Mathematical
Inequalities
& Applications
Volume 27, Number 4 (2024), 859–885 doi:10.7153/mia-2024-27-59

WEIGHTED NORM INEQUALITIES FOR SCHRÖDINGER OPERATORS ON VARIABLE LEBESGUE SPACES

ADRIÁN CABRAL

(*Communicated by I. Perić*)

Abstract. In this work we show that many operators from harmonic analysis associated with the semigroup generated by the Schrödinger operator $\mathscr{L} = -\Delta + V$ in $\mathbb{R}^n$, where $n > 2$ and the non–negative potential V belongs to the reverse Hölder class RH_q with $q > n/2$ – such as maximal operators, the Littlewood–Paley function, pseudo–differential operators, singular integrals, and their commutators – are bounded on the weighted variable Lebesgue space $L^{p(\cdot)}(w)$. We do so by applying the theory of weighted norm inequalities and extrapolation.

1. Introduction

In this paper, we consider the Schrödinger differential operator in $\mathbb{R}^n$ with $n > 2$, defined by

$$\mathscr{L} = -\Delta + V, \tag{1}$$

where $V \geqslant 0$ and belongs to a reverse-Hölder class RH_q for some exponent $q > n/2$, i.e., there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B V(x)^q dx\right)^{1/q} \leqslant \frac{C}{|B|}\int_B V(x)\,dx, \tag{2}$$

for every ball $B \subset \mathbb{R}^n$. By Hölder inequality we can get that $RH_q \subset RH_p$, for $q \geqslant p > 1$. One remarkable feature about the RH_q class is that, if $V \in RH_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on n and the constant C in (2), such that $V \in RH_{q+\varepsilon}$. Therefore, it is equivalent consider $q > n/2$ or $q \geqslant n/2$.

The fundamental results that laid the foundation for the development of the theory of harmonic analysis related to the Schrödinger operator under the hypotheses mentioned above, are found in the work of Z. Shen [28]. There, basic tools are introduced, such as the definition of the critical radius function ρ associated with the potential V, its properties, estimates of the fundamental solution, and the study of the behavior of operators such as the Riesz transform in this context, on Lebesgue spaces.

In recent years, a wide range of operators associated to $\mathscr{L}$ have been catching the attention of several authors (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 22, 23, 25, 29, 30, 31, 32]).

Mathematics subject classification (2020): Primary 42B25; Secondary 35J10.
Keywords and phrases: Norm inequalities, Schrödinger operator, variable Lebesgue spaces, extrapolation, weights.

In these works, the authors have addressed the boundedness results of such operators acting on different spaces, such as weighted L^p spaces, appropriate Hardy spaces, as well as regularity spaces like BMO. In this paper, we seek to extend these results to the case of weighted variable Lebesgue spaces $L^{p(\cdot)}(w)$. These spaces were introduced in [12, 14] as a natural generalization of the classical weighted Lebesgue spaces and the variable Lebesgue spaces $L^{p(\cdot)}$, which have been extensively studied for the past 30 years since the publication of [24]–see [13, 21] and the references they contain.

Given the different nature of these operators, the boundedness of each of them in classical $L^p(w)$ spaces has been approached in a different way. Here, rather than considering estimates for individual operators, we apply techniques from the theory of weighted normal inequalities and extrapolation to show that the boundedness of a wide variety of such operators follows from the boundedness of certain maximal operators on variable L^p spaces, and from known estimates on weighted Lebesgue spaces. For this purpose, we will use the natural bridge that exists between the spaces $L^p(w)$ and $L^{p(\cdot)}(w)$ given by the theory of Rubio de Francia extrapolation. Building on the results in [14], Cruz–Uribe and Wang proved (roughly) that if an operator T maps $L^p(w)$ to itself whenever w is in the Muckenhoupt A_p class, then T maps $L^{p(\cdot)}(w)$ to itself for all weights in $A_{p(\cdot)}$ (see [19]). In this paper, we explore a similar connection between $L^p(w)$ and $L^{p(\cdot)}(w)$ with weights in the classes A_p^ρ and $A_{p(\cdot)}^\rho$, respectively, which satisfy critical radius conditions and were originally introduced in [5] and [10].

The classical theory of extrapolation introduced by Rubio de Francia in [27] is a powerful tool in harmonic analysis, for a detailed treatment, see [17]. Extrapolation in the scale of the variable Lebesgue spaces was originally developed in [15] to prove unweighted inequalities and in [19] for the weighted case.

On the other hand, weighted inequalities in the scale of weighted variable Lebesgue spaces in the context of the Schrödinger operator using extrapolation techniques were also developed in [1] and [11]. In the first one, boundedness results are obtained for fractional operators associated with Schrödinger operator $\mathscr{L}$. These operators include fractional integrals and their respective commutators. Particularly, they obtain weighted inequalities of the type $L^{p(\cdot)}$–$L^{q(\cdot)}$ and estimates of the type $L^{p(\cdot)}$–Lipschitz variable integral spaces. While in the second, the authors study the boundedness on weighted variable Lebesgue spaces, $L^{p(\cdot)}(w)$, of operators that are singular integrals given by a kernel $K(x,y)$, which satisfies certain size and smoothness conditions with respect to the critical radius function ρ. These results can also be obtained from ours as we will see below (see Subsection 4.3).

The structure of this paper is as following. We start in Section 2 giving some definitions and notations related to variable Lebesgue spaces $L^{p(\cdot)}(w)$ and extrapolation results in a general framework of weights governed by a family of operators. The proofs are based on the techniques developed in [19]. In Section 3 we state and prove the auxiliary results which are important tools in order to prove the theorems stated in Section 2. Later, we deal with the proofs of the main results.

At the beginning of Section 4, we deal with the maximal operators and classes of weights appearing in the aforementioned papers (see [5] and [10]) and prove that these weights satisfy the hypotheses of the general theorems developed in Section 2. Finally, we show how to apply extrapolation to prove weighted norm inequalities for several different kinds of operators. Our examples, while not exhaustive, allow us to illustrate the applicability of extrapolation.

Throughout this paper, unless otherwise indicated, we will use C and c to denote constants, which are not necessarily the same at each occurrence. We will say that $A \lesssim B$ when there exists a constant $c > 0$ such that $A \leqslant cB$ and we will write $A \simeq B$ whenever $A \lesssim B$ and $B \lesssim A$.

2. A general setting of extrapolation

In this section we will state several general theorems on extrapolation in variable Lebesgue spaces with weights associated to a family of sublinear operators. We begin with some definitions and notations related to these spaces.

Let $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ be a measurable function. Given a measurable set $A \subset \mathbb{R}^n$ we define

$$p^-(A) := \operatorname*{ess\,inf}_{x\in A} p(x), \qquad p^+(A) := \operatorname*{ess\,sup}_{x\in A} p(x).$$

For simplicity we let p^- denote $p^-(\mathbb{R}^n)$ and p^+ denote $p^+(\mathbb{R}^n)$.

Given $p(\cdot)$, the conjugate exponent $p'(\cdot)$ is defined pointwise

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

where we let $p'(x) = \infty$ if $p(x) = 1$.

By $\mathscr{P}(\mathbb{R}^n)$ we will designate the collection of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ and by $\mathscr{P}^*(\mathbb{R}^n)$ the set of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that $p^+ < \infty$.

Given $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, we say that a measurable function f belongs to $L^{p(\cdot)}(\mathbb{R}^n)$ if for some $\lambda > 0$, the modular of f/λ associated with $p(\cdot)$, that is,

$$\rho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx,$$

is finite. A Luxemburg type norm can be defined in $L^{p(\cdot)}(\mathbb{R}^n)$ by taking

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leqslant 1\}.$$

These spaces are special cases of Musielak-Orlicz spaces (see [26]), and generalize the classical Lebesgue spaces. For more information see, for example [13, 21, 24].

In addition, we denote by $L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ the space of functions f such that $f\chi_B \in L^{p(\cdot)}(\mathbb{R}^n)$ for every ball $B \subset \mathbb{R}^n$.

In the classical $L^p(\mathbb{R}^n)$ spaces, $1 < p < \infty$, the norm can be characterized using the identity

$$\|f\|_p = \sup \int_{\mathbb{R}^n} f(x)g(x)\,dx,$$

where the supremum is considered over all functions g such that $g \in L^{p'}(\mathbb{R}^n)$ and $\|g\|_{p'} \leqslant 1$. Analogously, we have the following result for variable Lebesgue spaces.

LEMMA 1. ([13, Theorem 2.34]) *Let* $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, f *a measurable function and*

$$|||f|||_{p(\cdot)} = \sup \left\{ \int_{\mathbb{R}^n} f(x)g(x)\,dx : \|g\|_{p'(\cdot)} \leqslant 1 \right\}.$$

Then,

$$c|||f|||_{p(\cdot)} \leqslant \|f\|_{p(\cdot)} \leqslant C|||f|||_{p(\cdot)},$$

where the constants c *and* C *depend only on* $p(\cdot)$.

On the other hand, analogously to the previous case, Hölder's inequality is also valid for variable exponents but with a constant on the right-hand side of it.

LEMMA 2. ([21, Lema 3.2.20]) *Given* $s(\cdot), p(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, *be such that* $1/s(\cdot) = 1/p(\cdot) + 1/q(\cdot)$. *Then, for* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *and* $g \in L^{q(\cdot)}(\mathbb{R}^n)$

$$\|fg\|_{s(\cdot)} \leqslant 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

Moreover, if $s(\cdot) \equiv 1$, *the inequality above gives*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \leqslant 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

Another elementary but useful property of the classical Lebesgue norm is that it is homogeneous in the exponent, more precisely $\|f^s\|_p = \|f\|_{sp}^s$ for $1 < s < \infty$ and non-negative f. This property also extends to variable Lebesgue spaces as follows.

