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OPTIMAL ESTIMATES IN MUSIELAK–ORLICZ SPACES FOR A PARABOLIC SCHRÖDINGER EQUATION

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Abstract. Let $d \geqslant 2$ and u be a strong solution to the parabolic Schrödinger equation

$$u_t - \Delta u + Vu = f \qquad \text{in} \quad \mathbb{R}^d_T := \mathbb{R}^d \times (0,T].$$

We show that

$$\|u_t\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|D^2u\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|Vu\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} \leqslant C\,\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)}$$

under suitable conditions on the Musielak-Orlicz function φ and the potential V.

1. Introduction

In this paper, we investigate the parabolic Schrödinger equation of the form

$$u_t - \Delta u + Vu = f \qquad \text{in} \quad \mathbb{R}^d_T := \mathbb{R}^d \times (0,T], \tag{1}$$

where $d \geqslant 2$. Our main result states that an optimal estimate up to the second order holds for any strong solution to (1) within an appropriate framework. The emphasis is that this estimate takes place in Musielak-Orlicz spaces.

To make this precise, we need some preparation. First, we require the potential V to be in the reverse Hölder class $RH_\infty \equiv RH_\infty(\mathbb{R}^{d+1})$. This means $V \geqslant 0$ and

$$\mathbb{D} := \sup_{Q_r \subset \mathbb{R}^{d+1}} \left(\fint_{Q_r} V\,dz \right)^{-1} \left(\sup_{z \in Q_r} V(z) \right) < \infty, \tag{2}$$

where

$$Q_r := B_r \times (-r^2, r^2).$$

We call $\mathbb{D}$ the reverse Hölder constant of V.

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Next let φ be a Musielak-Orlicz function and define

$$W^{2,1}_{\varphi(\cdot)}(\mathbb{R}^d_T) := \left\{ g \in L^{\varphi(\cdot)}(\mathbb{R}^d_T) : |Dg|, |D^2 g|, g_t \in L^{\varphi(\cdot)}(\mathbb{R}^d_T) \right\},$$

which is a Banach space endowed with the norm

$$\|g\|_{W^{2,1}_{\varphi(\cdot)}(\mathbb{R}^d_T)} := \|g\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|Dg\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|D^2 g\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|g_t\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)}.$$

By a strong solution u to (1), we mean $u \in W^{2,1}_{\varphi(\cdot)}(\mathbb{R}^d_T)$ which satisfies (1) almost everywhere. Then the optimal estimate has the form

$$\|u_t\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|D^2 u\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|Vu\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} \leqslant C\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)}, \tag{3}$$

provided that $f \in L^{\varphi(\cdot)}(\mathbb{R}^d_T)$ and φ enjoys certain nice properties.

Prior to our consideration, it is known that an L^p-version and an Orlicz version of (3) holds. The details are as follows.

- [5, Theorems 1.1 and 4.1] assert that

  $$\|D^2 u\|_{L^p(\mathbb{R}^d_T)} + \|Vu\|_{L^p(\mathbb{R}^d_T)} \leqslant C\|f\|_{L^p(\mathbb{R}^d_T)}$$

  holds for a strong solution $u \in W^{2,1}_p(\mathbb{R}^d_T)$ of (1) whenever $f \in L^p(\mathbb{R}^d_T)$ and V belongs to the reverse Hölder class $RH_\sigma(\mathbb{R}^d)$ with

  $$d \geqslant 3, \quad 1 \leqslant p \leqslant \sigma, \quad \sigma > \frac{d}{2\mu} \quad \text{and} \quad 0 < \mu < \frac{d}{d+2}.$$

  By $V \in RH_\sigma(\mathbb{R}^d)$, it is understood that $V \geqslant 0$ and there exists a constant $C = C(\sigma, V) > 0$ such that

  $$\left(\frac{1}{|B_r|} \int_{B_r} V^\sigma \right)^{1/\sigma} \leqslant \frac{C}{|B_r|} \int_{B_r} V$$

  holds for every ball $B_r \subset \mathbb{R}^d$. As such V is independent of the time variable.

  A higher-order extension of [5] is done in [10]. The techniques in both [5] and [10] are taken from harmonic analysis.

- Let $d \geqslant 2$ and ϕ be an Orlicz function which is at the same time doubling (Δ_2) and reverse doubling (∇_2). Then [12, Theorem 1.5] states that

  $$\|u_t\|_{L^\phi(\mathbb{R}^d_T)} + \|D^2 u\|_{L^\phi(\mathbb{R}^d_T)} + \|Vu\|_{L^\phi(\mathbb{R}^d_T)} \leqslant C\|f\|_{L^\phi(\mathbb{R}^d_T)}$$

  holds for a strong solution $u \in W^{2,1}_\phi(\mathbb{R}^d_T)$ of (1), assuming $f \in L^\phi(\mathbb{R}^d_T)$ and $V \in RH_\infty(\mathbb{R}^{d+1})$. The proof is based on a PDEs' approach.

- In [3], the authors consider a strong solution u to a slightly different equation

$$u_t - \Delta u + Vu = f \qquad \text{in} \quad \mathbb{R}^{d+1}$$

and show that

$$\|u_t\|_{L^p(\mathbb{R}^d_T)} + \|D^2 u\|_{L^p(\mathbb{R}^d_T)} + \|Vu\|_{L^p(\mathbb{R}^d_T)} \leqslant C\|f\|_{L^p(\mathbb{R}^d_T)}$$

whenever $d \geqslant 1$, $1 < p < \infty$, $f \in L^p(\mathbb{R}^d_T)$ and $V \in RH_\sigma(\mathbb{R}^{d+1})$, in the sense that $V \geqslant 0$ and there exists a constant $C = C(\sigma, V) > 0$ such that

$$\left(\frac{1}{|Q_r|}\int_{Q_r} V^\sigma\right)^{1/\sigma} \leqslant \frac{C}{|Q_r|}\int_{Q_r} V$$

holds for every cylinder $Q_r \subset \mathbb{R}^{d+1}$. Their proof relies on harmonic analysis and operator theory.

In the setting of Musielak-Orlicz spaces, we will prove (3) using a PDEs' approach. Specifically, we make use of a covering lemma together with a comparison argument. These tools were also employed in [12] in the setting of Orlicz spaces. However, the ideas therein are peculiar to Orlicz spaces, which can not be easily adapted to Musielak-Orlicz spaces. Here we introduce new ideas to arrive at (3). Furthermore, our proof is rather self-contained.

Next we provide several basic definitions in Musielak-Orlicz spaces.

DEFINITION 1. Let $L \geqslant 1$ be a constant. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called *L-almost increasing* if

$$f(s_1) \leqslant L f(s_2) \qquad \text{for all } s_1 < s_2.$$

Likewise, f is called *L-almost decreasing* if

$$f(s_2) \leqslant L f(s_1) \qquad \text{for all } s_1 < s_2.$$

DEFINITION 2. A function $\varphi = \varphi(z,s) : \mathbb{R}^d_T \times [0,\infty) \longrightarrow [0,\infty)$ is call a *weak Φ-function*, denoted by $\varphi \in \Phi_w(\mathbb{R}^d_T)$, if it has the following four properties:

(i) φ is measurable in the z-variable.

