}$$

In fact, according to the theory of reflexive space, we know from (3.5) that there exist two subsequences of $\{f_j^p\}$ and $\{g_j^p\}$ (still denoted by $\{f_j^p\}$ and $\{g_j^p\}$) and two nonnegative functions $f,g \in L^1(\mathbb{S}^{2n+1})$ such that

$$f_j^p \rightharpoonup f^p \quad \text{and} \quad g_j^p \rightharpoonup g^p \quad \text{weakly in} \quad L^{\frac{1}{p}}(\mathbb{S}^{2n+1}).$$

Using the fact $1 \in L^{\frac{1}{1-p}}(\mathbb{S}^{2n+1})$, we get (3.8).

Step 3. We show

$$\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}} \frac{f(\xi)g(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\xi d\eta \leqslant \liminf_{j\to+\infty} \int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}} \frac{f_j(\xi)g_j(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\xi d\eta. \tag{3.9}$$

As in Lemma 3.2 of [16], we have that, as $j \to +\infty$,

$$\int_{\mathbb{S}^{2n+1}} \frac{g_j^p(\eta)g^{1-p}(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta \to \int_{\mathbb{S}^{2n+1}} \frac{g(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta \tag{3.10}$$

uniformly for $\xi \in \mathbb{S}^{2n+1}$. Then, for any $\varepsilon > 0$, there exists some $N > 0$ such that for any $j > N$,

$$\left|\int_{\mathbb{S}^{2n+1}} \frac{g_j^p(\eta)g^{1-p}(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta - \int_{\mathbb{S}^{2n+1}} \frac{g(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta\right| \leqslant \varepsilon$$

and

$$\begin{aligned}&\Big|\int_{\mathbb{S}^{2n+1}} f_j^p(\xi)f^{1-p}(\xi)\int_{\mathbb{S}^{2n+1}}\frac{g_j^p(\eta)g^{1-p}(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta d\xi\\&\quad-\int_{\mathbb{S}^{2n+1}} f_j^p(\xi)f^{1-p}(\xi)\int_{\mathbb{S}^{2n+1}}\frac{g(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta d\xi\Big|\\&\leqslant \varepsilon\int_{\mathbb{S}^{2n+1}} f_j^p(\xi)f^{1-p}(\xi)d\xi\leqslant C\varepsilon.\end{aligned}\tag{3.11}$$

On the other hand, noting $f^{1-p}(\xi)\in L^{1/(1-p)}(\mathbb{S}^{2n+1})$ and

$$\int_{\mathbb{S}^{2n+1}}|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2}g(\eta)d\eta\leqslant C\int_{\mathbb{S}^{2n+1}}g(\eta)d\eta\leqslant C,$$

we have by the weak convergence that, as $j\to+\infty$,

$$\begin{aligned}&\int_{\mathbb{S}^{2n+1}} f_j^p(\xi)f^{1-p}(\xi)\int_{\mathbb{S}^{2n+1}}\frac{g(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta d\xi\\&\to\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}\frac{f(\xi)g(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta d\xi.\end{aligned}\tag{3.12}$$

Combining (3.11) and (3.12), it holds that

$$\begin{aligned}&\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}\frac{f(\xi)g(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta d\xi\\&=\lim_{j\to+\infty}\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}\frac{f_j^p(\xi)f^{1-p}(\xi)g_j^p(\eta)g^{1-p}(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta d\xi\\&\leqslant\lim_{j\to+\infty}\Big(\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}\frac{f_j(\xi)g_j(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta d\xi\Big)^p\\&\qquad\cdot\Big(\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}\frac{f(\xi)g(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}}d\eta d\xi\Big)^{1-p}.\end{aligned}\tag{3.13}$$

Thus, (3.9) holds.

Combining Step 1, Step 2 with Step 3, we know that the function pair $(f,g)\in L^1(\mathbb{S}^{2n+1})\times L^1(\mathbb{S}^{2n+1})$ is a minimizer.

2. We present that f,g satisfy the Euler-Lagrange equations (3.4).

Because $0<p<1$, it brings some difficulties to deduce (3.4). To overcome it, we need to prove $f>0,\ g>0$ a.e. on $\mathbb{S}^{2n+1}$.