LEMMA 3. ([13, Proposition 2.18]) *Let* $p(\cdot) \in \mathscr{P}^*(\mathbb{R}^n)$, *so for all* s, $1/p^- \leqslant s < \infty$,

$$\||f|^s\|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s.$$

The following conditions on the exponent arise in connection with the boundedness of the Hardy–Littlewood maximal operator M in $L^{p(\cdot)}(\mathbb{R}^n)$ (see, for example, [20], [13] or [21]). We will say that $p(\cdot)$ is log–Hölder continuous, and we will write $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, if $p(\cdot) \in \mathscr{P}^*(\mathbb{R}^n)$ and if there are constants p_∞ and $C > 0$ such that

$$|p(x) - p(y)| \leqslant -\frac{C}{\log(|x-y|)}, \qquad x, y \in \mathbb{R}^n, \ |x-y| < 1/2, \tag{3}$$

and

$$|p(x) - p_\infty| \leqslant \frac{C}{\log(e+|x|)}, \qquad x \in \mathbb{R}^n. \tag{4}$$

By a weight we will mean a locally integrable function w defined on $\mathbb{R}^n$ such that $0 < w(x) < \infty$ almost everywhere. Given a weight w and $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, we define the weighted variable Lebesgue space $L^{p(\cdot)}(w)$ to be the set of all measurable functions f such that $fw \in L^{p(\cdot)}(\mathbb{R}^n)$, and we write

$$\|f\|_{L^{p(\cdot)}(w)} = \|f\|_{p(\cdot),w} = \|fw\|_{p(\cdot)}.$$

Thus, we say that an operator T is bounded on $L^{p(\cdot)}(w)$ if

$$\|Tfw\|_{p(\cdot)} \leqslant C\|fw\|_{p(\cdot)},$$

for all $f \in L^{p(\cdot)}(w)$.

Suppose now that we have a family of positive, sublinear operators $\{T_\theta\}_{\theta\in I}$, where I is a certain set of indexes. Associated to a fixed $\theta \in I$, we define the following families of weights,

- for $1 < p < \infty$, the U_p^θ family, as those weights w such that T_θ maps $L^p(w)$ onto itself, and we denote by $[w]_{p,\theta} = \|T_\theta\|_{L^p(w)}$ the usual operator norm;
- for $p = 1$, U_1^θ is the family of weights w such that for some constant $C > 0$, $T_\theta w \leqslant Cw$ a.e., and $[w]_{1,\theta}$ is defined as the infimum of those C satisfying the inequality;
- for $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, the $U_{p(\cdot)}^\theta$ family, as those weights w such that T_θ maps $L^{p(\cdot)}(w)$ onto itself.

We also call $U_p = \bigcup_{\theta\in I} U_p^\theta$, $U_\infty = \bigcup_{p\geqslant 1} U_p$ and $U_{p(\cdot)} = \bigcup_{\theta\in I} U_{p(\cdot)}^\theta$.

We will further assume that these families satisfy the following basic properties.

1. if $w_1 \in U_1^{\theta_1}$ and $w_2 \in U_1^{\theta_2}$ for some $\theta_2, \theta_2 \in I$ then for every $p \geqslant 1$ there exists $\theta = \theta(\theta_1, \theta_2, p)$ such that $w_1 w_2^{1-p} \in U_p^\theta$;
2. if $w \in U_{p(\cdot)}^\theta$, for some $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $\theta \in I$ then there exists $\theta' = \theta'(\theta, p(\cdot))$, such that $w^{-1} \in U_{p'(\cdot)}^{\theta'}$.

Although our goal is to use extrapolation to prove the boundedness of some specific operators, we will state our results in a more abstract way. Following the approach established in [16] (see also [17] and [15]) we will present our extrapolation theorems for pairs of measurable, non-negative functions (f,g) belonging to some family $\mathscr{F}$. Henceforth, if we write

$$\|f\|_X \leqslant C\|g\|_Y, \quad (f,g) \in \mathscr{F},$$

where X and Y are spaces of functions (i.e., weighted Lebesgue spaces, classical or variable), then we mean that this inequality is true for any pair $(f,g) \in \mathscr{F}$ such that the left–hand side of this inequality is finite.

We are now in a position to state our first extrapolation results. The first is a direct generalization of the classical Rubio de Francia extrapolation theorem to weighted variable Lebesgue spaces.

THEOREM 1. *Let $1 \leqslant p_0 < \infty$ and suppose that for all $w \in U_{p_0}$ it is verified that*

$$\int_{\mathbb{R}^n} f^{p_0} w\,dx \leqslant C \int_{\mathbb{R}^n} g^{p_0} w\,dx, \qquad (f,g) \in \mathscr{F}. \tag{5}$$

Then, if $w \in U_{p(\cdot)}$ *and* $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ *it follows that*

$$\|fw\|_{p(\cdot)} \leqslant C\|gw\|_{p(\cdot)}, \qquad (f,g) \in \mathscr{F}.$$

To state our next results we introduce a more general class of exponent functions. We say $p(\cdot) \in \mathscr{P}_0^*(\mathbb{R}^n)$ if $p(\cdot) : \mathbb{R}^n \longrightarrow (0,\infty)$ be a measurable function and $p^+ < \infty$. For such $p(\cdot)$ we define the norm $\|\cdot\|_{p(\cdot)}$ (actually a quasi-norm, see [18]) exactly as we do for $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$.

THEOREM 2. *Let* $0 < s < q_0 < \infty$ *and suppose that inequality*

$$\int_{\mathbb{R}^n} f^{q_0} w\,dx \leqslant C \int_{\mathbb{R}^n} g^{q_0} w\,dx, \qquad (f,g) \in \mathscr{F},$$

holds for every $w \in U_{q_0/s}$.

Then given $p(\cdot) \in \mathscr{P}_0^*(\mathbb{R}^n)$, *such that* $p^- \geqslant s$ *and* w *such that* $w^s \in U_{p(\cdot)/s}$, *it follows that*

$$\|fw\|_{p(\cdot)} \leqslant C\|gw\|_{p(\cdot)}, \qquad (f,g) \in \mathscr{F}.$$

We can also consider extrapolation theorems for weights in U_1 and U_∞ as follows.

THEOREM 3. *Suppose that for some* $0 < p_0 < \infty$ *and every* $w \in U_1$,

$$\int_{\mathbb{R}^n} f^{p_0} w\,dx \leqslant C \int_{\mathbb{R}^n} g^{p_0} w\,dx, \qquad (f,g) \in \mathscr{F},$$

Then, given $p(\cdot) \in \mathscr{P}_0^*(\mathbb{R}^n)$ *such that* $p^- > p_0$ *and* $w^{p_0} \in U_{p(\cdot)/p_0}$, *it follows that*

$$\|fw\|_{p(\cdot)} \leqslant C\|gw\|_{p(\cdot)}, \qquad (f,g) \in \mathscr{F}.$$

THEOREM 4. *Let* $0 < q_0 < \infty$ *and suppose that inequality*

$$\int_{\mathbb{R}^n} f^{q_0} w\,dx \leqslant C \int_{\mathbb{R}^n} g^{q_0} w\,dx, \qquad (f,g) \in \mathscr{F},$$

holds for every $w \in U_\infty$.

Then, given $p(\cdot) \in \mathscr{P}_0^*(\mathbb{R}^n)$, $0 < s \leqslant \min\{q_0, p^-\}$ *and* w *such that* $w^s \in U_{p(\cdot)/s}$, *it follows that*

$$\|fw\|_{p(\cdot)} \leqslant C\|gw\|_{p(\cdot)}, \qquad (f,g) \in \mathscr{F}.$$

3. Proof of main results

In this section we give the proofs of our main theorems. We will first consider the following variable version of the Rubio de Francia's extrapolation algorithm.

LEMMA 4. *Let* $r(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ *and suppose that* w *is a weight in* $U^{\theta}_{r(\cdot)}$ *for some* $\theta \in I$. *For a non-negative function* $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *such that* $T_\theta h(x) < \infty$ *a.e. we define*

$$\mathscr{R}h(x) = \sum_{k=0}^{\infty} \frac{(T_\theta)^k h(x)}{2^k \|T_\theta\|^k_{L^{r(\cdot)}(w)}}.$$

Then,

1. $h(x) \leqslant \mathscr{R}h(x), \quad a.e.\ x \in \mathbb{R}^n$.

2. $\|\mathscr{R}h\|_{L^{r(\cdot)}(w)} \leqslant 2\|h\|_{L^{r(\cdot)}(w)}$.

3. $\mathscr{R}h \in U_1$ *with* $[\mathscr{R}h]_{1,\theta} \leqslant 2\|T_\theta\|_{L^{r(\cdot)}(w)}$.

Proof. The proof is essentially the same as in the constant exponent case. Property 1. for $\mathscr{R}h$ is immediate, property 2. is deduced from the assumption that T_θ is bounded in $L^{r(\cdot)}(w)$, and finally, the property 3. follows from the fact that T_θ is sublinear and h is non-negative. □

As a consequence of the previous result we will prove the following corollary which will be used in the proof of several of our theorems.

COROLLARY 1. *Let $r(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and suppose that w is a weight in $U^\theta_{r(\cdot)}$ for some $\theta \in I$. Then, for each $\sigma \geqslant 0$ and every weights v, the operator $Hh = \mathscr{R}(hv^\sigma)v^{-\sigma}$ verifies that*

1. $h(x) \leqslant Hh(x), \quad a.e.\ x \in \mathbb{R}^n$;

2. *if* $\mu = v^\sigma w$, H *is bounded on* $L^{r(\cdot)}(\mu)$, *with* $\|H\|_{L^{r(\cdot)}(\mu)} \leqslant 2\|h\|_{L^{r(\cdot)}(\mu)}$;

3. $Hhv^\sigma \in U_1$.

Proof. Clearly, 1. and 3. are a consequence of 1. and 3. from the previous lemma. Finally,

$$\|Hh\|_{L^{r(\cdot)}(\mu)} = \|Hhv^\sigma w\|_{r(\cdot)} = \|\mathscr{R}(hv^\sigma)w\|_{r(\cdot)} \leqslant 2\|hv^\sigma w\|_{r(\cdot)} = 2\|h\|_{L^{r(\cdot)}(\mu)}. \quad \square$$

We begin with the proof of Theorem 1. The following argument extends the proof of Theorem 3.28 in [17] to the variable exponent setting.

Proof of Theorem 1. Let $(f,g) \in \mathscr{F}$ and $w \in U_{p(\cdot)}$ such that $\|fw\|_{p(\cdot)} < \infty$. Without loss of generality, we can also assume that $\|fw\|_{p(\cdot)} > 0$ and $\|gw\|_{p(\cdot)} < \infty$, since otherwise there is nothing to prove. We can also assume that $\|g\|_{L^{p(\cdot)}(w)} > 0$, since otherwise $g(x) = 0$ at almost every point and then, from the hypothesis we would also have $f(x) = 0$ at almost every point.

Let us consider

$$h_1 = \frac{f}{\|fw\|_{p(\cdot)}} + \frac{g}{\|gw\|_{p(\cdot)}}.$$

Clearly $h_1 \in L^{p(\cdot)}(w)$ and $\|h_1 w\|_{p(\cdot)} \leqslant 2$.