(ii) φ is non-decreasing and left-continuous in the s-variable.

(iii) For all $z \in \mathbb{R}^d_T$, one has

$$\varphi(z,0) = \lim_{s \to 0^+} \varphi(z,s) = 0 \qquad \text{and} \qquad \lim_{s\to\infty} \varphi(z,s) = \infty.$$

(iv) There exists a constant $L \geqslant 1$ such that the mapping

$$(0,\infty) \ni s \longmapsto \frac{\varphi(z,s)}{s}$$

is L-almost increasing for all $z \in \mathbb{R}^d_T$.

The set Φ_w is defined to consist of all weak Φ-functions which are independent of the z-variable.

To achieve our main result, further regularities are required on Φ_w functions. The following conditions are taken from [7].

Let $\varphi \in \Phi_w(\mathbb{R}^d_T)$. We denote

$$\varphi^+_U(s) := \sup_{z\in U} \varphi(z,s) \qquad \text{and} \qquad \varphi^-_U(s) := \inf_{z\in U} \varphi(z,s)$$

for each $s \in [0,\infty)$ and $U \subset \mathbb{R}^d_T$. If $U = \mathbb{R}^d_T$ we simply write

$$\varphi^\pm(s) := \varphi^\pm_{\mathbb{R}^d_T}(s)$$

for each $s \in [0,\infty)$.

The conditions to be imposed on φ are as follows.

(A0) There exists a constant $\beta \in (0,1)$ such that $\varphi^+(\beta) \leqslant 1 \leqslant \varphi^-(\beta^{-1})$.

(A1-φ^-) There exists a constant $\beta \in (0,1)$ such that

$$\varphi^+_Q(\beta s) \leqslant \varphi^-_Q(s)$$

for all $s \in [1, (\varphi^-)^{-1}(|Q|^{-1})]$ and for every cylinder $Q \subset \mathbb{R}^d_T$. Here $(\varphi^-)^{-1}$ is understood in the sense of [6, Definition 2.3.1]. To be specific, the function $(\varphi^-)^{-1} : [0,\infty) \longrightarrow [0,\infty)$ is the *left inverse* of φ^- which is given by

$$(\varphi^-)^{-1}(\tau) := \inf\{s \geqslant 0 : \varphi(s) \geqslant \tau\}. \tag{4}$$

$(\text{aInc})_p$ There exist constants $p \geqslant 1$ and $L \geqslant 1$ such that the mapping

$$(0,\infty) \ni s \longmapsto \frac{\varphi(z,s)}{s^p}$$

is L-almost increasing for all $z \in \mathbb{R}^d_T$.

$(\text{aDec})_q$ There exist constants $q \geqslant 1$ and $L \geqslant 1$ such that the mapping

$$(0,\infty) \ni s \longmapsto \frac{\varphi(z,s)}{s^q}$$

is L-almost decreasing for all $z \in \mathbb{R}^d_T$.

As a consequence of $(\text{aInc})_p$ and $(\text{aDec})_q$,

$$\alpha^p L^{-1} \varphi(z,s) \leqslant \varphi(z,\alpha s) \leqslant \alpha^q L \varphi(z,s) \tag{5}$$

for all $(z,s) \in \mathbb{R}^d_T \times [0,\infty)$ and $\alpha > 1$. This means that $\phi(z,\cdot)$ enjoys the Δ_2 and ∇_2 properties for each $z \in \mathbb{R}^d_T$, in the sense of [6, Definitions 2.2.6 and 2.4.14]. Furthermore, in view of [6, Proposition 2.3.7], we know that $(\varphi^-)^{-1}$ also satisfies $(\text{aInc})_{1/q}$ and $(\text{aDec})_{1/p}$. Similar to (5), one has

$$\alpha^{\frac{1}{q}} L^{\frac{1}{q}} (\varphi^-)^{-1}(\tau) \leqslant (\varphi^-)^{-1}(\alpha\tau) \leqslant \alpha^{\frac{1}{p}} L^{\frac{1}{p}} (\varphi^-)^{-1}(\tau) \tag{6}$$

for all $\tau \in [0,\infty)$ and $\alpha > 1$.

Next we introduce Musielak-Orlicz spaces.

DEFINITION 3. Let $\varphi \in \Phi_w(\mathbb{R}^d_T)$. We define

$$L^{\varphi(\cdot)}(\mathbb{R}^d_T) := \left\{ u \in L^0(\mathbb{R}^d_T) : \lim_{\lambda\to 0} \rho_{\varphi(\cdot)}(\lambda u) = 0 \right\},$$

where $L^0(\mathbb{R}^d_T)$ is the set of all measurable functions on $\mathbb{R}^d_T$ and $\rho_{\varphi(\cdot)}$ is the *modular* of φ given by

$$\rho_{\varphi(\cdot)}(u) := \int_{\mathbb{R}^d_T} \varphi(z,|u|)\,dz.$$

We equip $L^{\varphi(\cdot)}(\mathbb{R}^d_T)$ with the (quasi-)norm

$$\|u\|_{\varphi(\cdot)} := \|u\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} := \inf\left\{ \lambda > 0 : \rho_{\varphi(\cdot)}\left(\frac{u}{\lambda}\right) \leqslant 1 \right\}.$$

If φ is independent of the z-variable, in the sense that $\varphi(z,s) = \varphi(s)$, then we write $L^{\varphi}(\mathbb{R}^d_T)$ in place of $L^{\varphi(\cdot)}(\mathbb{R}^d_T)$.

Our main result is the following.

THEOREM 1. *Let $d \geqslant 2$ and $\varphi \in \Phi_w(\mathbb{R}^d_T)$ satisfy* (A0), (A1-φ^-), $(\text{aInc})_p$ *and* $(\text{aDec})_q$ *for some constants $1 < p \leqslant q < \infty$. Suppose $V \in RH_\infty$ is symmetric in the time variable, i.e., $V(x,t) = V(x,-t)$ for a.e. $(x,t) \in \mathbb{R}^{d+1}$. Let $f \in L^{\varphi(\cdot)}(\mathbb{R}^d_T)$. Let $u \in W^{2,1}_{\varphi(\cdot)}(\mathbb{R}^d_T)$ be a strong solution to*

$$u_t - \Delta u + Vu = f \qquad \text{in } \mathbb{R}^d_T.$$

Then there exists a constant $C = C(d,\varphi,p,q,L,\mathbb{D}) > 0$ such that

$$\|u_t\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|D^2 u\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|Vu\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} \leqslant C\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)}.$$

A remark is immediate.

REMARK 1. Some examples of $V \in RH_\infty(\mathbb{R}^{d+1})$ which is symmetric in the time variable are as follows.

- $V(x,t) = V(x)$ for a.e. $(x,t) \in \mathbb{R}^{d+1}$, where $V(x) \in RH_\infty(\mathbb{R}^d)$.

  This type of potential was considered in [5]. Also see [9] and [4].

- $V(x,t) = |(x,t)|^{\alpha}$ for a.e. $(x,t) \in \mathbb{R}^{d+1}$ and for some $\alpha \geqslant 0$.
  See [8, (0.12)] for more details.