For any positive $\varphi\in C^\infty(\mathbb{S}^{2n+1})$ and $t>0$ small, we have $f+t\varphi>0$ on $\mathbb{S}^{2n+1}$ and

$$\begin{aligned}&t\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}\varphi(\xi)g(\eta)|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2}d\xi d\eta\\&=\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}(f+t\varphi)(\xi)g(\eta)|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2}d\xi d\eta\\&\quad-\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}f(\xi)g(\eta)|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2}d\xi d\eta\end{aligned}$$

$$
\begin{aligned}
&\geqslant N_{Q,\alpha,p}\left(\|f+t\varphi\|_{L^p(\mathbb{S}^{2n+1})}-\|f\|_{L^p(\mathbb{S}^{2n+1})}\right)\\
&=N_{Q,\alpha,p}t\cdot\left(\int_{\mathbb{S}^{2n+1}}(f+\theta\varphi)^p d\xi\right)^{\frac{1}{p}-1}\int_{\mathbb{S}^{2n+1}}(f+\theta\varphi)^{p-1}\varphi d\xi \ (0<\theta<t), \qquad (3.14)
\end{aligned}
$$

where the mean value theorem was used between the fourth and fifth line. Then, by Fatou's lemma, it has

$$
\begin{aligned}
&\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}\varphi(\xi)g(\eta)|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2}d\xi d\eta\\
\geqslant& N_{Q,\alpha,p}\lim_{t\to 0^+}\left(\int_{\mathbb{S}^{2n+1}}(f+\theta\varphi)^p d\xi\right)^{\frac{1}{p}-1}\cdot\lim_{t\to 0^+}\int_{\mathbb{S}^{2n+1}}(f+\theta\varphi)^{p-1}\varphi d\xi\\
\geqslant& N_{Q,\alpha,p}\int_{\mathbb{S}^{2n+1}}f^{p-1}\varphi d\xi. \qquad (3.15)
\end{aligned}
$$

By now, we claim that $f>0$ a.e. on $\mathbb{S}^{2n+1}$. Otherwise, for any $\varepsilon>0$, there exists $\Omega_\varepsilon\subset\mathbb{S}^{2n+1}$ such that $|\Omega_\varepsilon|>0$ and

$$
f(\xi)<\varepsilon, \quad \forall\xi\in\Omega_\varepsilon.
$$

Then, it follows from (3.15) that

$$
\begin{aligned}
\varepsilon^{p-1}\int_{\Omega_\varepsilon}d\xi\leqslant&\int_{\Omega_\varepsilon}f^{p-1}d\xi\\
\leqslant&\frac{1}{N_{Q,\alpha,p}}\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}g(\eta)|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2}d\xi d\eta\\
\leqslant& C\int_{\mathbb{S}^{2n+1}}g(\eta)d\eta\leqslant C,
\end{aligned}
$$

which yields a contradiction as $\varepsilon>0$ small enough. Similarly, we also have $g>0$ a.e. on $\mathbb{S}^{2n+1}$. So, minimizer pair (f,g) is a weak solution of (3.4).

3. We finally prove that $(f,g)\in C^1(\mathbb{S}^{2n+1})\times C^1(\mathbb{S}^{2n+1})$.

Since $f,g\in L^1(\mathbb{S}^{2n+1})$ and $0<p<p_\alpha<1$, it is easy to prove from (3.4) that $f\geqslant C_6>0$ and $g\geqslant C_6>0$. Then, by (3.4), we have $f<C_7$ and $g<C_7$. Moreover, since $\alpha>Q\geqslant 4$, we have $f,g\in C^1(\mathbb{S}^{2n+1})$ and $\|f\|_{C^1(\mathbb{S}^{2n+1})},\|f\|_{C^1(\mathbb{S}^{2n+1})}\leqslant C_8<+\infty$. $\square$

4. Sharp HLS inequalities on $\mathbb{S}^{2n+1}$

LEMMA 4.1. *$N_{Q,\alpha,p}\to N_{Q,\alpha}$ as $p\to p_\alpha^-$. Further more, the corresponding minimizer pairs $\{f_p,g_p\}\in C^1(\mathbb{S}^{2n+1})\times C^1(\mathbb{S}^{2n+1})$ form a minimizing sequence for sharp constant $N_{Q,\alpha}$, namely,*

$$
N_{Q,\alpha}=\lim_{p\to p_\alpha^-}\frac{\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}f_p(\xi)g_p(\eta)|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2}d\xi d\eta}{\|f_p\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}\|g_p\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}}. \qquad (4.1)
$$

Proof. Let $\{f_p, g_p\} \in C^1(\mathbb{S}^{2n+1}) \times C^1(\mathbb{S}^{2n+1})$ be a pair of minimizer given by Proposition 2.1. Namely, $\{f_p, g_p\}$ satisfy $\|f_p\|_{L^p(\mathbb{S}^{2n+1})} = \|g_p\|_{L^p(\mathbb{S}^{2n+1})} = 1$ and

$$N_{Q,\alpha,p} = \int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{f_p(\xi) g_p(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}} d\xi d\eta.$$