Since $f \in L^{p(\cdot)}(w)$, by duality, there exists $h_2 \in L^{p'(\cdot)}(\mathbb{R}^n)$, $\|h_2\|_{p'(\cdot)} \leqslant 1$, such that

$$\|fw\|_{p(\cdot)} \lesssim \int_{\mathbb{R}^n} f w h_2 dx.$$

Let us consider the functions $H_1 = H(h_1) = \mathscr{R}(h_1)$ and $H_2 = H(h_2) = \mathscr{R}(h_2 w)w^{-1}$. Thus, by Corollary 1, being that $w \in U_{p(\cdot)}$ and $w^{-1} \in U_{p'(\cdot)}$ it follows that

1. $h_1(x) \leqslant H_1(x), \quad a.e.\ x \in \mathbb{R}^n$.

2. $\|H_1\|_{L^{p(\cdot)}(w)} \leqslant 2\|h_1\|_{L^{p(\cdot)}(w)}$.

3. $H_1 \in U_1$.

4. $h_2(x) \leqslant H_2(x), \quad a.e.\ x \in \mathbb{R}^n$.

5. $\|H_2\|_{L^{p'(\cdot)}} \leqslant 2\|h_2\|_{L^{p'(\cdot)}}$.

6. $H_2 w \in U_1$.

Therefore, accounting for 4. and Hölder's inequality for $p_0 > 1$ with respect to the measure $H_2 w\,dx$ we have

$$\begin{aligned}\|fw\|_{p(\cdot)} &\leqslant \int_{\mathbb{R}^n} f H_1^{-1/p_0'} H_1^{1/p_0'} H_2 w\,dx \\ &\leqslant \left(\int_{\mathbb{R}^n} f^{p_0} H_1^{-p_0/p_0'} H_2 w\,dx\right)^{1/p_0} \left(\int_{\mathbb{R}^n} H_1 H_2 w\,dx\right)^{1/p_0'} \\ &= I_1^{1/p_0} I_2^{1/p_0'}.\end{aligned}$$

Let us now verify that the factor I_2 is uniformly bounded. On the one hand, considering the variable Hölder's inequality, 2. and 5. we have that

$$I_2 \leqslant 2\|H_1 w\|_{p(\cdot)}\|H_2\|_{p'(\cdot)} \leqslant 8\|h_1 w\|_{p(\cdot)}\|h_2\|_{p'(\cdot)} \leqslant 16.$$

In order to use the hypothesis, we must jointly prove that I_1 is finite. Considering the definition of h_1, the properties of H_1 and H_2 and the variable Hölder's inequality it follows that

$$\begin{aligned}I_1 &\leqslant \int_{\mathbb{R}^n} f^{p_0} h_1^{-p_0/p_0'} H_2 w\,dx \\ &\leqslant \int_{\mathbb{R}^n} f^{p_0} \left(\frac{f}{\|fw\|_{p(\cdot)}}\right)^{1-p_0} H_2 w\,dx \\ &= \|fw\|_{p(\cdot)}^{p_0-1} \int_{\mathbb{R}^n} f H_2 w\,dx \\ &\leqslant 2\|fw\|_{p(\cdot)}^{p_0-1} \|fw\|_{p(\cdot)} \|H_2\|_{p'(\cdot)} \\ &\leqslant 4\|fw\|_{p(\cdot)}^{p_0} < \infty.\end{aligned}$$

To estimate I_1 we will apply the hypothesis with the weight $w_0 = H_1^{1-p_0} H_2 w$ the one that belongs to U_{p_0} because $H_1, H_2 w \in U_1$. With the same considerations as in the previous case, it follows that,

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} f^{p_0} H_1^{1-p_0} H_2\, w dx \\
&\leqslant C \int_{\mathbb{R}^n} g^{p_0} H_1^{1-p_0} H_2\, w dx \\
&\leqslant C \int_{\mathbb{R}^n} g^{p_0} \left(\frac{g}{\|gw\|_{p(\cdot)}} \right)^{1-p_0} H_2 w\, dx \\
&= C \|gw\|_{p(\cdot)}^{p_0-1} \int_{\mathbb{R}^n} g\, H_2 w\, dx \\
&\leqslant C \|gw\|_{p(\cdot)}^{p_0-1} \|gw\|_{p(\cdot)} \|H_2\|_{p'(\cdot)} \\
&\leqslant C \|gw\|_{p(\cdot)}^{p_0},
\end{aligned}
$$

and the proof is finished for the case $p_0 > 1$.

The $p_0 = 1$ case follows more directly. In fact, as $H_2 w \in U_1$, we get

$$
\begin{aligned}
\|fw\|_{p(\cdot)} &\leqslant \int_{\mathbb{R}^n} f\, H_2 w\, dx \\
&\leqslant C \int_{\mathbb{R}^n} g\, H_2 w\, dx \\
&\leqslant C \|gw\|_{p(\cdot)} \|H_2\|_{p'(\cdot)} \\
&\leqslant C \|gw\|_{p(\cdot)}. \quad \square
\end{aligned}
$$

Proof of Theorem 2. We consider the family $\mathscr{F}_0$, of those pairs (f^s, g^s) such that $(f,g) \in \mathscr{F}$. By the hypothesis, if $w_0 \in U_{q_0/s}$ and $(f,g) \in \mathscr{F}$, we have

$$
\int_{\mathbb{R}^n} (f^s)^{q_0/s} w_0\, dx = \int_{\mathbb{R}^n} f^{q_0} w_0\, dx \leqslant C \int_{\mathbb{R}^n} g^{q_0} w_0\, dx = C \int_{\mathbb{R}^n} (g^s)^{q_0/s} w_0\, dx.
$$

Therefore, we have proved inequality (5) with $p_0 = q_0/s$ for the family $\mathscr{F}_0$ and weights in $U_{q_0/s}$. In this way, applying Theorem 1, it follows

$$
\|f^s w\|_{q(\cdot)} \leqslant C \|g^s w\|_{q(\cdot)},
$$

for every $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $w \in U_{q(\cdot)}$.

Let now $p(\cdot) \in \mathscr{P}_0^*(\mathbb{R}^n)$ such that $p^- \geqslant s$ and w such that $w^s \in U_{p(\cdot)/s}$, it follows then, from Lemma 3,

$$
\|fw\|_{p(\cdot)} = \|f^s w^s\|_{\frac{p(\cdot)}{s}}^{1/s} \leqslant C \|g^s w^s\|_{\frac{p(\cdot)}{s}}^{1/s} = C \|gw\|_{p(\cdot)}. \quad \square
$$

Proof of Theorem 3. Fix $p(\cdot) \in \mathscr{P}_0^*(\mathbb{R}^n)$, with $p^- > p_0$ and $w^{p_0} \in U_{p(\cdot)/p_0}$. As before, we may assume without loss of generality that $\|fw\|_{p(\cdot)} > 0$ and $\|gw\|_{p(\cdot)} < \infty$.

Thus, by dilation and duality, there exists $h_2 \in L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$, $\|h_2\|_{(p(\cdot)/p_0)'} \leqslant 1$, such that

$$\|fw\|_{p(\cdot)}^{p_0} = \|f^{p_0}w^{p_0}\|_{p(\cdot)/p_0} \leqslant C\int_{\mathbb{R}^n} f^{p_0}w^{p_0}h_2\,dx.$$

We will consider the Corollary 1 to define an operator $H_2 = H(h_2) = \mathscr{R}(h_2w^{p_0})w^{-p_0}$. Thus, by Corollary 1, being that $w^{-p_0} \in U_{(p(\cdot)/p_0)'}$ it follows that

1. $h_2(x) \leqslant H_2(x)$, $\quad a.e.\ x \in \mathbb{R}^n$.
2. $\|H_2\|_{(p(\cdot)/p_0)'} \leqslant 2\|h_2\|_{(p(\cdot)/p_0)'}$.
3. $H_2 w^{p_0} \in U_1$.

From the above inequality and the item 1. it then follows that

$$\|fw\|_{p(\cdot)}^{p_0} \leqslant C\int_{\mathbb{R}^n} f^{p_0}w^{p_0}h_2\,dx \leqslant C\int_{\mathbb{R}^n} f^{p_0}w^{p_0}H_2\,dx.$$

To apply the hypothesis let us verify that the term on the right is finite. Given that $h_2 \in L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$, by Hölder inequality, dilation and item 2. we have that

$$\int_{\mathbb{R}^n} f^{p_0}w^{p_0}H_2\,dx \leqslant \|f^{p_0}w^{p_0}\|_{p(\cdot)/p_0}\|H_2\|_{(p(\cdot)/p_0)'} \leqslant 2\|fw\|_{p(\cdot)}^{p_0}\|h_2\|_{(p(\cdot)/p_0)'} < \infty.$$

Finally, using $w^{p_0}H_2 \in U_1$, it follows from the hypothesis that

$$\begin{aligned}\int_{\mathbb{R}^n} f^{p_0}w^{p_0}H_2\,dx &\leqslant C\int_{\mathbb{R}^n} g^{p_0}w^{p_0}H_2\,dx\\ &\leqslant C\|g^{p_0}w^{p_0}\|_{p(\cdot)/p_0}\|H_2\|_{(p(\cdot)/p_0)'}\\ &\leqslant C\|gw\|_{p(\cdot)}^{p_0}\|h_2\|_{(p(\cdot)/p_0)'}\\ &\leqslant C\|gw\|_{p(\cdot)}^{p_0}. \quad \square\end{aligned}$$

Proof of Theorem 4. Let $r \geqslant 1$ and consider the family $\mathscr{F}_0$, of those pairs $(f^{q_0/r}, g^{q_0/r})$ such that $(f,g) \in \mathscr{F}$. By the hypothesis, if $w_0 \in U_r$ and $(f,g) \in \mathscr{F}$, we have

$$\int_{\mathbb{R}^n}(f^{q_0/r})^r w_0\,dx = \int_{\mathbb{R}^n} f^{q_0}w_0\,dx \leqslant C\int_{\mathbb{R}^n} g^{q_0}w_0\,dx = C\int_{\mathbb{R}^n}(g^{q_0/r})^r w_0\,dx.$$

Therefore, we have proved inequality (5) with $p_0 = r$ for the family $\mathscr{F}_0$ and weights in U_r. In this way, applying Theorem 1, it follows

$$\|f^{q_0/r}w\|_{q(\cdot)} \leqslant C\|g^{q_0/r}w\|_{q(\cdot)},$$

for every $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $w \in U_{q(\cdot)}$.

