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PROOFS FOR ALZER'S CONJECTURE ON THE GENERALIZED LOGARITHMIC MEAN

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Abstract. One defines with K. B. Stolarsky in 1975 the generalized logarithmic mean of two positive numbers. In 1986, H. Alzer posed a conjecture on the generalized logarithmic mean, obtaining some partial results. Using properties of hyperbolic functions, two inequalities for the generalized inverse harmonic means are established in this paper. Based on these inequalities, a proof of Alzer's conjecture is given.

1. Introduction

In 1975, Stolarsky defined in [7] the generalized logarithmic mean of two distinct positive numbers a, b as

$$L_r(a,b) = \left(\frac{b^r - a^r}{r(b-a)}\right)^{\frac{1}{r-1}},$$

where $r \in [-\infty, +\infty]$ and $L_{-\infty}(a,b)$, $L_0(a,b)$, $L_1(a,b)$, $L_{+\infty}(a,b)$ are looked at as the corresponding limits:

$$L_{-\infty}(a,b) = \lim_{r\to-\infty} L_r(a,b) = \min(a,b),$$

$$L_0(a,b) = \lim_{r\to 0} L_r(a,b) = \frac{b-a}{\ln b - \ln a},$$

$$L_1(a,b) = \lim_{r\to 1} L_r(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},$$

$$L_{+\infty}(a,b) = \lim_{r\to+\infty} L_r(a,b) = \max(a,b).$$

Similarly, in this paper, the value of a function on its contact discontinuity point is always looked at as its corresponding limit. The generalized logarithmic mean has been studied by many researchers (see [2, 4, 6], for examples). The aim of this paper is to prove the following inequalities:

$$2L_0(a,b) < L_r(a,b) + L_{-r}(a,b) < a+b, \quad \forall r \in (0,+\infty); \;\; b > a > 0. \tag{1.1}$$

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This is a conjecture posed by Alzer [1] in 1986. Alzer himself proved that

$$L_1(a,b)+L_{-1}(a,b)>2L_0(a,b), \qquad \forall b>a>0$$

and the following result:

PROPOSITION 1.1. *For any $r\in(0,+\infty)$, $b>a>0$, it holds that*

$$ab<L_r(a,b)L_{-r}(a,b)<L_0^2(a,b).$$

We showed that Alzer's conjecture can be proved if a conjecture on the generalized inverse harmonic mean holds (see [5] and the inequality (1.3) below). Based on this observation, some special cases for Alzer's conjecture were proved. In this paper, we will prove Alzer's conjecture for general cases.

The generalized inverse harmonic mean of two positive numbers a,b is defined by

$$C_r(a,b)=\left(\frac{a^r+b^r}{a+b}\right)^{\frac{1}{r-1}},$$

where $r\in[-\infty,+\infty]$. One finds that the generalized inverse harmonic mean is a special case of Gini mean [3]:

$$G_{p,q}(a,b)=\left(\frac{a^p+b^p}{a^q+b^q}\right)^{\frac{1}{p-q}}.$$

We mention that

$$C_0(a,b)=L_2(a,b)=\frac{a+b}{2}, \quad C_{-1}(a,b)=L_{-1}(a,b)=\sqrt{ab}, \quad C_2(a,b)=\frac{a^2+b^2}{a+b}$$

are the arithmetic mean, the geometric mean and the inverse harmonic mean, respectively; and

$$C_{-\infty}(a,b)=\min(a,b), \quad C_1(a,b)=\left(a^ab^b\right)^{\frac{1}{b+a}}, \quad C_{+\infty}(a,b)=\max(a,b).$$

On the other hand, we have

$$L_r(a^2,b^2)=L_r(a,b)C_r(a,b), \qquad \forall r\in[-\infty,+\infty];\ \ a,b>0. \tag{1.2}$$

The above equality is crucial in proving (1.1). By using (1.2), it is observed in [5] that (1.1) can be proved if the following equalities hold (see the proofs of Alzer's conjecture in Section 5):

$$\begin{cases} C_r(a,b)+C_{-r}(a,b)>a+b, \\ C_r^2(a,b)+C_{-r}^2(a,b)<a^2+b^2, \end{cases} \qquad \forall r\in(0,+\infty);\ \ b>a>0. \tag{1.3}$$

To facilitate description, we rewrite (1.1) and (1.3) as

THEOREM 1.1. *It holds that*

$$L_r(a,b)+L_{-r}(a,b)>2L_0(a,b), \qquad \forall r\in(0,+\infty];\ \ b>a>0. \tag{1.4}$$

THEOREM 1.2. *It holds that*

$$L_r(a,b)+L_{-r}(a,b)<a+b, \qquad \forall r\in[0,+\infty);\ \ b>a>0. \tag{1.5}$$

THEOREM 1.3. *It holds that*

$$C_r(a,b)+C_{-r}(a,b)>a+b, \qquad \forall r\in(0,+\infty);\ \ b>a>0. \tag{1.6}$$

THEOREM 1.4. *It holds that*

$$C_r^2(a,b)+C_{-r}^2(a,b)<a^2+b^2, \qquad \forall r\in[0,+\infty);\ \ b>a>0. \tag{1.7}$$

We showed in [5] that Theorem 1.3 implies Theorem 1.1 while Theorem 1.4 implies Theorem 1.2. Unfortunately, theorems 1.3 and 1.4 are also difficult to prove, though some special cases were verified in [5]. It was proved there that theorems 1.3 and 1.1 hold when $r=1,2,\frac{1}{2},3,\frac{1}{3},\frac{3}{2},\frac{2}{3}$, while theorems 1.4 and 1.2 hold when $r\in[\frac{1}{7},7]$.