Write $\tilde{f}_p = \frac{f_p}{\|f_p\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}}$ and $\tilde{g}_p = \frac{g_p}{\|g_p\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}}$. Then

$$\begin{aligned} N_{Q,\alpha,p} =& \|f_p\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})} \|g_p\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})} \int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{\tilde{f}_p(\xi)\tilde{g}_p(\eta)}{|1-\xi\cdot\bar{\eta}|^{(Q-\alpha)/2}} d\xi d\eta \\ \geqslant& |\mathbb{S}^{2n+1}|^{2(1/p_\alpha - 1/p)} N_{Q,\alpha} \to N_{Q,\alpha}, \quad \text{as} \quad p \to p_\alpha^-, \end{aligned}$$

which implies that

$$\liminf_{p\to p_\alpha^+} N_{Q,\alpha,p} \geqslant N_{Q,\alpha}. \tag{4.2}$$

Let $\{f_k, g_k\}_{k=1}^{+\infty} \subset L^{p_\alpha}(\mathbb{S}^{2n+1}) \times L^{p_\alpha}(\mathbb{S}^{2n+1})$ be a pair of minimizing sequence of $N_{Q,\alpha}$, namely,

$$N_{Q,\alpha} = \lim_{k\to+\infty} \frac{\int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} f_k(\xi) g_k(\eta) |1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2} d\xi d\eta}{\|f_k\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})} \|g_k\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}}.$$

Write $\tilde{f}_k = \frac{f_k}{\|f_k\|_{L^p(\mathbb{S}^{2n+1})}}$ and $\tilde{g}_k = \frac{g_k}{\|g_k\|_{L^p(\mathbb{S}^{2n+1})}}$ for any $p \in (0, p_\alpha)$. It is easy to see

$$\begin{aligned} N_{Q,\alpha,p} \leqslant& \frac{\int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \tilde{f}_k(\xi) \tilde{g}_k(\eta) |1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2} d\xi d\eta}{\|\tilde{f}_k\|_{L^p(\mathbb{S}^{2n+1})} \|\tilde{g}_k\|_{L^p(\mathbb{S}^{2n+1})}} \\ =& \frac{\int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} f_k(\xi) g_k(\eta) |1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2} d\xi d\eta}{\|f_k\|_{L^p(\mathbb{S}^{2n+1})} \|g_k\|_{L^p(\mathbb{S}^{2n+1})}}. \end{aligned} \tag{4.3}$$

Sending p to p_α^- in (4.3), we get

$$\limsup_{p\to p_\alpha^-} N_{Q,\alpha,p} \leqslant \frac{\int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} f_k(\xi) g_k(\eta) |1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2} d\xi d\eta}{\|f_k\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})} \|g_k\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}}.$$

And then, letting $k \to +\infty$, we deduce

$$\limsup_{p\to p_\alpha^+} N_{Q,\alpha,p} \leqslant N_{Q,\alpha}. \tag{4.4}$$

Combining (4.2) with (4.4), we arrive at $\lim_{p\to p_\alpha^+} N_{Q,\alpha,p} = N_{Q,\alpha}$.

By the definition of $N_{Q,\alpha}$ and Hölder inequality,

$$\begin{aligned} N_{Q,\alpha} \leqslant& \frac{\int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} f_p(\xi) g_p(\eta) |1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2} d\xi d\eta}{\|f_p\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})} \|g_p\|_{L^{p_\alpha}(\mathbb{S}^{2n+1})}} \\ \leqslant& \frac{\int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} f_p(\xi) g_p(\eta) |1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2} d\xi d\eta}{|\mathbb{S}^{2n+1}|^{2(1/p_\alpha - 1/p)}} \\ \to& N_{Q,\alpha} \quad \text{as} \quad p \to p_\alpha^-. \end{aligned}$$

Hence, we deduce that (4.1) holds and the lemma is proved. □

Proof of Theorem 1.3. As in Lemma 4.1, take the minimizer $\{f_p, g_p\} \in C^1(\mathbb{S}^{2n+1}) \times C^1(\mathbb{S}^{2n+1})$ as a minimizing sequence for $N_{Q,\alpha}$. Then, $\{f_p, g_p\}$ satisfy (3.4). By the translation invariance, we assume, without loss of generality, that $f_p(\mathfrak{N}) = \max_{\xi \in \mathbb{S}^n} f_p(\xi)$ with $\mathfrak{N} = (0, \cdots, 0, 1)$.

Case 1: For some subsequence $p_j \to p_\alpha^-$, $\max\{\max_{\xi \in \mathbb{S}^{2n+1}} f_{p_j}, \max_{\xi \in \mathbb{S}^{2n+1}} g_{p_j}\}$ is uniformly bounded. Then, sequences $\{f_{p_j}\}$ and $\{g_{p_j}\}$ are uniformly bounded and equicontinuous on $\mathbb{S}^{2n+1}$. Moreover, by (3.4), there exists some positive constant C independent of p_j such that $f_{p_j}, g_{p_j} \geqslant C > 0$. So, by Arzelà-Ascoli theorem, there exist two subsequences of $\{f_{p_j}\}$ and $\{g_{p_j}\}$ (still denoted by $\{f_{p_j}\}$ and $\{g_{p_j}\}$) and two positive functions $f, g \in C^1(\mathbb{S}^{2n+1})$ such that

$$f_{p_j} \to f \quad \text{and} \quad g_{p_j} \to g \quad \text{uniformly on} \quad \mathbb{S}^{2n+1}.$$