Let now $p(\cdot) \in \mathscr{P}_0^*(\mathbb{R}^n)$, $0 < s \leqslant \min\{q_0, p^-\}$ and w such that $w^s \in U_{p(\cdot)/s}$. Taking $r = q_0/s$, it follows from Lemma 3

$$\|fw\|_{p(\cdot)} = \|f^{q_0/r}w^{q_0/r}\|_{\frac{r}{q_0}p(\cdot)}^{r/q_0} = \|f^{q_0/r}w^s\|_{\frac{p(\cdot)}{s}}^{1/s} \leqslant C\|g^{q_0/r}w^s\|_{\frac{p(\cdot)}{s}}^{1/s} = C\|gw\|_{p(\cdot)}. \quad \square$$

4. Applications: norm inequalities for operators

In this section we use extrapolation to prove norm inequalities for a great variety of operators on the weighted variable Lebesgue spaces in the Schrödinger context.

First, we will deal with maximal operators and classes of weights that have recently arised in conection to the Schrödinger operators (see [2] and [10]). These classes fit the general context above and allow us to obtain the applications we seek.

We call a *critical radius function* to any positive function ρ with the property that there exist constants c_ρ, $N_\rho \geqslant 1$ such that

$$c_\rho^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_\rho} \leqslant \rho(y) \leqslant c_\rho\,\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{N_\rho}{N_\rho+1}}, \tag{6}$$

for every $x,y \in \mathbb{R}^n$. In particular, according to [28, Lemma 1.4], if $V \in RH_q$, with $q > n/2$, the associated function ρ_V defined by

$$\rho_V(x) = \sup\left\{r>0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V \leqslant 1\right\}, \tag{7}$$

verifies (6).

Given a critical radius function ρ, for each $\theta \geqslant 0$ we define the maximal operator M_ρ^θ, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, as

$$M_\rho^\theta f(x) = \sup_{r>0}\left(1+\frac{r}{\rho(x)}\right)^{-\theta}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy. \tag{8}$$

We say that the weight w belongs to the $A_p^{\rho,\theta}$ class , for $1<p<\infty$ and $\theta \geqslant 0$, if there exists a constant $C>0$ such that the inequality

$$\left(\frac{1}{|B|}\int_B w\,dy\right)^{1/p}\left(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}}\,dy\right)^{1/p'} \leqslant C\left(1+\frac{r}{\rho(x)}\right)^{\theta}, \tag{9}$$

holds for every ball $B=B(x,r)\subset\mathbb{R}^n$. For the case $p=1$, we will say that w belongs to the class $A_1^{\rho,\theta}$ if there exists a constant $C>0$ such that the inequality

$$\frac{1}{|B|}\int_B w\,dy \leqslant C\Big(\inf_{x\in B} w(x)\Big)\left(1+\frac{r}{\rho(x)}\right)^{\theta},$$

holds for every ball $B=B(x,r)\subset\mathbb{R}^n$. We denote $A_p^\rho = \cup_{\theta\geqslant 0}A_p^{\rho,\theta}$ and $A_\infty^\rho = \cup_{p\geqslant 1}A_p^\rho$.

Also, given $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ and $\theta\geqslant 0$, we will say that $A_{p(\cdot)}^{\rho,\theta}$ if the inequality

$$\|w\chi_B\|_{p(\cdot)}\|w^{-1}\chi_B\|_{p'(\cdot)} \leqslant C|B|\left(1+\frac{r}{\rho(x)}\right)^{\theta}, \tag{10}$$

holds for all balls $B=B(x,r)\subset\mathbb{R}^n$. We denote $A_{p(\cdot)}^\rho = \cup_{\theta\geqslant 0}A_{p(\cdot)}^{\rho,\theta}$.

REMARK 1. It follows from the above definition that if $w \in A^{\rho}_{p(\cdot)}$, then $w^{-1} \in A^{\rho}_{p'(\cdot)}$.

The following results shows the connection between the weights $A^{\rho,\theta}_{p}$ and $A^{\rho,\theta}_{p(\cdot)}$ and the boundedness of the operators M^{θ}_{ρ} in $L^{p}(w)$ and $L^{p(\cdot)}(w)$ respectively.

THEOREM 5. ([2, Proposition 3]) *Let $1 < p < \infty$. Then, a weight w belongs to A^{ρ}_{p} if and only if there exists $\theta \geqslant 0$ such that M^{θ}_{ρ} is bounded on $L^{p}(w)$.*

THEOREM 6. ([10, Theorem 5]) *Let $p \in \mathscr{P}^{\log}(\mathbb{R}^{n})$ with $p^{-} > 1$. Then, a weight w belongs to $A^{\rho}_{p(\cdot)}$ if and only if there exists $\theta > 0$ such that M^{θ}_{ρ} is bounded on $L^{p(\cdot)}(w)$.*

If for each $\theta \geqslant 0$ we denote $M^{\theta}_{\rho} = T_{\theta}$, we see from the above results, that A^{ρ}_{p} coincides with the U_{p} class of Section 2, for every $1 < p < \infty$, as well as the coincidence between classes $A^{\rho}_{p(\cdot)}$ and $U_{p(\cdot)}$ whenever $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^{n})$ with $p^{-} > 1$. However, it may not be true that for a fixed θ the U^{θ}_{p} class associated with M^{θ}_{ρ} coincides with $A^{\rho,\theta}_{p}$. The same is true for the classes $U^{\theta}_{p(\cdot)}$. However, it is straightforward to verify $A^{\rho,\theta}_{1} = U^{\theta}_{1}$ (see [2]).

On the other hand, it is straightforward to check, from their definition, that classes A^{ρ}_{p} and $A^{\rho}_{p(\cdot)}$ satisfy properties 1 and 2 requested to the classes U_{p} and $U_{p(\cdot)}$ (see for example [2] and [10]).

From the above, we are able to apply the theorems stated in Section 2 for classes A^{ρ}_{p} and $A^{\rho}_{p(\cdot)}$. First we will see how to prove that an operator T is bounded in $L^{p(\cdot)}(w)$ using Theorem 1. These same ideas can be used to apply the other theorems.

The key point in applying Theorem 1 is to consider an appropriate $\mathscr{F}$ family. This usually requires a density argument, since we need pairs of functions (f,g) such that f lies both the appropriate weighted space to apply the hypothesis and in the weighted variable Lebesgue space in which we want to obtain the thesis.

The dense subsets of $L^{p}(w)$ are well known, for example, smooth functions and bounded functions of compact support. These sets are also dense in $L^{p(\cdot)}(\mathbb{R}^{n})$ and in $L^{p(\cdot)}(w)$ (see, for example, [13] and [19]). More specifically, in [19] it is proved that if $p(\cdot) \in \mathscr{P}(\mathbb{R}^{n})$ with $p^{+} < \infty$ and $w \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^{n})$, then L^{∞}_{c}, the set of bounded functions of compact support, and C^{∞}_{c}, the smooth functions of compact support, are dense in $L^{p(\cdot)}(w)$.

Suppose now that for all $w_0 \in A^{\rho}_{p_0}$ it is verified that

$$\|Tf w_0\|_{p_0} \leqslant C\|f w_0\|_{p_0}. \tag{11}$$

We want to show that given a $w \in A^{\rho}_{p(\cdot)}$, T is bounded on $L^{p(\cdot)}(w)$. Since $w \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^{n})$, by a standard density argument (see [13], Theorem 5.39) it is sufficient to show that

$$\|Tf w\|_{p(\cdot)} \leqslant C\|f w\|_{p(\cdot)},$$

for all $f \in L_c^\infty$. Although intuitively, it can be thought to define $\mathscr{F}$ as

$$\mathscr{F} = \{(|Tf|,|f|) : f \in L_c^\infty\},$$

it is not known a priori that Tf is in $L^{p(\cdot)}(w)$. To overcome this, we again proceed by approximation and define $(Tf)_n = \min\{|Tf|,n\}\chi_{B(0,n)}$. Given that $w \in L_{\mathrm{loc}}^{p(\cdot)}$, it follows that $(Tf)_n \in L^{p(\cdot)}(w)$. On the other hand, it is clear that (11) is verified with $|Tf|$ replaced by $(Tf)_n$. Therefore, if we define

$$\mathscr{F} = \{((Tf)_n,|f|) : f \in L_c^\infty, n \geqslant 1\},$$

we can apply Theorem 1 and Fatou's Lemma in this context (see [13], Theorem 2.61) and conclude that for any $f \in L_c^\infty$

$$\|Tfw\|_{p(\cdot)} \leqslant \liminf_{n\to\infty} \|(Tf)_n w\|_{p(\cdot)} \leqslant C\|fw\|_{p(\cdot)}.$$

For some of the applications we will consider the space BMO_ρ^θ, defined in [6], as the set of locally integrable functions b such that for $\theta > 0$,

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_{B(x,r)}|\,dy \leqslant C\left(1+\frac{r}{\rho(x)}\right)^\theta, \tag{12}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. The infimum of the constant in (12) gives a norm for $b \in BMO_\rho^\theta$, denoted by $\|b\|_{BMO_\rho^\theta}$. We write $BMO_\rho^\infty = \bigcup_{\theta>0} BMO_\rho^\theta$.

From the above definition (12), it is clear that $BMO \subset BMO_\rho^\theta \subset BMO_\rho^\sigma$ for $0 < \theta \leqslant \sigma$, and hence $BMO \subset BMO_\rho^\infty$. Moreover, it is in general a larger class. For instance, when ρ is constant the functions $b_j(x) = |x_j|$, $1 \leqslant j \leqslant n$, belong to BMO_ρ^∞ but not to BMO. Also, when $\rho(x) \simeq \frac{1}{1+|x|}$, we may take $b(x) = |x|^2$.

4.1. Maximal operator of the diffusion semi-group

The maximal operator of the diffusion semi-group is defined by

$$T^*f(x) = \sup_{t>0} |e^{-t\mathscr{L}} f(x)| = \sup_{t>0}\left|\int_{\mathbb{R}^n} k_t(x,y) f(y)\,dy\right|,$$

and its commutator

$$T_b^*f(x) = \sup_{t>0}\left|\int_{\mathbb{R}^n} k_t(x,y)(b(x)-b(y)) f(y)\,dy\right|,$$

where k_t is the kernel of the operator $e^{-t\mathscr{L}}$, $t > 0$.

By combining Theorem 2 in [5], Theorem 1.2 in [32] and Theorem 1 together, we obtain the following result.