The symmetric assumption on V is due to a reflection principle that we apply later in the course of the proof. The assumption can be removed in particular situations. See Remark 2 below.

The paper is planned as follows. We summarize a sufficient background in Section 2. Theorem 1 is proved in Section 3.

Throughout assumption We always assume $d \geqslant 2$.

Notation In the whole paper, we employ the following set of notation:

- $Q_r = B_r \times (-r^2, r^2)$ for every $r > 0$.
- $Q_r(y,s) = B_r(y) \times (s - r^2, s + r^2)$ for every $y \in \mathbb{R}^d$, $r > 0$ and $s > 0$.
- $\kappa Q_r(y,s) = Q_{\kappa r}(y,s) = B_{\kappa r}(y) \times (s - (\kappa r)^2, s + (\kappa r)^2)$ for every $y \in \mathbb{R}^d$, $r > 0$, $s > 0$ and $\kappa > 0$.
- For each $A \subset \mathbb{R}^d_T$, we write
$$f_A := \fint_A f(z)\,dz = \frac{1}{|A|}\int_A f(z)\,dz.$$
- $V(Q) = \int_Q V(z)\,dz$ for each cylinder $Q \subset \mathbb{R}^d_T$.
- ϕ is used to denote an Orlicz function, whereas φ is used for a Musielak-Orlicz function. In notation, we write $\phi \in \Phi_w$ and $\varphi \in \Phi_w(\mathbb{R}^d_T)$.

2. Preliminaries

We collect the required background for proving the main theorem in this section.

2.1. Properties of Φ_w-functions

The following Jensen-type inequality will be useful in the sequel.

LEMMA 1. ([7, Lemma 2.3]) *Let $\phi \in \Phi_w$ satisfy* (aInc)$_p$ *for some $p \geqslant 1$. Then there exists a constant $C = C(p,L) > 0$ such that*
$$\phi\left(C\left(\fint_Q |f|^p\,dz\right)^{\frac{1}{p}}\right) \leqslant \fint_Q \phi(|f|)\,dz$$
for all cylinders $Q \subset \mathbb{R}^d_T$.

2.2. Background estimates

We report an local uniform boundedness for a weak solution to a Schrödinger equation.

LEMMA 2. ([12, Lemma 2.5]) *Let $\rho > 0$ and $V \in RH_\infty$. Let h be a weak solution to*

$$h_t - \Delta h + Vh = 0 \qquad \text{in } Q_{2\rho}.$$

Then there exists a constant $C = C(d) > 0$ such that

$$\|h\|_{L^\infty(Q_\rho)} \leqslant \frac{C}{V(Q_{2\rho})} \int_{Q_{2\rho}} V|h|.$$

Also recall the Hessian estimate in Orlicz space.

PROPOSITION 1. *Let $\phi \in \Phi_w$ satisfy* (aInc)$_p$ *and* (aDec)$_q$ *for some constants $1 < p \leqslant q < \infty$. Let $V \in RH_\infty$ and $f \in L^\phi(\mathbb{R}^d_T)$. Let $u \in W^{2,\phi}(\mathbb{R}^d_T)$ be a strong solution to*

$$u_t - \Delta u + Vu = f \qquad \text{in } \mathbb{R}^d_T.$$

Then there exists a constant $C = C(d, \phi, p, q, L, \mathbb{D}) > 0$ such that

$$\|u_t\|_{L^\phi(\mathbb{R}^d_T)} + \|D^2 u\|_{L^\phi(\mathbb{R}^d_T)} + \|Vu\|_{L^\phi(\mathbb{R}^d_T)} \leqslant C\|f\|_{L^\phi(\mathbb{R}^d_T)}.$$

Proof. We observe that $\phi \in \Delta_2 \cap \nabla_2$ in view of (5), in the sense of [6, Definitions 2.2.6 and 2.4.14]. Hence the claim follows at once from [12, Theorem 1.5]. □

The following comparison estimate is a combination of [2, Lemma 4.4, Corollary 4.5 and (4.29)] together.

LEMMA 3. *Let $\gamma > 1$ and $g \in L^\gamma(Q_4)$. Then for each $\varepsilon \in (0,1)$ there exists a constant $\delta = \delta(n, \gamma, \varepsilon) > 0$ such that if $\ell \in W^{2,\gamma}(Q_4)$ is a solution to*

$$\ell_t - \Delta \ell = g \qquad \text{in } Q_4$$

with

$$\fint_{Q_4} |\ell_t|^\gamma \, dz + \fint_{Q_4} |D^2 \ell|^\gamma \, dz \leqslant 1 \qquad \text{and} \qquad \fint_{Q_4} |g|^\gamma \, dz \leqslant \delta,$$

then there exists a solution $v \in W^{2,\gamma}(Q_4)$ to

$$v_t - \Delta v = 0 \qquad \text{in } Q_4$$

with the following properties:

(i) $\displaystyle\fint_{Q_1} |\ell_t - v_t|^\gamma dz + \fint_{Q_1} |D^2\ell - D^2 v|^\gamma \, dz \leqslant \varepsilon,$

(ii) $\displaystyle\fint_{Q_4} |v_t|^\gamma \, dz + \fint_{Q_4} |D^2 v|^\gamma \, dz \leqslant 1$ *and*

(iii) $\|v_t\|_{L^\infty(Q_1)} + \|D^2 v\|_{L^\infty(Q_1)} \leqslant C$, *where $C = C(d, \gamma) > 0$.*

3. Proof of main result

In this section, we prove Theorem 1. We adopt the assumptions of Theorem 1 hereafter.

To begin with, let u be a strong solution to (1). Since $u \in W^{2,1}_{\varphi(\cdot)}(\mathbb{R}^d_T)$, [6, Lemma 6.1.6] and the Sobolev embeddings together yield

$$u \in C\left([0,T]; W^{2,p}(\mathbb{R}^d)\right).$$

Consequently,

$$u(x,0) := \lim_{t \to 0^+} u(x,t) \tag{7}$$

exists in $\mathbb{R}$ for a.e. $x \in \mathbb{R}^d$. Define $\overline{u} : \mathbb{R}^d \times [-T,T] \longrightarrow \mathbb{R}$ by

$$\overline{u}(x,t) := \begin{cases} u(x,t) & \text{if } (x,t) \in \mathbb{R}^d \times [0,T], \\ u(x,-t) & \text{if } (x,t) \in \mathbb{R}^d \times [-T,0). \end{cases}$$

That is, $\overline{u}$ is an extension of u to $\mathbb{R}^d \times [-T,T]$. Similarly, let $\overline{f} : \mathbb{R}^d \times [-T,T] \longrightarrow \mathbb{R}$ be defined by

$$\overline{f}(x,t) := \begin{cases} f(x,t) & \text{if } (x,t) \in \mathbb{R}^d \times (0,T], \\ f(x,-t) & \text{if } (x,t) \in \mathbb{R}^d \times [-T,0), \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\overline{u} \in W^{2,1}_{\varphi(\cdot)}(\mathbb{R}^d \times [-T,T])$ and $\overline{f} \in L^{\varphi(\cdot)}(\mathbb{R}^d \times [-T,T])$. Moreover, the assumption that V is symmetric in the time variable assures that $\overline{u}$ is a strong solution to

$$\overline{u}_t - \Delta \overline{u} + V \overline{u} = \overline{f} \qquad \text{in} \quad \mathbb{R}^d \times [-T,T]. \tag{8}$$

For an ease of notation, we will identify u and f with $\overline{u}$ and $\overline{f}$ respectively in what follows.