Then,

$$\int_{\mathbb{S}^{2n+1}} f^{p_\alpha}(\xi) d\xi = \lim_{p_j \to p_\alpha} \int_{\mathbb{S}^{2n+1}} f_{p_j}^{p_j}(\xi) d\xi = 1,$$
$$\int_{\mathbb{S}^{2n+1}} g^{p_\alpha}(\xi) d\xi = \lim_{p_j \to p_\alpha} \int_{\mathbb{S}^{2n+1}} g_{p_j}^{p_j}(\xi) d\xi = 1.$$

Furthermore, by (3.4) and Lemma 4.1,

$$\begin{cases} N_{Q,\alpha} f^{p_\alpha - 1}(\xi) = \int_{\mathbb{S}^{2n+1}} |1 - \xi \cdot \bar{\eta}|^{(\alpha - Q)/2} g(\eta) d\eta, \\ N_{Q,\alpha} g^{p_\alpha - 1}(\xi) = \int_{\mathbb{S}^{2n+1}} |1 - \xi \cdot \bar{\eta}|^{(\alpha - Q)/2} g(\eta) d\eta, \end{cases} \tag{4.5}$$

as $j \to +\infty$. Namely, $\{f, g\}$ are minimizers.

Case 2: For any subsequence $p_j \to p_\alpha^-$, $f_{p_j}(\mathfrak{N}) \to +\infty$ or $\max_{\xi \in \mathbb{S}^n} g_{p_j} \to +\infty$. Without loss of generality, we assume $f_{p_j}(\mathfrak{N}) \to +\infty$.

Case 2a: $\limsup_{j \to +\infty} \frac{f_{p_j}(\mathfrak{N})}{\max_{\xi \in \mathbb{S}^{2n+1}} g_{p_j}} = +\infty$. Then, there exists a subsequence of $\{p_j\}$ (still denoted by $\{p_j\}$) such that $f_{p_j}(\mathfrak{N}) \to +\infty$ and $\frac{f_{p_j}(\mathfrak{N})}{\max_{\xi \in \mathbb{S}^{2n+1}} g_{p_j}} \to +\infty$. Let $\phi_j = f_{p_j}^{p_j - 1}$ and $\psi_j = g_{p_j}^{p_j - 1}$. Then, ϕ_j and ψ_j satisfy

$$\int_{\mathbb{S}^{2n+1}} \phi_j^{q_j} d\xi = \int_{\mathbb{S}^{2n+1}} \psi_j^{q_j} d\xi = 1 \tag{4.6}$$

and by (3.4),

$$\begin{cases} N_{Q,\alpha,p_j} \phi_j(\xi) = \int_{\mathbb{S}^{2n+1}} |1 - \xi \cdot \bar{\eta}|^{(\alpha - Q)/2} \psi_j^{q_j - 1}(\eta) d\eta, \\ N_{Q,\alpha,p_j} \psi_j(\xi) = \int_{\mathbb{S}^{2n+1}} |1 - \xi \cdot \bar{\eta}|^{(\alpha - Q)/2} \phi_j^{q_j - 1}(\eta) d\eta, \end{cases} \tag{4.7}$$

where $\frac{1}{p_j}+\frac{1}{q_j}=1$. Applying Cayley transformation and dilations on $\mathbb{H}^n$, we get from (4.7) that

$$\begin{cases} \frac{N_{Q,\alpha,p_j}\phi_j(\mathscr{C}(\delta_\lambda(u)))}{\left((1+|\lambda z|^2)^2+(\lambda^2 t)^2\right)^{\frac{Q-\alpha}{4}}} = 2^{\frac{Q+\alpha-2}{2}}\lambda^\alpha \int_{\mathbb{H}^n} \frac{\left((1+|\lambda z'|^2)^2+(\lambda^2 t')^2\right)^{-\frac{Q+\alpha}{4}}\psi_j(\mathscr{C}(\delta_\lambda(v)))^{q_j-1}}{|u^{-1}v|^{Q-\alpha}}dv, \\ \frac{N_{Q,\alpha,p_j}\psi_j(\mathscr{C}(\delta_\lambda(u)))}{\left((1+|\lambda z|^2)^2+(\lambda^2 t)^2\right)^{\frac{Q-\alpha}{4}}} = 2^{\frac{Q+\alpha-2}{2}}\lambda^\alpha \int_{\mathbb{H}^n} \frac{\left((1+|\lambda z'|^2)^2+(\lambda^2 t')^2\right)^{-\frac{Q+\alpha}{4}}\phi_j(\mathscr{C}(\delta_\lambda(v)))^{q_j-1}}{|u^{-1}v|^{Q-\alpha}}dv. \end{cases} \tag{4.8}$$