THEOREM 7. *Let $b \in BMO_\rho^\infty$ and $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$. Then, for every $w \in A_{p(\cdot)}^\rho$ there exists $C > 0$ such that*

$$\|T^* f w\|_{p(\cdot)} \leqslant C \|f w\|_{p(\cdot)},$$

and

$$\|T_b^* f w\|_{p(\cdot)} \leqslant C \|b\|_{BMO_\rho^\infty} \|f w\|_{p(\cdot)}.$$

4.2. The Littlewood–Paley function and the area function

We first introduce some notations. For $(x,t) \in \mathbb{R}^n \times (0,\infty)$, let

$$Q_t f(x) = t^2 \left(\frac{d}{ds} e^{-s\mathscr{L}} \Big|_{s=t^2} f \right)(x).$$

In [23] the authors introduce a Littlewood–Paley function associated to $\mathscr{L}$ which can be written as

$$g_Q(f)(x) = \left(\int_0^\infty |Q_t f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

On the other hand, in [32] they introduce the area function S_Q related to Schrödinger operators as

$$S_Q f(x) = \left(\int_0^\infty \int_{|x-y|<t} |Q_t f(y)|^2 \, dy \frac{dt}{t^{n+1}} \right)^{1/2}.$$

For these operators, we have the following result, taking into account Theorem 5 in [5], Theorem 1.1 in [32] and Theorem 1.

THEOREM 8. *Let $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$. Then, if $w \in A_{p(\cdot)}^\rho$ there exists $C > 0$ such that*

$$\|g_Q f w\|_{p(\cdot)} \leqslant C \|f w\|_{p(\cdot)},$$

and

$$\|S_Q f w\|_{p(\cdot)} \leqslant C \|f w\|_{p(\cdot)}.$$

The commutator of g_Q and S_Q with $b \in BMO_\rho^\infty$ is defined by

$$g_{Q,b} f(x) = \left(\int_0^\infty |Q_t((b(x) - b(\cdot))f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

and

$$S_{Q,b} f(x) = \left(\int_0^\infty \int_{|x-y|<t} |Q_t((b(x) - b(\cdot))f)(y)|^2 \, dy \frac{dt}{t^{n+1}} \right)^{1/2}.$$

The following theorem is a consequence of Theorem 1.1 in [31] and Theorem 1.2 in [32].

THEOREM 9. *Let $b \in BMO_\rho^\infty$ and $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$. Then, if $w \in A^\rho_{p(\cdot)}$ there exists $C > 0$ such that*

$$\|g_{Q,b} f w\|_{p(\cdot)} \leqslant C\|b\|_{BMO_\rho^\infty}\|f w\|_{p(\cdot)},$$

and

$$\|S_{Q,b} f w\|_{p(\cdot)} \leqslant C\|b\|_{BMO_\rho^\infty}\|f w\|_{p(\cdot)}.$$

4.3. Schrödinger type singular integrals

In our next application, instead of considering a specific operator, we will contemplate families of operators and then take into account particular cases.

In [2] a class of operators resembling those of Calderón–Zygmund theory, but adapted to the Schrödinger context, is introduced (see also [3] and [4]). This type of operators were also considered in [7] and [25], where conditions are given to obtain their boundedness in regularity spaces in the context of Schrödinger with and without weights, respectively.

In [11] it was shown that the so-called Schrödinger–Calderón–Zygmund operators are bounded on $L^{p(\cdot)}(w)$ using also extrapolation techniques but different from those studied here. These results can also be obtained from ours as we will see below.

We will consider two different types of operator families that will allow us to categorize several types of operators that appear in the Schrödinger context according to the regularity of the potential V.

We shall call Schrödinger–Calderón–Zygmund operator of type (∞, δ) for $0 < \delta \leqslant 1$ to an operator T such that

1. T is bounded in L^p for some $1 < p < \infty$.

2. T has an associated kernel $K : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, in the sense that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \quad f \in L_c^\infty \text{ and a.e. } x \notin \operatorname{supp} f.$$

   Further, for each $N > 0$ there exists a constant $C_N > 0$ such that

$$|K(x,y)| \leqslant C_N \frac{1}{|x-y|^d}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}, \tag{13}$$

   for any $x \neq y$, and there exists $C > 0$ such that

$$|K(x,y) - K(x_0,y)| \leqslant C\frac{|x-x_0|^\delta}{|x-y|^{d+\delta}}, \tag{14}$$

   for every $x, y \in \mathbb{R}^n$, whenever $|x - x_0| < \frac{|x-y|}{2}$.

On the other hand, we will say that a linear operator T is a Schrödinger–Calderón–Zygmund operator of type (s, δ), for $1 < s < \infty$ and $0 < \delta \leqslant 1$, if

1. T is bounded from $L^{s'}$ into $L^{s',\infty}$.

2. T has an associated kernel $K:\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}$, in the sense that

$$Tf(x)=\int_{\mathbb{R}^d}K(x,y)f(y)\,dy,\quad f\in L^\infty_c \text{ and a.e. } x\notin \operatorname{supp} f.$$

Further, for each $N>0$ there exists a constant $C_N>0$ such that

$$\left(\frac{1}{R^n}\int_{R<|x_0-y|<2R}|K(x,y)|^s dy\right)^{1/s}\leqslant C_N R^{-n}\left(1+\frac{R}{\rho(x)}\right)^{-N}, \tag{15}$$

for any $|x-x_0|<R/2$, and there exists $C>0$ such that

$$\left(\frac{1}{R^n}\int_{R<|x_0-y|<2R}|K(x,y)-K(x_0,y)|^s dy\right)^{1/s}\leqslant CR^{-n}\left(\frac{r}{R}\right)^{\delta}, \tag{16}$$

for every $|x-x_0|<r<\rho(x_0)$ and $r<R/2$.

REMARK 2. We also get that if T is a Schrödinger–Calderón–Zygmund operator of type (∞,δ), then T is a Schrödinger–Calderón–Zygmund operator of type (s,δ), for any $1<s<\infty$.

From the results of [2] and [4] we have the following theorem concerning strong weighted inequalities for Schrödinger–Calderón–Zygmund operator of type (s,δ) for $1<s\leqslant\infty$.

THEOREM 10. ([11], Theorem 20) *Let $0<\delta\leqslant 1$, $1<s\leqslant\infty$ and T be an Schrödinger–Calderón–Zygmund operator of type (s,δ). Then T is bounded on $L^p(w)$ for every $s'<p<\infty$ and any $w\in A^{\rho}_{p/s'}$.*

From the previous theorem and Theorems 1 and 2, the following fundamental results follows.

THEOREM 11. *Let $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^->1$. If T is a Schrödinger–Calderón–Zygmund operator of type (∞,δ), then*

$$\|Tfw\|_{p(\cdot)}\leqslant C\|fw\|_{p(\cdot)},$$

holds for every weight $w\in A^{\rho}_{p(\cdot)}$. Moreover, its adjoint operator T^* is also bounded on $L^{p(\cdot)}(w)$ for every $w\in A^{\rho}_{p(\cdot)}$.

Proof. The first statement follows directly from Theorems 1 and 10.

Regarding the second statement, by Lemma 1 and Lemma 2 we obtain that

$$\begin{aligned}
\|T^*fw\|_{p(\cdot)} &\lesssim \sup_{\|g\|_{p'(\cdot)}\leqslant 1} \left|\int_{\mathbb{R}^d} T^*f(x)\,g(x)w(x)\,dx\right| \\
&= \sup_{\|g\|_{p'(\cdot)}\leqslant 1} \left|\int_{\mathbb{R}^d} f(x)\,T(gw)(x)\,dx\right| \\
&\lesssim \sup_{\|g\|_{p'(\cdot)}\leqslant 1} \|fw\|_{p(\cdot)}\,\|T(gw)\,w^{-1}\|_{p'(\cdot)} \\
&\leqslant \|fw\|_{p(\cdot)} \sup_{\|g\|_{p'(\cdot)}\leqslant 1} \|T(gw)\,w^{-1}\|_{p'(\cdot)}.
\end{aligned}$$

Since that for $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$, it is $p^+<\infty$, it follows that $(p')^-=(p^+)'>1$. Also, condition $w\in A^{\rho}_{p(\cdot)}$ implies that $w^{-1}\in A^{\rho}_{p'(\cdot)}$. According to what has already been proven in the first part, it follows that

$$\begin{aligned}
\|T^*fw\|_{p(\cdot)} &\lesssim \|fw\|_{p(\cdot)} \sup_{\|g\|_{p'(\cdot)}\leqslant 1} \|T(gw)\,w^{-1}\|_{p'(\cdot)} \\
&\lesssim \|fw\|_{p(\cdot)} \sup_{\|g\|_{p'(\cdot)}\leqslant 1} \|g\|_{p'(\cdot)} \\
&\leqslant \|fw\|_{p(\cdot)}. \quad \square
\end{aligned}$$

Let us now see the corresponding result for Schrödinger–Calderón–Zygmund operator of type (s,δ).

THEOREM 12. *Let $1<s<\infty$ and $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^->s'$. If T is a Schrödinger–Calderón–Zygmund operator of type (s,δ), then*

$$\|Tfw\|_{p(\cdot)}\leqslant C\|fw\|_{p(\cdot)},$$

holds for every weight w such that $w^{s'}\in A^{\rho}_{p(\cdot)/s'}$. Moreover, its adjoint operator T^* is bounded on $L^{p(\cdot)}(w)$ for every $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $1<p^+<s$ and any w such that $w^{-s'}\in A^{\rho}_{p'(\cdot)/s'}$.

The proof of this theorem follows the same steps as in the previous case and will be omitted.

Hereafter, for shortness we will use the notation SCZ to mean a Schrödinger–Calderón–Zygmund operator when ρ is the critical radius function derived from the potential V.

4.3.1. Schrödinger–Riesz transforms

In this subsection we consider the singular integral operators known as the first and second order Riesz–Schrödinger transforms, given by $R_1=\nabla\mathscr{L}^{-1/2}$ and $R_2=\nabla^2\mathscr{L}^{-1}$ respectively, together with their adjoints $R_1^*=\mathscr{L}^{-1/2}\nabla$ and $R_2^*=\mathscr{L}^{-1}\nabla^2$.

Let $V \in RH_q$ with $q > n/2$. We will analyze the operators R_1, R_1^*, R_2 and R_2^* according to the regularity of the potential V.

Considering what was proved in [11] (Theorems 26, 27, and 28) and in [8] (Proposition 3), the following result follows.

PROPOSITION 1. *Let $V \in RH_q$. Then, we have*

1. *If $q \geqslant n$, the operators R_1 and R_1^* are SCZ operators of type (∞, δ) with $\delta = 1 - n/q$ and $\delta = 1$ respectively.*

2. *If $n/2 < q < n$, the operator R_1^* is a SCZ operator of type (s, δ) with s such that $1/s = 1/q - 1/n$ and $\delta = 2 - n/q$.*

3. *If $q > n/2$, the operator R_2^* is a SCZ operator of type (q, δ) with $\delta = \min\{1, 2 - n/q\}$.*

4. *If $q > n/2$ and ρ verifies a local smoothness condition*

$$|V(x) - V(y)| \lesssim \frac{|x-y|^\alpha}{\rho(x)^{\alpha+2}}, \tag{17}$$

for every $x, y \in \mathbb{R}$ such that $|x-y| < \rho(x)$ and some $0 < \alpha \leqslant 1$, then R_2 is a SCZ operator of type (∞, α).