REMARK 2. We use the symmetry of V in (8) above. However, if either

$$u(x,0) = 0 \qquad \text{for a.e. } x \in \mathbb{R}^d$$

in (7) or $\mathbb{R}^d_T$ is replaced by $\mathbb{R}^{d+1}$ in (1), then the symmetry assumption on V can be removed. The former case was employed in [2, p. 2294] to enable the zero extension of u in the time variable. In the latter case, our main result is in agreement with [3, Theorem 3.6].

Let

$$1 < \nu := \sqrt{p} < p \leqslant q$$

and $\delta \in (0,1)$ be a sufficiently small constant to be specified later. We set

$$\psi(z,s) := \varphi(z,s)^{\frac{1}{\nu}}, \qquad \psi^+(s) := \sup_{z \in \mathbb{R}^d_T} \psi(z,s) \qquad \text{and} \qquad \psi^-(s) := \inf_{z \in \mathbb{R}^d_T} \psi(z,s)$$

for each $s \in [0,\infty)$.

For each $\zeta \in \mathbb{R}^d_T$, define $G_\zeta : (0,\infty) \longrightarrow [0,\infty)$ by

$$G_\zeta(\tau) := \fint_{Q_\tau(\zeta)} \Big[\psi(z,|u_t|) + \psi(z,|D^2u|) + \psi(z,V|u|) + \frac{1}{\delta}\,\psi(z,|f|)\Big]\,dz.$$

Set

$$E(\lambda) := \big\{z \in \mathbb{R}^d_T : \psi(z,|u_t|) + \psi(z,|D^2u|) + \psi(z,V|u|) > \lambda\big\}$$

for each $\lambda > 0$.

The next fundamental lemma provides a covering of $E(\lambda)$ by cylinders of suitable sizes. For an ease of notation, therein we write

$$[w > k] := \{z \in \mathbb{R}^d_T : w(z) > k\}$$

for each measurable function w on $\mathbb{R}^d_T$ and $k \in \mathbb{R}$. Also denote

$$\lambda_0 := \int_{\mathbb{R}^d_T} \Big[\psi(z,|u_t|) + \psi(z,|D^2u|) + \psi(z,V|u|) + \frac{1}{\delta}\,\psi(z,|f|)\Big]\,dz.$$

LEMMA 4. *For all $\lambda > 0$ there exists a disjoint family $\{Q_{\tau_k}(z_k)\}_{k\in\mathbb{N}}$ with $z_k \in E(\lambda)$ and $\tau_k > 0$ such that*

$$G_{z_k}(\tau_k) = \lambda \qquad \text{and} \qquad G_{z_k}(\tau_k) < \lambda \quad \text{for all } \tau > \tau_k. \tag{9}$$

Moreover,

$$E(\lambda) \subset \bigcup_{k\in\mathbb{N}} Q_{5\tau_k}(z_k) \cup \text{negligible set} \tag{10}$$

and

$$\begin{aligned} |Q_{\tau_k}(z_k)| \leqslant \frac{2}{\lambda}\Bigg(&\int_{Q_{\tau_k}(z_k)\cap\left[\psi(\cdot,|u_t|)+\psi(\cdot,|D^2u|)+\psi(\cdot,V|u|)>\frac{\lambda}{4}\right]} \Big\{\psi(z,|u_t|) + \psi(z,|D^2u|) \\ &\qquad + \psi(z,V|u|)\Big\}\,dz \\ &+ \frac{1}{\delta}\int_{Q_{\tau_k}(z_k)\cap\left[\psi(\cdot,|f|)>\frac{\delta\lambda}{4}\right]} \psi(z,|f|)\,dz\Bigg) \end{aligned} \tag{11}$$

for all $k \in \mathbb{N}$.

Proof. We perform the exit time argument on the functional G_ζ which was first introduced in [1].

Let $\lambda > 0$ and $\zeta \in E(\lambda)$. Let $\tau_0 > 0$ be such that

$$\lambda\,|Q_{\tau_0}(\zeta)| = 2\lambda_0.$$

Then for all $\tau \geqslant \tau_0$, one has

$$\begin{aligned} G_\zeta(\tau) &\leqslant \frac{2}{|Q_\tau(\zeta)|} \int_{\mathbb{R}^d_T} \Big[\psi(z,|u_t|)+\psi(z,|D^2u|)+\psi(z,V|u|)+\frac{1}{\delta}\,\psi(z,|f|)\Big]\,dz \\ &\leqslant \frac{2}{|Q_{\tau_0}(\zeta)|} \int_{\mathbb{R}^d_T} \Big[\psi(z,|u_t|)+\psi(z,|D^2u|)+\psi(z,V|u|)+\frac{1}{\delta}\,\psi(z,|f|)\Big]\,dz \\ &= \frac{2\lambda\,\lambda_0}{\lambda\,|Q_{\tau_0}(\zeta)|} = \lambda. \end{aligned}$$

On the other hand, Lebesgue's differentiation theorem implies

$$\lim_{\tau\to 0^+} G_\zeta(\tau) = \lim_{\tau\to 0^+} \fint_{Q_\tau(\zeta)} \Big[\psi(z,|u_t|)+\psi(z,|D^2u|)+\psi(z,V|u|)+\frac{1}{\delta}\,\psi(z,|f|)\Big]\,dz > \lambda.$$

Therefore, the continuity of G_ζ yields that there exists a $\tau_\zeta > 0$ such that

$$G_\zeta(\tau_\zeta)=\lambda \qquad \text{and} \qquad G_\zeta(\tau)<\lambda \quad \text{for all } \tau>\tau_\zeta.$$

This in combination with Vitali's covering lemma asserts the existence of a disjoint family $\{Q_{\tau_k}(z_k)\}_{k\in\mathbb{N}}$ with $z_k \in E(\lambda)$ and $\tau_k > 0$ such that (9) and (10) hold.

Next since $G_{z_k}(\tau_k)=\lambda$ by (9), we may conclude that

$$\begin{aligned} |Q_{\tau_k}(z_k)| \leqslant \frac{1}{\lambda}\Bigg(\int_{Q_{\tau_k}(z_k)\cap\left[\psi(\cdot,|u_t|)+\psi(\cdot,|D^2u|)+\psi(\cdot,V|u|)>\frac{\lambda}{4}\right]} & \psi(z,|u_t|)+\psi(z,|D^2u|) \\ &+\psi(z,V|u|)\,dz \\ +\frac{1}{\delta}\int_{Q_{\tau_k}(z_k)\cap\left[\psi(\cdot,|f|)>\frac{\delta\lambda}{4}\right]} \psi(z,|f|)\,dz + \frac{\lambda}{2}\,|Q_{\tau_k}(z_k)| \Bigg)&, \end{aligned}$$

from which (11) follows at once. □

We are ready to prove Theorem 1.

Proof of Theorem 1. We divide the proof into five steps as follows.