Take $\lambda=\lambda_j$ satisfying $\lambda_j^{\alpha/(q_j-2)}\phi_j(\mathscr{C}(0))=1$ and denote

$$\begin{cases} \Phi_j(u) = \frac{\lambda_j^{\alpha/(q_j-2)}}{\left((1+|\lambda z|^2)^2+(\lambda^2 t)^2\right)^{\frac{Q-\alpha}{4}}}\phi_j(\mathscr{C}(\delta_{\lambda_j}(u))), \\ \Psi_j(u) = \frac{\lambda_j^{\alpha/(q_j-2)}}{\left((1+|\lambda z|^2)^2+(\lambda^2 t)^2\right)^{\frac{Q-\alpha}{4}}}\psi_j(\mathscr{C}(\delta_{\lambda_j}(u))). \end{cases} \tag{4.9}$$

Then, Φ_j,Ψ_j satisfy the following renormalized equations

$$\begin{cases} N_{Q,\alpha,p_j}\Phi_j(u) = 2^{\frac{Q+\alpha-2}{2}}\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(v)}{|u^{-1}v|^{Q-\alpha}}dv, \\ N_{Q,\alpha,p_j}\Psi_j(u) = 2^{\frac{Q+\alpha-2}{2}}\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Phi_j^{q_j-1}(v)}{|u^{-1}v|^{Q-\alpha}}dv. \end{cases} \tag{4.10}$$

Moreover, $\Phi_j(u)\geqslant\Phi_j(0)=1$ and

$$\Psi_j(u)\geqslant\lambda_j^{\alpha/(q_j-2)}\min_{\xi\in\mathbb{S}^{2n+1}}\psi_j=\frac{\min_{\xi\in\mathbb{S}^{2n+1}}\psi_j}{\phi_j(\mathscr{S}(0))}\to+\infty \tag{4.11}$$

uniformly for any u as $j\to+\infty$.

Claim. There exist $C_1,C_2>0$ such that, for any $u\in\mathbb{H}^n$, when $j\to\infty$,

$$0<C_1(1+|u|^{\alpha-Q})\leqslant\Phi_j(u)\leqslant C_2(1+|u|^{\alpha-Q}) \text{ uniformly.} \tag{4.12}$$

Once the claim holds,

$$\begin{aligned} N_{Q,\alpha,p_j}\Psi_j(0) =& 2^{\frac{Q+\alpha-2}{2}}\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Phi_j^{q_j-1}(v)}{|v|^{Q-\alpha}}dv \\ \leqslant& C\int_{\mathbb{H}^n}|v|^{\alpha-Q}(1+|v|^{\alpha-Q})^{q_j-1}dv\leqslant C, \end{aligned}$$

which contradicts with (4.11). This shows that Case 2a does not appear.

Now, we give the proof of the claim (4.12). Noting that

$$N_{Q,\alpha,p_j}=2^{\frac{Q+\alpha-2}{2}}\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(v)}{|v|^{Q-\alpha}}dv\leqslant C<+\infty \tag{4.13}$$

uniformly as $j\to\infty$, we obtain from (4.11) that as $j\to\infty$ and $|u|\geqslant 1$,

$$\begin{aligned}&\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\\ \leqslant&\int_{|v|\leqslant 1}C\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}dv\\ &+\int_{|v|>1}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(v)}{|v|^{Q-\alpha}}dv\leqslant C<+\infty \end{aligned}\tag{4.14}$$

and

$$\begin{aligned}&\int_{\mathbb{H}^n}\frac{|u^{-1}v|^{\alpha-Q}}{|u|^{\alpha-Q}}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\\ \leqslant&C\int_{\mathbb{H}^n}(1+|v|^{\alpha-Q})\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\\ \leqslant&C<+\infty\end{aligned}\tag{4.15}$$

uniformly. By dominated convergence theorem,

$$\begin{aligned}&\lim_{|u|\to+\infty}\frac{\Phi_j(u)}{|u|^{\alpha-Q}}\\ =&\frac{2^{\frac{Q+\alpha-2}{2}}}{N_{Q,\alpha,p_j}}\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\leqslant C.\end{aligned}\tag{4.16}$$

On the other hand, if we can prove

$$\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\geqslant C'>0\ \text{ as }\ j\to\infty,\tag{4.17}$$

then we have the claim (4.12). By contradiction, we assume that (4.17) does not hold. Then, there exists a subsequence (still denoted as $\{\Psi_j\}$) such that

$$\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\to 0,\quad \text{as}\quad j\to+\infty.\tag{4.18}$$