Now, as a consequence of the last results, Theorem 11 and Theorem 12, we establish the following result concerning the bounding of Riesz–Schrödinger transforms of order 1 and 2.

THEOREM 13. *Let $V \in RH_q$ and $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, it follows then that*

1. *If $q \geqslant n$ and $p^- > 1$, the operators R_1 and R_1^* are bounded on $L^{p(\cdot)}(w)$ for every $w \in A^\rho_{p(\cdot)}$.*

2. *If $n/2 < q < n$, s is such that $1/s = 1/q - 1/n$ and $p^- > s'$, the operator R_1^* is bounded on $L^{p(\cdot)}(w)$ for every weight w such that $w^{s'} \in A^\rho_{p(\cdot)/s'}$, while the if $1 < p^+ < s$, the operator R_1 is bounded on $L^{p(\cdot)}(w)$ for every weight w such that $w^{-s'} \in A^\rho_{p'(\cdot)/s'}$.*

3. *If $q > n/2$ and $p^- > q'$, the operator R_2^* is bounded on $L^{p(\cdot)}(w)$ for every weight w such that $w^{q'} \in A^\rho_{p(\cdot)/q'}$, while the if $1 < p^+ < q$, the operators R_2 is bounded on $L^{p(\cdot)}(w)$ for every weight w such that $w^{-q'} \in A^\rho_{p'(\cdot)/q'}$.*

4. *If $q > n/2$, $p^- > 1$ and V satisfies (17) for some $0 < \alpha \leqslant 1$, the operators R_2 and R_2^* are bounded on $L^{p(\cdot)}(w)$ for every $w \in A^\rho_{p(\cdot)}$.*

4.3.2. Schrödinger–Riesz transforms involving V

We now consider the operators $M_\gamma = \mathscr{L}^{-\gamma}V^\gamma$ for $0<\gamma<n/2$ and $N_\gamma = \mathscr{L}^{-\gamma}\nabla V^{\gamma-1/2}$ for $1/2<\gamma\leqslant 1$. In [7] the authors prove the following results directly related to the operators in question.

PROPOSITION 2. *Let $V \in RH_q$ with $q>n/2$. Then,*

1. *The operator N_γ is a SCZ operator of type (s,δ) for $1/2<\gamma\leqslant 1$, with $\delta = \{1, 2-n/q\}$ and s such that $\frac{1}{s} = \left(\frac{1}{q}-\frac{1}{n}\right)^+ + \frac{2\gamma-1}{2q}$, where $\left(\frac{1}{q}-\frac{1}{n}\right)^+ = \max\{\frac{1}{q}-\frac{1}{n}, 0\}$.*
2. *The operator M_γ is a SCZ operator of type (s,δ) for $0<\gamma<n/2$, with $\delta < \{1, 2-n/q\}$ and $s = q/\gamma$.*

As an application of Theorem 12 and the above proposition we get boundedness properties for these operators on $L^{p(\cdot)}(w)$.

THEOREM 14. *Let $V \in RH_q$ with $q>n/2$, $1/2<\gamma\leqslant 1$ and s such that $\frac{1}{s} = \left(\frac{1}{q}-\frac{1}{n}\right)^+ + \frac{2\gamma-1}{2q}$. Then, the operator N_γ is bounded on $L^{p(\cdot)}(w)$ for every $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^- > s'$ and every weight w such that $w^{s'} \in A^\rho_{p(\cdot)/s'}$. Moreover, its adjoint operator $V^{\gamma-1/2}\nabla\mathscr{L}^{-\gamma}$ is bounded on $L^{p(\cdot)}(w)$ for every $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $1<p^+<s$ and any w such that $w^{-s'} \in A^\rho_{p'(\cdot)/s'}$.*

THEOREM 15. *Let $V \in RH_q$ with $q>n/2$, $0<\gamma<n/2$ and $s=q/\gamma$. Then, the operator M_γ is bounded on $L^{p(\cdot)}(w)$ for every $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^->s'$ and every weight w such that $w^{s'} \in A^\rho_{p(\cdot)/s'}$. Moreover, its adjoint operator $V^\gamma\mathscr{L}^{-\gamma}$ is bounded on $L^{p(\cdot)}(w)$ for every $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $1<p^+<s$ and any w such that $w^{-s'} \in A^\rho_{p'(\cdot)/s'}$.*

We can now apply these general results to some of the operators considered by Shen in [28].

THEOREM 16. *Let $V \in RH_q$ with $q>n/2$ and s such that $s=2q$ if $q\geqslant n$ or $\frac{1}{s} = \frac{3}{2q}-\frac{1}{n}$ if $n/2<q<n$. Then, the operator $\mathscr{L}^{-1}\nabla V^{1/2}$ is bounded on $L^{p(\cdot)}(w)$ for every $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^->s'$ and every weight w such that $w^{s'}\in A^\rho_{p(\cdot)/s'}$.*

Moreover, the operator $V^{1/2}\nabla\mathscr{L}^{-1}$ is bounded on $L^{p(\cdot)}(w)$ for every $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $1<p^+<s$ and any w such that $w^{-s'}\in A^\rho_{p'(\cdot)/s'}$.

THEOREM 17. *Let $V\in RH_q$ with $q>n/2$. Then, the operator $\mathscr{L}^{-1/2}V^{1/2}$ is bounded on $L^{p(\cdot)}(w)$ for every $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^->(2q)'$ and every weight w such that $w^{(2q)'}\in A^\rho_{p(\cdot)/(2q)'}$.*

Moreover, the operator $V^{1/2}\mathscr{L}^{-1/2}$ *is bounded on* $L^{p(\cdot)}(w)$ *for every* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *with* $1 < p^+ < 2q$ *and any* w *such that* $w^{-(2q)'} \in A^{\rho}_{p'(\cdot)/(2q)'}$.

THEOREM 18. *Let* $V \in RH_q$ *with* $q > n/2$. *Then, the operator* $\mathscr{L}^{-1}V$ *is bounded on* $L^{p(\cdot)}(w)$ *for every* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *with* $p^- > q'$ *and every weight* w *such that* $w^{q'} \in A^{\rho}_{p(\cdot)/q'}$.

Moreover, the operator $V\mathscr{L}^{-1}$ *is bounded on* $L^{p(\cdot)}(w)$ *for every* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *with* $1 < p^+ < q$ *and any* w *such that* $w^{-q'} \in A^{\rho}_{p'(\cdot)/q'}$.

4.3.3. Operators of the type $\mathscr{L}^{i\gamma}$

In this subsection we consider the power operators $\mathscr{L}^{i\gamma}$ for $\gamma \in \mathbb{R}$. From what has been done in [28] it is easy to verify that the following theorem holds.

THEOREM 19. *Let* $V \in RH_q$ *with* $q > n/2$. *Then, the operator* $\mathscr{L}^{i\gamma}$ *with* $\gamma \in \mathbb{R}$ *is a SCZ operator of type* (∞, δ). *Moreover,* $\mathscr{L}^{i\gamma}$ *is bounded on* $L^{p(\cdot)}(w)$ *for every* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *with* $p^- > 1$ *and any* $w \in A^{\rho}_{p(\cdot)}$.

4.4. Commutators of Schrödinger type singular integrals

We will now consider the commutator T_b for a Schrödinger–Calderón–Zygmund operators T of type (s, δ) for $1 < s \leqslant \infty$ and $0 < \delta \leqslant 1$. Let us remind that by commutator of a linear operator T with multiplication by a function $b \in L^1_{\text{loc}}$, called symbol, means

$$T_b f(x) = [T, b]f(x) = T(bf)(x) - b(x)Tf(x).$$

In Theorem 1 of [9], the following result regarding the boundedness of the commutator of an SCZ operator of type (s, ∞) for $1 < s \leqslant \infty$ is established.

PROPOSITION 3. *Let* T *be a SCZ operators of type* (s, δ) *for* $1 < s \leqslant \infty$, $0 < \delta \leqslant 1$ *and let* $b \in BMO^{\infty}_{\rho}$. *Then the commutators* $[T, b]$ *is bounded operator on* $L^p(w)$ *for any* $p > s'$ *and every* $w \in A^{\rho}_{p/s'}$.

Therefore, from what we have seen in the previous subsection we obtain the following consequences.

THEOREM 20. *Let* $V \in RH_q$, $b \in BMO^{\infty}_{\rho}$ *and* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, *it follows then that*

1. *If* $q \geqslant n$ *and* $p^- > 1$, *the operators* $[R_1, b]$ *and* $[R_1^*, b]$ *are bounded on* $L^{p(\cdot)}(w)$ *for every* $w \in A^{\rho}_{p(\cdot)}$.
2. *If* $n/2 < q < n$, s *is such that* $1/s = 1/q - 1/n$ *and* $p^- > s'$, *the operator* $[R_1^*, b]$ *is bounded on* $L^{p(\cdot)}(w)$ *for every weight* w *such that* $w^{s'} \in A^{\rho}_{p(\cdot)/s'}$.