In this paper, we will give proofs for theorems 1.3 and 1.4 in sections 3 and 4. Section 5 will be devoted to prove theorems 1.1 and 1.2. In Section 2, some preliminaries are listed. In Section 6, some corollaries of the theorems 1.1–1.4 will be given.

2. Preliminaries

First, we recall some basic properties of $L_r(a,b)$ and $C_r(a,b)$.

PROPOSITION 2.1. *Assume* $a,b>0$, $r\in[-\infty,+\infty]$. *Then*

(i) $L_r(a,b)$ *is symmetric, that is,* $L_r(a,b)=L_r(b,a)$.

(ii) *For any* $\alpha>0$, $L_r(\alpha a,\alpha b)=\alpha L_r(a,b)$.

(iii) *For any* $-\infty<s<r<+\infty$, $b>a>0$, *it holds that*

$$\min(a,b)<L_s(a,b)<L_r(a,b)<\max(a,b). \tag{2.1}$$

The proof of the above proposition can be found in [7]. One can establish similar result for C_r easily as in the following.

PROPOSITION 2.2. *Assume* $a,b>0$, $r\in[-\infty,+\infty]$. *Then*

(i) $C_r(a,b)$ *is symmetric, that is,* $C_r(a,b)=C_r(b,a)$.

(ii) *For any* $\alpha>0$, $C_r(\alpha a,\alpha b)=\alpha C_r(a,b)$.

(iii) *For any* $-\infty < s < r < +\infty$, $b > a > 0$, *it holds that*

$$\min(a,b) < C_s(a,b) < C_r(a,b) < \max(a,b). \tag{2.2}$$

Proof. We prove only the inequality $C_s(a,b) < C_r(a,b)$ in (2.2). It suffices to prove it in the case of $b > a = 1$. We have

$$\ln C_r(1,b) = \frac{1}{r-1}\int_1^r \frac{b^t \ln b}{b^t+1}\,dt = \int_0^1 \left(1 - \frac{1}{b^{1+t(r-1)}+1}\right)\ln b\,dt. \tag{2.3}$$

This implies that $C_r(1,b)$ is strictly increasing in $r \in (-\infty,+\infty)$ and we get the conclusion. $\square$

By the way, when $r,s \neq 1$, it is not difficult to get (2.2) by Hölder's inequality and categorical discussions. In addition, we can deduce (2.1) from (2.2) by a similar discussion in the proof of Theorem 6.2.

Unless said otherwise, in the rest of the paper we assume the hypothesis $b > a > 0$ *and denote* $t = \frac{1}{2}\ln\frac{b}{a}$. Then $\frac{b}{a} = e^{2t}$ and

$$\begin{aligned} C_r(a,b) =& \left(\frac{a^r+b^r}{a+b}\right)^{\frac{1}{r-1}} = \sqrt{ab}\left(\frac{\left(\frac{a}{b}\right)^{\frac{r}{2}} + \left(\frac{b}{a}\right)^{\frac{r}{2}}}{\left(\frac{a}{b}\right)^{\frac{1}{2}} + \left(\frac{b}{a}\right)^{\frac{1}{2}}}\right)^{\frac{1}{r-1}} \\ =& \sqrt{ab}\left(\frac{\cosh rt}{\cosh t}\right)^{\frac{1}{r-1}}, \qquad r \in (-\infty,+\infty). \end{aligned}$$

Consequently, Theorem 1.3 is equivalent to

$$\left(\frac{\cosh rt}{\cosh t}\right)^{\frac{1}{r-1}} + \left(\frac{\cosh t}{\cosh rt}\right)^{\frac{1}{r+1}} > 2\cosh t, \qquad \forall r \in (0,+\infty);\ \ t > 0. \tag{2.4}$$

While Theorem 1.4 is equivalent to

$$\left(\frac{\cosh rt}{\cosh t}\right)^{\frac{2}{r-1}} + \left(\frac{\cosh t}{\cosh rt}\right)^{\frac{2}{r+1}} < 2\cosh 2t, \qquad \forall r \in [0,+\infty);\ \ t > 0. \tag{2.5}$$

Noting that $2\cosh^2 t = \cosh 2t + 1$, (2.5) is equivalent to

$$\left(\frac{\cosh rt+1}{\cosh t+1}\right)^{\frac{1}{r-1}} + \left(\frac{\cosh t+1}{\cosh rt+1}\right)^{\frac{1}{r+1}} < 2\cosh t, \qquad \forall r \in [0,+\infty);\ \ t > 0. \tag{2.6}$$

Concerning the hyperbolic functions, we have

LEMMA 2.1. *Assume* $r > 1$, $t > 0$. *Then*

(i) $\sinh t > t$.