For any $u\in B(0,1)$,

$$\begin{aligned}1\leqslant&\Phi_j(u)=\frac{2^{\frac{Q+\alpha-2}{2}}}{N_{Q,\alpha,p_j}}\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(v)}{|u^{-1}v|^{Q-\alpha}}dv\\ \leqslant&\frac{2^{\frac{Q+\alpha-2}{2}}}{N_{Q,\alpha,p_j}}\Bigg((4\gamma)^{\alpha-Q}\int_{|v|\leqslant 3}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\\ &+\int_{|v|>3}\left(\frac{4}{3}\gamma\right)^{\alpha-Q}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(v)}{|v|^{Q-\alpha}}dv\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{2^{\frac{Q+\alpha-2}{2}}}{N_{Q,\alpha,p_j}}(4\gamma)^{\alpha-Q}\int_{|v|\leqslant 3}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\\
&\qquad+\left(\frac{4}{3}\gamma\right)^{\alpha-Q}.
\end{aligned}
$$

From (4.18), there exists $N_0>0$ such that

$$
1\leqslant\Phi_j(u)\leqslant 1+\left(\frac{4}{3}\gamma\right)^{\alpha-Q}
$$

for $j\geqslant N_0$. Then, for $|u|\geqslant 3$, if follows from (4.10) that

$$
\begin{aligned}
\Psi_j(u)\geqslant&\frac{2^{\frac{Q+\alpha-2}{2}}}{N_{Q,\alpha,p_j}}\int_{|v|\leqslant 1}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Phi_j^{q_j-1}(v)}{|u^{-1}v|^{Q-\alpha}}dv\\
\geqslant&C\int_{|v|\leqslant 1}|u|^{\alpha-Q}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}dv\\
\geqslant&C|u|^{\alpha-Q},
\end{aligned}
\tag{4.19}
$$

for $j\geqslant N_0$. We used the fact in the last inequality: as $j\to\infty$,

$$
\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\to 1\quad\text{uniformly on } B(0,1).
$$

Letting p_j close to p_α and choosing $R\gg 3$, it follows from (4.19) that

$$
\begin{aligned}
N_{Q,\alpha,p_j}=&2^{\frac{Q+\alpha-2}{2}}\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(v)}{|v|^{Q-\alpha}}dv\\
\leqslant&CR^{\alpha-Q}\int_{|v|\leqslant R}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\\
&+C\int_{|v|>R}|v|^{\alpha-Q}\cdot|v|^{(\alpha-Q)(q_j-1)}dy\\
\leqslant&CR^{\alpha-Q}\int_{|v|\leqslant R}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\Psi_j^{q_j-1}(v)dv\\
&+CR^{(\alpha-Q)q_j+Q}.
\end{aligned}
$$

Taking firstly R large enough and then letting $j\to+\infty$, we have $N_{Q,\alpha,p_j}\to 0$, which is contradiction with $N_{Q,\alpha,p_j}\to N_{Q,\alpha}$. Hence, (4.17) holds.

Case 2b: $\limsup_{j\to+\infty}\frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^{2n+1}}g_{p_j}}=0$. Then, there exists a subsequence of $\{p_j\}$ (still denoted as $\{p_j\}$) such that $f_{p_j}(\mathfrak{N})\to+\infty$ and $\frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^{2n+1}}g_{p_j}}\to 0$, which implies that $\max_{\xi\in\mathbb{S}^{2n+1}}g_{p_j}\to+\infty$. Similar to Case 2a, we can show that Case 2b does not appear.

Case 2c: $\limsup_{j\to+\infty}\frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^{2n+1}}g_{p_j}}=c_0\in(0,+\infty)$. Then, there exists a subsequence of $\{p_j\}$ (still denoted as $\{p_j\}$) such that $f_{p_j}(\mathfrak{N})\to+\infty$, $\max g_{p_j}\to+\infty$ and $\frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^{2n+1}}g_{p_j}}\to c_0\in(0,+\infty)$. As Case 2a, choose a sequence of function pairs $\{\Phi_j,\Psi_j\}$ defined as (4.9), which satisfies (4.10), $\Phi_j(u)\geqslant\Phi_j(0)=1$ and

$$\Psi_j(u)\geqslant\lambda_j^{\alpha/(q_j-2)}\min_{\xi\in\mathbb{S}^{2n+1}}\psi_j=\frac{\min_{\xi\in\mathbb{S}^{2n+1}}\psi_j}{\phi_j(\mathscr{C}(0))}\to c_0^{1-p_\alpha}\in(0,+\infty) \tag{4.20}$$

uniformly for any u as $j\to+\infty$. So, $\{\Psi_j(u)\}$ have uniformly lower bound $C>0$.

Repeating the proof of (4.12), there exist two positive constants C_1 and C_2 such that, as $j\to+\infty$,

$$0<C_1(1+|u|^{\alpha-Q})\leqslant\Phi_j(u)\leqslant C_2(1+|u|^{\alpha-Q}), \tag{4.21}$$

$$0<C_1(1+|u|^{\alpha-Q})\leqslant\Psi_j(u)\leqslant C_2(1+|u|^{\alpha-Q}) \tag{4.22}$$

uniformly for any u.