Step 1: We perform a scaling argument.
Set

$$\tilde u = \frac{u}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)}} \qquad \text{and} \qquad \tilde f = \frac{f}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)}}.$$

Then

$$\|\tilde f\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} = 1 \tag{12}$$

and $\tilde u$ is a solution to

$$\tilde u_t - \Delta\tilde u + V\tilde u = \tilde f \qquad \text{in } \mathbb{R}^d_T.$$

The claim is then equivalent to showing that

$$\|\tilde u_t\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|D^2\tilde u\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} + \|V\tilde u\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)} \leqslant C \tag{13}$$

since $\|\cdot\|_{L^{\varphi(\cdot)}(\mathbb{R}^d_T)}$ is a quasi-norm in view of [6, Lemma 3.2.2] and the quasi-norm is understood in the sense of [6, p. 7].

Observe that Proposition 1 gives

$$\|\tilde{u}_t\|_{L^{\varphi^-}(\mathbb{R}^d_T)} + \|D^2\tilde{u}\|_{L^{\varphi^-}(\mathbb{R}^d_T)} + \|V\tilde{u}\|_{L^{\varphi^-}(\mathbb{R}^d_T)} \leqslant C. \tag{14}$$

Step 2: Let $\lambda > 0$ and $\{Q_{\tau_k}(z_k)\}_{k\in\mathbb{N}}$ be given by Lemma 4 with u and f being replaced by $\tilde{u}$ and $\tilde{f}$ respectively. For short, we will write $Q_k = Q_{\tau_k}(z_k)$ for each $k\in\mathbb{N}$. Let $k\in\mathbb{N}$. We will estimate the sizes of $V\tilde{u}$ and $\tilde{f}$ in $L^\nu(20Q_k)$, where $\nu = \sqrt{p}$.

Recall that we set $\psi = \varphi^{\frac{1}{\nu}}$. Since $\varphi \in \Phi_w(\mathbb{R}^d_T)$ satisfies $(\mathrm{aInc})_p$ and $(\mathrm{aDec})_q$, we deduce that $\psi \in \Phi_w(\mathbb{R}^d_T)$ satisfies $(\mathrm{aInc})_\nu$ and $(\mathrm{aDec})_{q/\nu}$. Then it follows from (9) that

$$\fint_{20Q_k} \psi(z,|\tilde{u}_t|) + \psi(z,|D^2\tilde{u}|) + \psi(z,V|\tilde{u}|)\,dz \leqslant \lambda \qquad \text{and} \qquad \fint_{20Q_k} \psi(z,|\tilde{f}|)\,dz \leqslant \delta\lambda. \tag{15}$$

Define

$$\varphi_k^-(s) := \inf_{z\in 20Q_k} \varphi(z,s) \qquad \text{and} \qquad \varphi_k^+(s) := \sup_{z\in 20Q_k} \varphi(z,s)$$

and likewise

$$\psi_k^-(s) := \inf_{z\in 20Q_k} \psi(z,s) \qquad \text{and} \qquad \psi_k^+(s) := \sup_{z\in 20Q_k} \psi(z,s)$$

for each $s\in[0,\infty)$.

Observe that ψ_k^- satisfies $(\mathrm{aInc})_\nu$. Therefore, in view of Lemma 1 and (5) we deduce that

$$\begin{aligned}\psi_k^-\left(\left[\fint_{20Q_k} |D^2\tilde{u}|^\nu\,dz\right]^{\frac{1}{\nu}}\right) &\leqslant C\fint_{20Q_k} \psi_k^-(|D^2\tilde{u}|)\,dz \\ &\leqslant C\fint_{20Q_k} \psi(x,|D^2\tilde{u}|)\,dz \leqslant C\lambda.\end{aligned}$$

By the same token,

$$\begin{aligned}\left[\fint_{20Q_k} |D^2\tilde{u}|^\nu\,dz\right]^{\frac{1}{\nu}} &\leqslant C(\varphi^-)^{-1}\left(\fint_{20Q_k} \varphi^-(|D^2\tilde{u}|)\,dz\right) \\ &\leqslant C(\varphi^-)^{-1}\left(|20Q_k|^{-1}\int_{\mathbb{R}^d_T} \varphi^-(|D^2\tilde{u}|)\,dz\right) \\ &\leqslant C_0(\varphi^-)^{-1}\left(|Q_k|^{-1}\right)\end{aligned}$$

for some $C_0 \geqslant \beta^{-1}$, where $(\varphi^-)^{-1}$ is the left inverse of φ^- given by (4), the constant β is given by (A1-φ^-) and we used (14) together with (6) in the last step.

Using (A1-φ^-), we have

$$\begin{aligned}\varphi_k^+\left(\left[\fint_{20Q_k}|D^2\tilde{u}|^\nu\,dz\right]^{\frac{1}{\nu}}\right) &\leqslant C\,\varphi_k^+\left(\frac{1}{C_0}\left[\fint_{20Q_k}|D^2\tilde{u}|^\nu\,dz\right]^{\frac{1}{\nu}}\right)\\ &\leqslant C\,\varphi_k^-\left(\left[\fint_{20Q_k}|D^2\tilde{u}|^\nu\,dz\right]^{\frac{1}{\nu}}\right)\\ &\leqslant C\lambda^\nu.\end{aligned}$$

This coupled with (5) yields that

$$\left[\fint_{20Q_k}|D^2\tilde{u}|^\nu\,dz\right]^{\frac{1}{\nu}} \leqslant (\psi_k^+)^{-1}(c\lambda) \leqslant C_1\,(\psi_k^+)^{-1}(\lambda), \tag{16}$$

where $C_1 = C_1(d,p,q,L) > 0$ and $(\psi_k^+)^{-1}$ is the left inverse of ψ_k^+ defined as in (4).

Analogously we also have

$$\left[\fint_{20Q_k}|\tilde{u}_t|^\nu\,dz\right]^{\frac{1}{\nu}} \leqslant (\psi_k^+)^{-1}(c\lambda) \leqslant C_1\,(\psi_k^+)^{-1}(\lambda), \tag{17}$$

$$\left[\fint_{20Q_k}|V\tilde{u}|^\nu\,dz\right]^{\frac{1}{\nu}} \leqslant (\psi_k^+)^{-1}(c\lambda) \leqslant C_1\,(\psi_k^+)^{-1}(\lambda) \tag{18}$$

and

$$\left[\fint_{20Q_k}|\tilde{f}|^\nu\,dz\right]^{\frac{1}{\nu}} \leqslant (\psi_k^+)^{-1}(\delta\lambda) \leqslant C_1\,\delta^{\frac{\nu}{q}}\,(\psi_k^+)^{-1}(\lambda). \tag{19}$$

Step 3: We estimate the sizes of upper-level sets involving the potential term. We infer from (19) that $\tilde{f} \in L^\nu(20Q_k)$. In the sequel, for each $Q \Subset \mathbb{R}^{d+1}$ set

$$W_\nu^{j,1}(Q) := W^{1,\nu}(0,T;W^{j,\nu}(Q)), \qquad j \in \{1,2\}.$$

Let $W \in W_\nu^{2,1}(\mathbb{R}_T^d)$ be a strong solution to

$$W_t - \Delta W + VW = \tilde{f}\,\mathbb{1}_{20Q_k} \qquad \text{in } \mathbb{R}^{d+1},$$

where

$$\mathbb{1}_{20Q_k}(z) := \begin{cases}1 & \text{if } z \in 20Q_k,\\ 0 & \text{otherwise.}\end{cases}$$

Also set

$$H = \tilde{u} - W.$$

Then H satisfies

$$H_t - \Delta H + VH = 0 \qquad \text{in } 20Q_k.$$

We aim to estimate $\|VH\|_{L^\infty(10Q_k)}$.