3. *If $q > n/2$ and $p^- > q'$, the operator $[R_2^*, b]$ is bounded on $L^{p(\cdot)}(w)$ for every weight w such that $w^{q'} \in A^{\rho}_{p(\cdot)/q'}$.*

4. *If $q > n/2$, $p^- > 1$ and V satisfies (17) for some $0 < \alpha \leqslant 1$, the operators $[R_2, b]$ and $[R_2^*, b]$ are bounded on $L^{p(\cdot)}(w)$ for every $w \in A^{\rho}_{p(\cdot)}$.*

THEOREM 21. *Let $V \in RH_q$, $b \in BMO^{\infty}_{\rho}$ and $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, it follows that*

1. *If $q > n/2$ and s such that either $s = 2q$ when $q \geqslant n$ or $\frac{1}{s} = \frac{3}{2q} - \frac{1}{n}$ when $n/2 < q < n$, and denote by $T_1 = \mathscr{L}^{-1}\nabla V^{1/2}$, then the operator $[T_1, b]$ is bounded on $L^{p(\cdot)}(w)$ for every weight w such that $w^{s'} \in A^{\rho}_{p(\cdot)/s'}$ provided that $p^- > s'$.*

2. *If $q > n/2$, $p^- > (2q)'$ and denote by $T_2 = \mathscr{L}^{-1/2}V^{1/2}$, then the operator $[T_2, b]$ is bounded on $L^{p(\cdot)}(w)$ for every weight w such that $w^{(2q)'} \in A^{\rho}_{p(\cdot)/(2q)'}$.*

3. *If $q > n/2$, $p^- > q'$ and denote by $T_3 = \mathscr{L}^{-1}V$, then the operator $[T_3, b]$ is bounded on $L^{p(\cdot)}(w)$ for every weight w such that $w^{q'} \in A^{\rho}_{p(\cdot)/q'}$.*

THEOREM 22. *Let $V \in RH_q$ with $q > n/2$ and $b \in BMO^{\infty}_{\rho}$. Then, the operator $[\mathscr{L}^{i\gamma}, b]$, with $\gamma \in \mathbb{R}$, is bounded on $L^{p(\cdot)}(w)$ for every $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$ and any $w \in A^{\rho}_{p(\cdot)}$.*

4.5. Pseudo–differential operators

Let m be real number. Following [33], a symbol in $S^m_{1,\delta}$ is a smooth function $\sigma(x,\xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α and β the following estimate holds

$$|D^{\alpha}_x D^{\beta}_{\xi}\sigma(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-\beta-\delta|\alpha|}, \tag{18}$$

where $C_{\alpha,\beta} > 0$ is independent of x and ξ. The operator T given by

$$Tf(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)e^{2\pi i x\cdot\xi}\hat{f}(\xi)\,d\xi,$$

is called a pseudo–differential operator with symbol $\sigma(x,\xi) \in S^m_{1,\delta}$, where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f. Denote by $L^m_{1,\delta}$ the class of pseudo–differential operators with symbols in $S^m_{1,\delta}$.

In this case, if we consider $V = c$ with $c > 0$, we have that $\tilde{\rho} = \rho_V \cong 1$, and then from Theorem 1.1 and Theorem 1.2 in [30], together with Theorem 1, we have

THEOREM 23. *Let $b \in BMO$, $T \in L^1_{1,0}$ and $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $p^- > 1$. Then, if $w \in A^{\tilde{\rho}}_{p(\cdot)}$ there exists $C > 0$ such that*

$$\|Tfw\|_{p(\cdot)} \leqslant C\|fw\|_{p(\cdot)},$$

and

$$\|T_b fw\|_{p(\cdot)} \leqslant C\|b\|_{BMO}\|fw\|_{p(\cdot)}.$$

4.6. Coifman-Fefferman type inequalities

As in [3] we consider classes of local weights associated to a critical radius function ρ. Given $p>1$ the class $A_p^{\rho,\mathrm{loc}}$ is defined as the set of weights w such that

$$\left(\int_B w\right)^{1/p}\left(\int_B w^{-\frac{1}{p-1}}\right)^{1/p'} \leqslant C|B|, \tag{19}$$

for every ball $B\in\mathscr{B}_\rho$, where $\mathscr{B}_\rho$ denote the family of subcritical balls of $\mathbb{R}^n$, i.e., the set of balls $B(x,r)$ with $x\in\mathbb{R}^n$ and $r\leqslant\rho(x)$.

For the case $p=1$, the class $A_1^{\rho,\mathrm{loc}}$ is defined as those weights w satisfying

$$\frac{1}{|B|}\int_B w \leqslant C\inf_B w, \tag{20}$$

for every ball $B\in\mathscr{B}_\rho$, where the infimum should be understood as an essential infimum with respect to the Lebesgue measure. Notice that $A_p^{\rho}\subset A_p^{\rho,\mathrm{loc}}$ for any $p\geqslant 1$. We denote $A_\infty^{\rho,\mathrm{loc}}=\bigcup_{p\geqslant 1}A_p^{\rho,\mathrm{loc}}$.

To give the precise statement of our following results we need to introduce the following maximal operator. Given a critical radius function ρ, $1\leqslant s<\infty$, $0\leqslant\alpha<n$ and $\theta\geqslant 0$ we define

$$M_s^{\alpha,\theta}f(x)=\sup_{B(x_0,r)\ni x}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta}|B(x_0,r)|^{\alpha/n}\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}|f|^s\right)^{1/s}.$$

We shall drop the parameter α in the above notation when $\alpha=0$ and the parameter s when $s=1$. With these definitions we will consider the following results, which were proved in [3] (see Theorems 5 and 6 there).

PROPOSITION 4. *Let $1<s<\infty$ and T be a weak type (s',s') operator with kernel K satisfying,*

1. *For each $N>0$ there exists C_N such that*

$$\left(\int_{R<|y-x_0|\leqslant 2R}|K(x,y)|^s dy\right)^{1/s}\leqslant C_N R^{-d/s'}\left(\frac{\rho(x_0)}{R}\right)^N, \tag{21}$$

for every $x\in B=B(x_0,\rho(x_0))$, and $R>2\rho(x_0)$.

2. *There exists a constant C such that*

$$\sum_{k\geqslant 1}(2^k r)^{d/s'}\left(\int_{B_{k+1}\setminus B_k}|K(x,y)-K(x_0,y)|^s dy\right)^{1/s}\leqslant C, \tag{22}$$

for every ball $B=B(x_0,r)$ and every $x\in B$, with $r\leqslant\rho(x_0)$ and $B_k=2^kB$, $k\in\mathbb{N}$.

Then, if $0<p<\infty$ *and* $\theta>0$*, there exists a constant* C *such that*

$$\int_{\mathbb{R}^n}|Tf(x)|^p w(x)dx \leqslant C\int_{\mathbb{R}^n}|M^{\theta}_{s'}f(x)|^p w(x)dx,$$

for every $w\in A_\infty^{\rho,\mathrm{loc}}$ *and* $f\in L^{s'}_{\mathrm{loc}}(\mathbb{R}^n)$.

PROPOSITION 5. *Let* T_α *be a weak type* $(1,n/(n-\alpha))$ *operator with* $0\leqslant\alpha<n$ *and kernel* K_α *satisfying,*

1. *For each* $N>0$ *there exists* C *such that*

$$|K_\alpha(x,y)| \leqslant \frac{C}{|x-y|^{n-\alpha}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}, \tag{23}$$

   for every $x,y\in\mathbb{R}^n$.

2. *There exist constants* C *and* $\lambda>0$ *such that*

$$|K_\alpha(x,y)-K_\alpha(x_0,y)| \leqslant C\frac{|x-x_0|^\lambda}{|x-y|^{n-\alpha+\lambda}}, \tag{24}$$

   for every $x,y\in\mathbb{R}^n$*, whenever* $|x-x_0|<\frac{|x-y|}{2}$.

Then, if $0<p<\infty$ *and* $\theta>0$*, there exists a constant* C *such that*

$$\int_{\mathbb{R}^n}|T_\alpha f(x)|^p w(x)dx \leqslant C\int_{\mathbb{R}^n}|M^{\alpha,\theta}f(x)|^p w(x)dx,$$

for every $w\in A_\infty^{\rho,\mathrm{loc}}$ *and* $f\in L^{1}_{\mathrm{loc}}(\mathbb{R}^n)$.

In order to state the following results of this subsection, we introduce the following notations. Given a Young function φ and a locally integrable function f we consider the φ-average over a ball B defined as

$$\|f\|_{\varphi,B}=\inf\left\{\lambda>0:\ \frac{1}{|B|}\int_B\varphi\left(\frac{|f|}{\lambda}\right)\leqslant 1\right\}.$$

For $0\leqslant\alpha<n$, $\sigma>0$ and Young function φ, we define

$$M^{\alpha,\sigma}_{\varphi}f(x)=\sup_{x\in B=B(z,r)}\left(1+\frac{r}{\rho(z)}\right)^{-\sigma}|B|^{\alpha/n}\|f\|_{\varphi,B}.$$

Observe that when $\varphi(t)=t^s$, with $s\geqslant 1$, the maximal function $M^{\alpha,\sigma}_{\varphi}$ coincides with $M^{\alpha,\sigma}_{s}$ defined previously.

It follows then, as a consequence of Theorems 7 and 8 in [3] the following.

PROPOSITION 6. *Let $1 < s < \infty$ and suppose T is an integral operator of weak type (s', s') with associated kernel K satisfying (21) and that for every $N \geqslant 0$ there exists C_N such that*

$$\sum_{k \geqslant 1} k(2^k r)^{d/s'} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^N \left(\int_{2^{k+1}B \setminus 2^k B} |K(x,y) - K(x_0,y)|^s dy\right)^{1/s} \leqslant C_N, \qquad (25)$$

a.e. $x \in B$, for every ball $B \in \mathscr{B}_\rho$.

Then, if $0 < p < \infty$, $w \in A_\infty^{\rho,\mathrm{loc}}$ and $b \in BMO_\rho^\infty$, for any $\sigma > 0$ there exists a constant C such that

$$\int_{\mathbb{R}^n} |[T,b]f(x)|^p w(x)dx \leqslant C\|b\|_{BMO_\rho^\infty} \int_{\mathbb{R}^n} |M_\psi^\sigma f(x)|^p w(x)dx,$$

for every f bounded and with compact support, where $\psi(t) = t^{s'} \log(1+t)^{s'}$.

PROPOSITION 7. *Let $0 \leqslant \alpha < n$ and suppose T_α is an integral operator of weak type $(1, n/(n-\alpha))$ with associated kernel K_α satisfying (23) and that for each $M > 0$ and $0 < \lambda < 1$ there exist a constant C such that*

$$|K_\alpha(y,z) - K_\alpha(x,z)| \leqslant C \frac{|y-x|^\lambda}{|y-z|^{n-\alpha+\lambda}} \left(1 + \frac{|y-z|}{\rho(y)}\right)^{-M}, \qquad (26)$$

whenever $|x-y| < \frac{1}{2}|y-z|$.

Then, if $0 < p < \infty$, $w \in A_\infty^{\rho,\mathrm{loc}}$ and $b \in BMO_\rho^\infty$, for any $\sigma > 0$ there exists a constant C such that

$$\int_{\mathbb{R}^n} |[T_\alpha,b]f(x)|^p w(x)dx \leqslant C\|b\|_{BMO_\rho^\infty} \int_{\mathbb{R}^n} |M_\psi^{\alpha,\sigma} f(x)|^p w(x)dx,$$

for every f bounded and with compact support, where $\psi(t) = t\log(1+t)$.

As a consequence of the above propositions, Theorems 9, 10 and 11 in [3] (and their proofs) and Theorem 4, we have the following results.