For any given constant $R_0>0$ and any $u\in B(0,R_0)$, as $j\to+\infty$, we have by (4.22) that

$$\begin{aligned}
&2^{-\frac{Q+\alpha-2}{2}}N_{Q,\alpha,p_j}\Phi_j(u) &(4.23)\\
=&\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(v)}{|u^{-1}v|^{Q-\alpha}}dv\\
=&\int_{\mathbb{H}^n}\left((1+|\lambda_j(z+z')|^2)^2+(\lambda_j^2(t+t'+2\mathrm{Im}(z\cdot\overline{z'})))^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(uv)}{|v|^{Q-\alpha}}dv\\
\leqslant&\int_{|v|\leqslant 2R_0}|v|^{\alpha-Q}C_1^{q_j-1}dv+\int_{|v|>2R_0}|v|^{\alpha-Q}C_1^{q_j-1}|uv|^{(\alpha-Q)(q_j-1)}dv\\
\leqslant&C(2R_0)^\alpha+C\int_{|v|>2R_0}|v|^{(\alpha-Q)q_j}dv\leqslant C, &(4.24)
\end{aligned}$$

namely, $\Phi_j(u)$ is uniformly bounded on $B(0,R_0)$. Similarly, $\Psi_j(u)$ is also uniformly bounded on $B(0,R_0)$.

Noting $\alpha>Q\geqslant 4$ and arguing as (4.23), we have that, as $j\to+\infty$,

$$\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(v)}{|u^{-1}v|^{Q-\alpha+1}}dv\leqslant C$$

and

$$\int_{\mathbb{H}^n}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}\frac{\Psi_j^{q_j-1}(v)}{|u^{-1}v|^{Q-\alpha+2}}dv\leqslant C$$

uniformly for any $u \in B(0,R_0)$. So, for any $u \in B(0,R_0)$ and $l,k = 1,2,\cdots,2n$, a direct computation yields

$$T_lT_k\Phi_j(u) = \frac{2^{\frac{Q+\alpha-2}{2}}}{N(Q,\alpha,p_j)}\int_{\mathbb{H}^n} T_lT_k(|u^{-1}v|^{\alpha-Q})\frac{\Psi_j^{q_j-1}(v)}{\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{Q-\alpha}{4}(q_\alpha-q_j)}}dv,$$

where $T_l = X_l, T_{l+n} = Y_l$ for $l = 1,2,\cdots,n$ and

$$X_l = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}, \quad Y_l = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t}, \quad l = 1,2,\cdots,n$$

are the left invariant vector fields on the Heisenberg group. Moreover, we know that $\Phi_j \in C^1(B(0,R_0))$ by Theorem 20.1 of [22]. Since the arbitrariness of R_0, we know that $\Phi_j(u) \in C^1(\mathbb{H}^n)$ and $\|\Phi_j\|_{C^1(B(0,R_0))}$ is uniformly bounded. Similarly, we can obtain that $\Psi_j(u) \in C^1(\mathbb{H}^n)$ and $\|\Psi_j\|_{C^1(B(0,R_0))}$ is uniformly bounded.

By Arzelà-Ascoli theorem, there exist two subsequences of $\{\Phi_j\}$ and $\{\Psi_j\}$ (still denoted as $\{\Phi_j\}$ and $\{\Psi_j\}$) and two functions $U,V \in C^1(\mathbb{H}^n)$ such that

$$\Phi_j \to U \quad \text{and} \quad \Psi_j \to V \quad \text{uniformly on} \quad B(0,R_0). \tag{4.25}$$

Moreover, by (4.21) and (4.22), it holds

$$0 < C_1(1+|u|^{\alpha-Q}) \leqslant U(u) \leqslant C_2(1+|u|^{\alpha-Q}), \tag{4.26}$$

$$0 < C_1(1+|u|^{\alpha-Q}) \leqslant V(u) \leqslant C_2(1+|u|^{\alpha-Q}). \tag{4.27}$$

By the arbitrariness of R_0, we prove that $U(u)$ and $V(u)$ satisfy

$$\begin{cases} N_{Q,\alpha}U(u) = 2^{\frac{Q+\alpha-2}{2}}\int_{\mathbb{H}^n}|u^{-1}v|^{\alpha-Q}V^{q_\alpha-1}(v)dv & \text{in} \quad \mathbb{H}^n, \\ N_{Q,\alpha}V(u) = 2^{\frac{Q+\alpha-2}{2}}\int_{\mathbb{H}^n}|u^{-1}v|^{\alpha-Q}U^{q_\alpha-1}(v)dv & \text{in} \quad \mathbb{H}^n. \end{cases} \tag{4.28}$$