As such [3, Theorem 3.6] asserts that

$$\int_{20Q_k} (V|W|)^\nu\,dz \leqslant \int_{\mathbb{R}^{d+1}} (V|W|)^\nu\,dz \leqslant C \int_{\mathbb{R}^{d+1}} |\tilde{f}\,\mathbb{1}_{20Q_k}|^\nu\,dz = C \int_{20Q_k} |\tilde{f}|^\nu\,dz.$$

Hence

$$⨍_{20Q_k} (V|W|)^\nu\,dz \leqslant C ⨍_{20Q_k} |\tilde{f}|^\nu\,dz \leqslant [C C_1\,\delta^{\frac{\nu}{q}}\,(\psi_k^+)^{-1}(\lambda)]^\nu, \tag{20}$$

where we used (19) in the last step.

As a consequence,

$$\begin{aligned}\left(⨍_{20Q_k} (V|H|)^\nu\,dz\right)^{\frac{1}{\nu}} &\leqslant \left(\int_{20Q_k} (V|W|)^\nu\,dz\right)^{\frac{1}{\nu}} + \left(\int_{20Q_k} V|\tilde{u}|\,dz\right)^{\frac{1}{\nu}}\\ &\leqslant C C_1\,\delta^{\frac{\nu}{q}}\,(\psi_k^+)^{-1}(\lambda) + C_1\,(\psi_k^+)^{-1}(\lambda)\\ &< C C_1\,(\psi_k^+)^{-1}(\lambda) + C_1\,(\psi_k^+)^{-1}(\lambda) =: C_2\,(\psi_k^+)^{-1}(\lambda)\end{aligned}$$

in view of (18) and (20). Consequently, Lemma 2 and Hölder's inequality give

$$\begin{aligned}\sup_{10Q_k} V|H| &\leqslant C(d)\left(\sup_{10Q_k} V\right)\left(\frac{1}{V(20Q_k)}\int_{20Q_k} V|H|\,dz\right)\\ &\leqslant C(d)\,C_2\,(\psi_k^+)^{-1}(\lambda)\left(\sup_{10Q_k} V\right)\left(⨍_{20Q_k} V\,dz\right)^{-1}\\ &\leqslant C(d)\,C_2\,(\psi_k^+)^{-1}(\lambda)\,\mathbb{D} \leqslant (\psi_k^+)^{-1}(C_3\,\lambda),\end{aligned} \tag{21}$$

where $\mathbb{D}$ is the reverse Hölder constant of V given by (2) and we used (5) for an appropriate constant $C_3 > 0$ in the last step.

Set

$$K := (2C_3)^{\frac{q}{\nu}}\,L^{\frac{1}{\nu}}.$$

Then

$$(\psi_k^+)^{-1}(K\lambda) = (\varphi_k^+)^{-1}\,(K^\nu\lambda) \geqslant 2C_3\,(\psi_k^+)^{-1}(\lambda).$$

With (21) in mind, we have

$$\begin{aligned}&|\{x \in 10Q_k : \psi(z, V|\tilde{u}|) > K\lambda\}|\\ &\leqslant \left|\left\{z \in 10Q_k : V|\tilde{u}| > (\psi_k^+)^{-1}\,(K\lambda)\right\}\right|\\ &\leqslant \left|\left\{z \in 10Q_k : V|\tilde{u}| > 2\,(\psi_k^+)^{-1}(C_3\lambda)\right\}\right|\\ &\leqslant |\{z \in 10Q_k : V|W| > (\psi_k^+)^{-1}(C_3\lambda)\}| + |\{z \in 10Q_k : V|H| > (\psi_k^+)^{-1}(C_3\lambda)\}|\\ &= |\{z \in 10Q_k : V|W| > (\psi_k^+)^{-1}(C_3\lambda)\}|\\ &\leqslant \frac{1}{[(\psi_k^+)^{-1}(C_3\lambda)]^\nu}\int_{10Q_k} (V|W|)^\nu\,dz\\ &\leqslant \frac{1}{[(\psi_k^+)^{-1}(C_3\lambda)]^\nu}\,[C_1\,\delta^{\frac{\nu}{q}}\,(\psi_k^+)^{-1}(\lambda)]^\nu\,|10Q_k| \leqslant C\,\delta^{\frac{\nu^2}{q}}\,|Q_k|\end{aligned}$$

$$\leqslant \frac{C}{\lambda}\delta^{\frac{\nu^2}{q}}\left(\int_{Q_k\cap\left[\psi(\cdot,|u_t|)+\psi(\cdot,|D^2u|)+\psi(\cdot,V|u|)>\frac{\lambda}{4}\right]}\left[\psi(\cdot,|u_t|)+\psi(z,|D^2u|)\right.\right.$$
$$\left.+\psi(z,V|u|)\right]dz$$
$$\left.+\frac{1}{\delta}\int_{Q_k\cap\left[\psi(\cdot,|f|)>\frac{\delta\lambda}{4}\right]}\psi(z,|f|)\,dz\right).$$

Step 4: We estimate the sizes of upper-level sets involving the principal term. Let $w\in W^{2,1}_\nu(20Q_k)\cap\mathring{W}^{1,1}_\nu(20Q_k)$ be the strong solution to

$$\begin{cases} w_t-\Delta w+Vw=\tilde{f} & \text{in } 20Q_k,\\ w=0 & \text{on } \partial_p(20Q_k),\end{cases}$$

where $\partial_p(20Q_k)$ denotes the parabolic boundary of $20Q_k$ given by

$$\partial_p(20Q_k):=\big(\partial(20B_k)\times[-\tau_k,\tau_k]\big)\cup\big(20B_k\times\{0\}\big)$$

and $\mathring{W}^{1,1}_\nu(20Q_k)$ is the closure of $C^\infty_c((20Q_k)^0)$ in $W^{1,1}_\nu(20Q_k)$ with

$$(20Q_k)^0:=20Q_k\setminus\big(20B_k\times\{0\}\big).$$

Also set

$$h=W-w,$$

where W is defined in Step 3. Then $h\in W^{2,1}_\nu(20Q_k)$ satisfies

$$\begin{cases} h_t-\Delta h+Vh=0 & \text{in } 20Q_k,\\ h=W & \text{on } \partial_p(20Q_k).\end{cases}$$

We aim to estimate $\|h_t\|_{L^\infty(10Q_k)}+\|D^2h\|_{L^\infty(10Q_k)}$.