THEOREM 24. *Let $V \in RH_q$ with $q > n$, $b \in BMO_\rho^\infty$, $\sigma > 0$ and $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$. Then, for every weight w such that $w^s \in A_{p(\cdot)/s}$ with $p^- > s$,*

$$\|R_1 f w\|_{p(\cdot)} + \|R_1^* f w\|_{p(\cdot)} \leqslant C\|M^\sigma f w\|_{p(\cdot)},$$

and

$$\|[R_1,b] f w\|_{p(\cdot)} + \|[R_1^*,b] f w\|_{p(\cdot)} \leqslant C\|M_\psi^\sigma f w\|_{p(\cdot)},$$

where $\psi(t) = t\log(1+t)$.

THEOREM 25. *Let $V \in RH_q$ with $q > n/2$, $b \in BMO_\rho^\infty$, $\sigma > 0$ and $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$. If $T_1 = R_1^*$, $T_2 = \mathscr{L}^{-1/2}V^{1/2}$ and $T_3 = \mathscr{L}^{-1}V$, then for every weight w such that $w^{s_j} \in A_{p(\cdot)/s_j}$ with $p^- > s_j$,*

$$\|T_j f w\|_{p(\cdot)} \leqslant C\|M_{s_j'}^\sigma f w\|_{p(\cdot)}, \quad j = 1,2,3;$$

and

$$\|[T_j,b]f\,w\|_{p(\cdot)} \leqslant C\|M^{\sigma}_{\psi_j} f\,w\|_{p(\cdot)}, \quad j=1,2,3;$$

where $\psi_j(t)=t^{s_j'}\log(1+t)^{s_j'}$, *with* $1/s_1=1/q-1/n$, $s_2=2q$ *and* $s_3=q$.

THEOREM 26. *Let* $V\in RH_q$ *with* $q>n$, $b\in BMO^{\infty}_{\rho}$, $\sigma>0$ *and* $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$. *Then, if* $T=\mathscr{L}^{-i\gamma}$ *with* $\gamma\in\mathbb{R}$, *for every weight* w *such that* $w^s\in A_{p(\cdot)/s}$ *with* $p^->s$,

$$\|Tf\,w\|_{p(\cdot)} \leqslant C\|M^{\sigma} f\,w\|_{p(\cdot)},$$

and

$$\|[T,b]f\,w\|_{p(\cdot)} \leqslant C\|M^{\sigma}_{\psi} f\,w\|_{p(\cdot)},$$

where $\psi(t)=t\log(1+t)$.

Finally, as the last application of this subsection we will consider the fractional integral associated to $\mathscr{L}$ defined for $0<\alpha<n$ as

$$I_\alpha f(x)=\mathscr{L}^{-\alpha/2}f(x)=\int_0^\infty e^{-t\mathscr{L}}f(x)\,t^{\alpha/2}\frac{dt}{t}=\int_{\mathbb{R}^n}K_\alpha(x,y)f(y)\,dy,$$

where $K_\alpha(x,y)=\int_0^\infty k_t(x,y)\,t^{\alpha/2}\,\frac{dt}{t}$, and $e^{-t\mathscr{L}}$, $t>0$ is the heat semigroup associated to $\mathscr{L}$.

As a consequence of Propositions 5 and 7, the Proposition 8 and the Theorem 12 in [3] and Theorem 4, we have the following result.

THEOREM 27. *Let* $V\in RH_q$ *with* $q>n/2$, $0<\alpha<n$, $b\in BMO^{\infty}_{\rho}$, $\sigma>0$ *and* $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$. *Then, for every weight* w *such that* $w^s\in A_{p(\cdot)/s}$ *with* $p^->s$,

$$\|I_\alpha f\,w\|_{p(\cdot)} \leqslant C\|M^{\alpha,\sigma} f\,w\|_{p(\cdot)},$$

and

$$\|[I_\alpha,b]f\,w\|_{p(\cdot)} \leqslant C\|M^{\alpha,\sigma}_{\psi} f\,w\|_{p(\cdot)},$$

where $\psi(t)=t\log(1+t)$.

REFERENCES

[1] R. AYALA AND A. CABRAL, *Boundedness of fractional operators associated with Schrödinger operators on weighted variable Lebesgue spaces via extrapolation*, Revista de la Unión Matemática Argentina, **66**, 1 (2023), 35–67.

[2] B. BONGIOANNI, A. CABRAL, AND E. HARBOURE, *Extrapolation for classes of weights related to a family of operators and applications*, Potential Anal., **38**, 4 (2013), 1207–1232.

[3] B. BONGIOANNI, A. CABRAL, AND E. HARBOURE, *Lerner's inequality associated to a critical radius function and applications*, J. Math. Anal. Appl., **407**, 1 (2013), 35–55.

[4] B. BONGIOANNI, A. CABRAL, AND E. HARBOURE, *Schrödinger type singular integrals: weighted estimates for* $p=1$, Math. Nachr. **289**, 11–12 (2016), 1341–1369.

[5] B. BONGIOANNI, E. HARBOURE, AND O. SALINAS, *Classes of weights related to Schrödinger operators*, J. Math. Anal. Appl., **373**, 2 (2011), 563–579.

[6] B. BONGIOANNI, E. HARBOURE, AND O. SALINAS, *Commutators of Riesz transforms related to Schrödinger operators*, J. Fourier Anal. Appl., **17**, 1 (2011), 115–134.

[7] B. BONGIOANNI, E. HARBOURE, AND P. QUIJANO, *Weighted Inequalities for Schrödinger Type Singular Integrals*, J. Fourier Anal. Appl., **25**, 3 (2019), 595–632.

[8] B. BONGIOANNI, E. HARBOURE AND P. QUIJANO, *Behaviour of Schrödinger Riesz transforms over smoothness spaces*, J. Math. Anal. Appl., **517**, 2 (2023), https://doi.org/10.1016/j.jmaa.2022.126613.

[9] B. BONGIOANNI, E. HARBOURE, AND P. QUIJANO, *Boundedness and compactness for commutators of singular integrals related to a critical radius function*, Preprints del IMAL, http://www.imal.santafe-conicet.gov.ar/publicaciones/preprints/2021-0052.pdf, (2021).

[10] A. CABRAL, *Weighted norm inequalities for the maximal functions associated to a critical radius function on variable Lebesgue spaces*, J. Math. Anal. Appl., **516** (2022), https://doi.org/10.1016/j.jmaa.2022.126521.

[11] A. CABRAL, *Weighted inequalities for Schrödinger type Singular Integrals on variable Lebesgue spaces*, Tunisian Journal of Mathematics, **6**, 2 (2024), 321–342.

[12] D. CRUZ-URIBE, L. DIENING AND P. HÄSTÖ, *The maximal operator on weighted variable Lebesgue spaces*, Frac. Calc. Appl. Anal., **14**, 3 (2011), 361–374.

[13] D. CRUZ-URIBE AND A. FIORENZA, *Variable Lebesgue Spaces*, Foundations and harmonic analysis. Series: Applied and Numerical Harmonic Analysis, IX. Springer, Heidelberg, (2013).

[14] D. CRUZ-URIBE, A. FIORENZA AND C. J. NEUGEBAUER, *Weighted norm inequalities for the maximal operator on variable Lebesgue spaces*, J. Math. Anal. Appl., **394**, 2 (2012), 744–760.

[15] D. CRUZ-URIBE, A FIORENZA, J. M. MARTELL AND C. PÉREZ, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math., **31**, 1 (2006), 239–264.

[16] D. CRUZ-URIBE, J. MARTELL, AND C. PÉREZ, *Extrapolation from A_∞ weights and applications*, J. Funct. Anal., **312**, 2 (2004), 412–439.

[17] D. CRUZ-URIBE, J. MARTELL, AND C. PÉREZ, *Weights, extrapolation and the theory of Rubio de Francia,* vol. 215 of Operator Theory: Advances and Applications, Birkhäuser/Springer Basel AG, Basel, (2011).

[18] D. CRUZ-URIBE, AND L.-A. WANG, *Variable Hardy spaces*, Indiana Univ. Math., **63**, 2 (2014), 447–493.

[19] D. CRUZ-URIBE, AND L.-A. WANG, *Extrapolation and weighted norm inequalities in the variable Lebesgue spaces*, Trans. Amer. Math. Soc., **369**, 2 (2017), 1205–1235.

[20] L. DIENING, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl., **7**, 2 (2004), 245–254.

[21] L. DIENING, P. HARJULEHTO, P. HÄSTÖ AND M. RŮŽIČKA, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer: Heidelberg, (2011).

[22] J. DZIUBAŃSKI AND J. ZIENKIEWICZ, *H^p spaces associated with Schrödinger operator with potential from reverse Hölder classes*, Colloq. Math., **98**, 1 (2003), 5–38.

[23] J. DZIUBAŃSKI, G. GARRIGÓS, T. MARTÍNEZ, J. L. TORREA AND J. ZIENKIEWICZ, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*. Math. Z., **249**, 2 (2005), 329–356.

[24] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J., **41**, 4 (1991), 592–618.

[25] T. MA, P. R. STINGA, J. L. TORREA, AND C. ZHANG, *Regularity estimates in Hölder spaces for Schrödinger operators via a $T1$ theorem*, Ann. Mat. Pura Appl., **193**, 2 (2014), 561–589.

[26] J. MUSIELAK, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034, Springer-Verlag, Berlin, (1983).

[27] J. L. RUBIO DE FRANCIA, *Factorization and extrapolation of weights*, Bull. Amer. Math. Soc., **7**, 2 (1982), 393–395.

[28] Z. SHEN, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble), **45**, 2 (1995), 513–546.

[29] L. TANG, *Weighted norm inequalities for Schrödinger type operators*, Forum Math., **27**, 4 (2015), 2491–2532.

[30] L. TANG, *Weighted norm inequalities for pseudo–differential operators with smooth symbols and their commutators*, Journal of Functional Analysis, **262**, 4 (2012), 1603–1629.

[31] L. TANG, *Weighted norm inequalities for commutators of Littlewood–Paley functions related to Schrödinger operators*, Prepint, arXiv:1109.0100, (2011).
[32] L. TANG, J. WANG AND H. ZHU, *Weighted norm inequalities for area functions related to Schrödinger operators*, Real and Complex Analysis, **4**, 1 (2019), 38–50.
[33] M. TAYLOR, *Pseudodifferential Operators and Nonlinear PDE*, Birkhäuser, Boston, (1991).

(Received January 9, 2024)

Adrián Cabral
Instituto de Modelado e Innovación Tecnológica
CONICET-UNNE, and Departamento de Matemática
Facultad de Ciencias Exactas y Naturales y Agrimensura
UNNE, 3400, Corrientes, Argentina
e-mail: enrique.cabral@comunidad.unne.edu.ar