Since

$$\begin{aligned} 1 &= \int_{\mathbb{S}^{2n+1}} \phi_j^{q_j}(\xi)d\xi \\ &= 2^{Q-1}\int_{\mathbb{H}^n} \Phi_j^{q_j}(u)\lambda_j^{Q-\frac{\alpha q_j}{q_j-2}}\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}dv \\ &\leqslant 2^{Q-1}\int_{\mathbb{H}^n} \Phi_j^{q_j}(u)\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}dv \end{aligned}$$

and

$$\Phi_j^{q_j}(u)\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)} \to U^{q_\alpha}(u)$$

uniformly on any compact domain, it follows from (4.21) that

$$\int_{\mathbb{H}^n} U^{q_\alpha}du = \lim_{j\to+\infty}\int_{\mathbb{H}^n} \Phi_j^{q_j}(u)\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)}du \geqslant 2^{1-Q}.$$

Similarly, by (4.22) it also holds

$$\int_{\mathbb{H}^n} V^{q_\alpha} du = \lim_{j\to+\infty} \int_{\mathbb{H}^n} \Psi_j^{q_j}(u)\left((1+|\lambda_j z'|^2)^2+(\lambda_j^2 t')^2\right)^{\frac{\alpha-Q}{4}(q_\alpha-q_j)} du \geqslant 2^{1-Q}.$$

Let $F(u)=U^{q_\alpha-1}(u)$ and $G(u)=V^{q_\alpha-1}(u)$, we have $\int_{\mathbb{H}^n} F^{p_\alpha} du \geqslant 2^{1-Q}$, $\int_{\mathbb{H}^n} G^{p_\alpha} du \geqslant 2^{1-Q}$ and F,G satisfy

$$\begin{cases} N_{Q,\alpha} F^{p_\alpha-1}(u) = 2^{\frac{Q+\alpha-2}{2}} \int_{\mathbb{H}^n} |u^{-1}v|^{\alpha-Q} G(v)dv & \text{in} \quad \mathbb{H}^n, \\ N_{Q,\alpha} G^{p_\alpha-1}(u) = 2^{\frac{Q+\alpha-2}{2}} \int_{\mathbb{H}^n} |u^{-1}v|^{\alpha-Q} F(v)dv & \text{in} \quad \mathbb{H}^n. \end{cases}$$

Combining $2>p_\alpha$ wtih Cayley transformation, it holds

$$\begin{aligned} N_{Q,\alpha}^2 &= \frac{\left(2^{\frac{Q+\alpha-2}{2}} \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} F(u)|u^{-1}v|^{\alpha-Q} G(v)dvdu\right)^2}{\int_{\mathbb{H}^n} F^{p_\alpha} du \int_{\mathbb{H}^n} G^{p_\alpha} du} \\ &\geqslant \frac{\left(2^{\frac{Q+\alpha-2}{2}} \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} F(u)|u^{-1}v|^{\alpha-Q} G(v)dvdu\right)^2}{2^{2-2Q}\left(2^{Q-1}\int_{\mathbb{H}^n} F^{p_\alpha} du\right)^{2/p_\alpha}\left(2^{Q-1}\int_{\mathbb{H}^n} G^{p_\alpha} du\right)^{2/p_\alpha}} \\ &= \frac{\left(2^{\frac{Q+\alpha-2}{2}} 2^{-\frac{n(Q-\alpha)}{Q}} \int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}} f(\xi)|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2} g(\eta)d\eta d\xi\right)^2}{2^{2-2Q}2^{\frac{4(Q-1)}{p_\alpha}}\left(\int_{\mathbb{S}^{2n+1}} f^{p_\alpha} d\xi\right)^{2/p_\alpha}\left(\int_{\mathbb{S}^{2n+1}} g^{p_\alpha} d\xi\right)^{2/p_\alpha}} \\ &= \frac{\left(\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}} f(\xi)|1-\xi\cdot\bar{\eta}|^{(\alpha-Q)/2} g(\eta)d\eta d\xi\right)^2}{\left(\int_{\mathbb{S}^{2n+1}} f^{p_\alpha} d\xi\right)^{2/p_\alpha}\left(\int_{\mathbb{S}^{2n+1}} g^{p_\alpha} d\xi\right)^{2/p_\alpha}}, \end{aligned}$$

where

$$F(u) = f(\mathscr{C}(u))J_{\mathscr{C}}(u)^{1/p_\alpha}, \quad G(u) = g(\mathscr{C}(u))J_{\mathscr{C}}(u)^{1/p_\alpha}.$$

Hence, $\{f(\xi),g(\xi)\}$ is a pair of minimizer of sharp constant $N_{Q,\alpha}$. Furthermore, they satisfy the Euler-Lagrange equations (4.5).

By (4.26) and (4.27), there exists a positive constant C such that

$$0 < \frac{1}{C} \leqslant f,g \leqslant C.$$

Since $\alpha > Q \geqslant 4$, we know by (4.5) that $f,g \in C^1(\mathbb{S}^{2n+1})$. $\square$

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