Observe that

$$\left(⨍_{20Q_k}|w_t|^\nu\,dz\right)^{\frac{1}{\nu}}+\left(⨍_{20Q_k}|D^2w|^\nu\,dz\right)^{\frac{1}{\nu}}\leqslant C\left(⨍_{20Q_k}|\tilde{f}|^\nu\,dz\right)^{\frac{1}{\nu}}$$
$$\leqslant CC_1\,\delta^{\frac{\nu}{q}}\,(\psi_k^+)^{-1}(\lambda), \tag{22}$$

where we used [11, Theorem 9.2.1 and Remark 9.2.2] in the first step and (19) in the second step. This in turn yields

$$\left(⨍_{20Q_k}(V|w|)^\nu\,dz\right)^{\frac{1}{\nu}}=\left(⨍_{20Q_k}|\tilde{f}+\Delta w-w_t|^\nu\,dz\right)^{\frac{1}{\nu}}$$
$$\leqslant\left(⨍_{20Q_k}|\tilde{f}|^\nu\,dz\right)^{\frac{1}{\nu}}+\left(⨍_{20Q_k}|D^2w|^\nu\,dz\right)^{\frac{1}{\nu}}+\left(⨍_{20Q_k}|w_t|^\nu\,dz\right)^{\frac{1}{\nu}}$$
$$\leqslant CC_1\,\delta^{\frac{\nu}{q}}\,(\psi_k^+)^{-1}(\lambda). \tag{23}$$

At the same time, it follows from [3, Theorem 3.6] that

$$\begin{aligned}\int_{20Q_k} |W_t|^\nu + |D^2W|^\nu + (V|W|)^\nu \, dz &\leqslant \int_{\mathbb{R}^{d+1}} |W_t|^\nu + |D^2W|^\nu + (V|W|)^\nu \, dz \\ &\leqslant C \int_{\mathbb{R}^{d+1}} |\tilde{f} \mathbb{1}_{20Q_k}|^\nu \, dz = C \int_{20Q_k} |\tilde{f}|^\nu \, dz,\end{aligned}$$

whence

$$\begin{aligned}\left(\fint_{20Q_k} |W_t|^\nu \, dz\right)^{\frac{1}{\nu}} + \left(\fint_{20Q_k} |D^2W|^\nu \, dz\right)^{\frac{1}{\nu}} + \left(\fint_{20Q_k} (V|W|)^\nu \, dz\right)^{\frac{1}{\nu}} & \\ \leqslant C \left(\fint_{20Q_k} |\tilde{f}|^\nu \, dz\right)^{\frac{1}{\nu}} & \\ \leqslant C C_1 \, \delta^{\frac{\nu}{q}} \, (\psi_k^+)^{-1}(\lambda). & \qquad (24)\end{aligned}$$

As a by-product,

$$\begin{aligned}&\left(\fint_{20Q_k} |H_t|^\nu \, dz\right)^{\frac{1}{\nu}} + \left(\fint_{20Q_k} |D^2H|^\nu \, dz\right)^{\frac{1}{\nu}} \\ &\leqslant \left(\fint_{20Q_k} |\tilde{u}_t|^\nu \, dz\right)^{\frac{1}{\nu}} + \left(\fint_{20Q_k} |D^2\tilde{u}|^\nu \, dz\right)^{\frac{1}{\nu}} + \left(\fint_{20Q_k} |W_t|^\nu \, dz\right)^{\frac{1}{\nu}} \\ &\qquad + \left(\fint_{20Q_k} |D^2W|^\nu \, dz\right)^{\frac{1}{\nu}} \\ &\leqslant C_1 \, (\psi_k^+)^{-1}(\lambda) + C C_1 \, \delta^{\frac{\nu}{q}} \, (\psi_k^+)^{-1}(\lambda) \\ &< C C_1 \, (\psi_k^+)^{-1}(\lambda).\end{aligned}$$

Combining (22), (23) and (24) together, we arrive at

$$\begin{aligned}\left(\fint_{20Q_k} (V|h|)^\nu \, dz\right)^{\frac{1}{\nu}} &= \left(\fint_{20Q_k} (V|W-w|)^\nu \, dz\right)^{\frac{1}{\nu}} \\ &\leqslant \left(\fint_{20Q_k} (V|W|)^\nu \, dz\right)^{\frac{1}{\nu}} + \left(\fint_{20Q_k} (V|w|)^\nu \, dz\right)^{\frac{1}{\nu}} \\ &\leqslant C C_1 \, \delta^{\frac{\nu}{q}} \, (\psi_k^+)^{-1}(\lambda),\end{aligned}$$

$$\begin{aligned}\left(\fint_{20Q_k} |h_t|^\nu \, dz\right)^{\frac{1}{\nu}} &= \left(\fint_{20Q_k} |W_t - w_t|^\nu \, dz\right)^{\frac{1}{\nu}} \\ &\leqslant \left(\fint_{20B_k} |W_t|^\nu \, dz\right)^{\frac{1}{\nu}} + \left(\fint_{20Q_k} |w_t|^\nu \, dz\right)^{\frac{1}{\nu}} \\ &\leqslant C C_1 \, \delta^{\frac{\nu}{q}} \, (\psi_k^+)^{-1}(\lambda) \leqslant C \, (\psi_k^+)^{-1}(\lambda)\end{aligned}$$

and

$$\left(\fint_{20Q_k} |D^2 h|^\nu\, dz\right)^{\frac{1}{\nu}} = \left(\fint_{20Q_k} |D^2 W - D^2 w|^\nu\, dz\right)^{\frac{1}{\nu}}$$
$$\leqslant \left(\fint_{20Q_k} |D^2 W|^\nu\, dz\right)^{\frac{1}{\nu}} + \left(\fint_{20Q_k} |D^2 w|^\nu\, dz\right)^{\frac{1}{\nu}}$$
$$\leqslant C C_1\, \delta^{\frac{\nu}{q}}\, (\psi_k^+)^{-1}(\lambda) \leqslant C\, (\psi_k^+)^{-1}(\lambda).$$

Consequently, we may apply Lemma 3 with

$$\ell(z) = \frac{h(z_k + 5\tau_k z)}{C C_1\, (5\tau_k)^2\, (\psi_k^+)^{-1}(\lambda)}$$

and

$$g(z) = \frac{-(Vh)(z_k + 5\tau_k z)}{C C_1\, (\psi_k^+)^{-1}(\lambda)}$$

to see that for each $\varepsilon > 0$ there exists a sequence $\{v_k\}_{k\in\mathbb{N}}$ verifying

$$\fint_{5Q_k} |h_t - (v_k)_t|^\nu\, dz + \fint_{5Q_k} |D^2 h - D^2 v_k|^\nu\, dz \leqslant \varepsilon\, [C C_1\, (\psi_k^+)^{-1}(\lambda)]^\nu$$

and

$$\|(v_k)_t\|_{L^\infty(5Q_k)} + \|D^2 v_k\|_{L^\infty(5Q_k)} \leqslant C_4\, (\psi_k^+)^{-1}(\lambda),$$

where $\delta = \delta(d,p,\varepsilon) > 0$ is sufficiently small and $C_4 = C_4(d,p) > 0$.

Next we set

$$K' := (4C_4)^{\frac{q}{\nu}}\, L^{\frac{1}{\nu}}.$$

Then

$$(\psi_k^+)^{-1}(K'\lambda) = (\varphi_k^+)^{-1}\big((K')^\nu \lambda\big) \geqslant 4C_4\, (\psi_k^+)^{-1}(\lambda).$$

Furthermore,

$$\begin{aligned}
&|\{z \in 5Q_k : \psi(z, |\tilde{u}_t|) + \psi(z, |D^2\tilde{u}|) > K'\lambda\}| \\
&\leqslant \big|\big\{z \in 5Q_k : \psi(z, |\tilde{u}_t|) + |D^2\tilde{u}| > (\psi_k^+)^{-1}(K'\lambda)\big\}\big| \\
&\leqslant \big|\big\{z \in 5Q_k : |\tilde{u}_t| + |D^2\tilde{u}| > 2\, (\psi_k^+)^{-1}(C_4\lambda)\big\}\big| \\
&\leqslant |\{z \in 5Q_k : |H_t| + |D^2 H| > (\psi_k^+)^{-1}(C_4\lambda)\}| \\
&\qquad + |\{z \in 10Q_k : |w_t| + |D^2 w| > (\psi_k^+)^{-1}(C_4\lambda)\}| \\
&\qquad + |\{z \in 5Q_k : |h_t - (v_k)_t| + |D^2 h - D^2 v_k| > (\psi_k^+)^{-1}(C_4\lambda)\}| \\
&\qquad + |\{z \in 5Q_k : |(v_k)_t| + |D^2 v_k| > (\psi_k^+)^{-1}(C_4\lambda)\}|
\end{aligned}$$

$$
\begin{aligned}
&= |\{z \in 5Q_k : |H_t| + |D^2 H| > (\psi_k^+)^{-1}(C_4\lambda)\}| \\
&\qquad + |\{z \in 5Q_k : |w_t| + |D^2 w| > (\psi_k^+)^{-1}(C_4\lambda)\}| \\
&\qquad + |\{z \in 5Q_k : |h_t - (v_k)_t| + |D^2 h - D^2 v_k| > (\psi_k^+)^{-1}(C_4\lambda)\}| \\
&\leqslant \frac{1}{[(\psi_k^+)^{-1}(C_4\lambda)]^\nu} \int_{5Q_k} |H_t|^\nu + |D^2 H|^\nu + |w_t|^\nu + |D^2 w|^\nu \, dz \\
&\qquad + \frac{1}{[(\psi_k^+)^{-1}(C_4\lambda)]^\nu} \int_{5Q_k} |h_t - (v_k)_t|^\nu + |D^2 h - D^2 v_k|^\nu \, dz \\
&\leqslant \frac{1}{[(\psi_k^+)^{-1}(C_4\lambda)]^\nu} [C C_1 \, \delta^{\frac{\nu}{q}} (\psi_k^+)^{-1}(\lambda)]^\nu |5Q_k| \\
&\qquad + \frac{1}{[(\psi_k^+)^{-1}(C_4\lambda)]^\nu} \varepsilon [C C_1 (\psi_k^+)^{-1}(\lambda)]^\nu |5Q_k| \\
&\leqslant C(\delta^{\frac{\nu^2}{q}} + \varepsilon) |Q_k| \\
&\leqslant \frac{C}{\lambda} (\delta^{\frac{\nu^2}{q}} + \varepsilon) \Bigg(\int_{Q_k \cap \left[\psi(\cdot,|\tilde{u}_t|) + \psi(\cdot,|D^2\tilde{u}|) + \psi(\cdot,V|\tilde{u}|) > \frac{\lambda}{4}\right]} \psi(z, |D^2\tilde{u}|) + \psi(z, |D^2\tilde{u}|) \\
&\qquad\qquad + \psi(z, V|\tilde{u}|) \, dz \\
&\qquad\qquad + \frac{1}{\delta} \int_{Q_k \cap \left[\psi(\cdot,|\tilde{f}|) > \frac{\lambda\delta}{4}\right]} \psi(z, |\tilde{f}|) \, dz \Bigg).
\end{aligned}
$$

Step 5: We derive the Hessian estimates.

Set $K'' = \max\{K, K'\}$. Then

$$
\begin{aligned}
&\int_{\mathbb{R}^d_T} \varphi(z, |\tilde{u}_t|) + \varphi(z, |D^2\tilde{u}|) + \varphi(z, |V\tilde{u}|) \, dz \\
&= \nu (K'')^\nu \int_0^\infty \lambda^{\nu-1} |E(K''\lambda)| \, d\lambda \\
&= \nu (K'')^\nu \sum_{k\in\mathbb{N}} \left(\int_0^\infty \lambda^{\nu-1} |5Q_k \cap E(K''\lambda)| \, d\lambda \right) \\
&\leqslant C\nu (K'')^\nu (\delta^{\frac{\nu^2}{q}} + \varepsilon) \\
&\quad \times \sum_{k\in\mathbb{N}} \Bigg(\int_0^\infty \lambda^{\nu-2} \Bigg[\int_{Q_k \cap \left[\psi(\cdot,|\tilde{u}_t|) + \psi(\cdot,|D^2\tilde{u}|) + \psi(\cdot,V|\tilde{u}|) > \frac{\lambda}{4}\right]} \psi(z, |\tilde{u}_t|) + \psi(z, |D^2\tilde{u}|) \\
&\qquad\qquad + \psi(z, V|\tilde{u}|) \, dz \\
&\qquad\qquad + \frac{1}{\delta} \int_{Q_k \cap \left[\psi(\cdot,|\tilde{f}|) > \frac{\delta\lambda}{4}\right]} \psi(z, |\tilde{f}|) \, dz \Bigg] d\lambda \Bigg)
\end{aligned}
$$

$$\leqslant C(\delta^{\frac{\nu^2}{q}}+\varepsilon)\left(\int_{\mathbb{R}^d_T}\varphi(z,|\tilde{u}_t|)+\varphi(z,|D^2\tilde{u}|)+\varphi(z,V|\tilde{u}|)\,dz+\frac{1}{\delta^\nu}\int_{\mathbb{R}^d_T}\varphi(z,|\tilde{f}|)\,dz\right)$$

$$\leqslant C_5\,(\delta^{\frac{\nu^2}{q}}+\varepsilon)\left(\int_{\mathbb{R}^d_T}\varphi(z,|\tilde{u}_t|)+\varphi(z,|D^2\tilde{u}|)+\varphi(z,V|\tilde{u}|)\,dz+\frac{1}{\delta^\nu}\right),$$

where we used (10) in the third step as well as the identity

$$\int_{\mathbb{R}^d_T}|h|^s\,dz=(s-1)\int_0^\infty\lambda^{s-2}\int_{[|h|>\lambda]}|h|\,dz\,d\lambda \qquad \text{for each } s>1$$

and (12) in the last two steps respectively.

At this point, we choose $\varepsilon,\delta>0$ to be sufficiently small such that

$$C_5\,(\delta^{\frac{\nu^2}{q}}+\varepsilon)=\frac{1}{2}$$

to arrive at

$$\int_{\mathbb{R}^d_T}\varphi(z,|\tilde{u}_t|)+\varphi(z,|D^2\tilde{u}|)+\varphi(z,|V\tilde{u}|)\,dz\leqslant C,$$

which is (13) as required. □

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