(ii) $r\tanh t > \tanh rt > \tanh t > 0$.

(iii) $\frac{\cosh^2 t}{\cosh 2t}$ *is decreasing strictly on* $[0,+\infty)$.

(iv) *There exists a* $t_r \in (0,+\infty)$ *such that* $r\cosh t - \cosh rt$ *is positive in* $(0,t_r)$ *and negative in* $(t_r,+\infty)$.

(v) *There exists a* $t_1 \in (0,+\infty)$ *such that* $\cosh t - t\sinh t$ *is positive in* $(0,t_1)$ *and negative in* $(t_1,+\infty)$.

Proof. The proof is easy. First, it is easy to see that both $\sinh t$ and $\cosh t$ are strictly increasing on $[0,+\infty)$. In the following, let $r>1$, $t>0$.

(i) We have

$$\sinh t - t = \int_0^t (\cosh s - 1)\,ds > 0.$$

(ii) It holds that

$$\begin{aligned} r\tanh t &= \int_0^t \frac{r}{\cosh^2 s}\,ds > \int_0^t \frac{r}{\cosh^2 rs}\,ds \\ &= \tanh rt = \int_0^{rt} \frac{1}{\cosh^2 s}\,ds > \int_0^t \frac{1}{\cosh^2 s}\,ds \\ &= \tanh t > 0. \end{aligned}$$

(iii) One can get the conclusion from

$$\frac{\cosh^2 t}{\cosh 2t} = \frac{\cosh 2t + 1}{2\cosh 2t} = \frac{1}{2} + \frac{1}{2\cosh 2t}.$$

(iv) By the well known series $\cosh t = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}$, we find that

$$r\cosh t - \cosh rt = (r-1) + r(1-r)\frac{t^2}{2!} + \cdots + r(1-r^{2n-1})\frac{t^{2n}}{(2n)!} + \cdots.$$

As $r>1$, the first term is positive. All the other terms define functions which evidently are strictly decreasing on $[0,+\infty)$. The claim follows.

(v) Since, $(\cosh t - t\sinh t)' = -t\cosh t < 0$, the result holds obviously. □

The above properties will be mainly used in proving Theorem 1.3.

The proof of Theorem 1.4 for $r \in (0,1)$ can be gotten from the case $r>1$. When $1 \leqslant r \leqslant 3$, the proof is easy. We assume $r \geqslant 3$ in the rest of this section. In this situation, the proof of Theorem 1.4 is based on analysing some monotonicity properties related to the following functions f,u,v,φ, which will be defined in a table below (see Table: Functions and Constants). A number of other functions and constants are introduced to get these monotonicity properties. We list them also in the same table. All the functions and constants are meant to be applied for $x \geqslant 1$, $r \geqslant 1$, often only for $x>1$, and $r \geqslant 3$.

$f(x)=x^{\frac{1}{r-1}}+x^{-\frac{1}{r+1}}$	$\varphi(x)=\begin{cases}\frac{f(x)-2}{1-x^{-1}} & \text{if } x>1\\ \frac{2}{r^2-1} & \text{if } x=1\end{cases}$
$u(x)=\frac{r}{r-1}x^{\frac{1}{r-1}}+1$	$u'(x)=\frac{r}{(r-1)^2}x^{\frac{1}{r-1}-1}$
$v(x)=\frac{r}{r+1}x^{-\frac{1}{r+1}}+1$	$v'(x)=-\frac{r}{(r+1)^2}x^{-\frac{1}{r+1}-1}$
$u(1)=\frac{2r-1}{r-1}$	$U(x)=x^{\frac{1}{r-1}}+\sigma x^{-\frac{1}{r+1}}$
$v(1)=\frac{2r+1}{r+1}$	$V(x)=(1-\sigma)x^{\frac{r}{r+1}}$
$\theta_0=\frac{1}{r}$	$\psi(x)=\frac{r-1}{r}u'(x)+(1+B)v'(x)-\varphi'(x)$
$B=\frac{(r+2)\sigma}{2r}-\frac{1}{2}=\frac{r^2-2r-1}{r(r-1)^2}$	$\Phi_1(x)=\frac{u^2(x)}{v(x)u'(x)}\psi(x)-1$
$\delta=-\frac{v'(1)}{u'(1)}=\frac{(r-1)^2}{(r+1)^2}$	$\Phi_2(x)=\frac{u(x)}{v(x)}\left[(\frac{1}{v(x)}-\frac{r-1}{r})u(x)+\delta\frac{u^2(x)}{v^2(x)}x^{-\frac{1}{r-1}-\frac{1}{r+1}}\right]$
$\rho_0=\frac{1}{v(1)}(1+\delta\frac{u(1)}{v(1)})=\frac{4r^2+2}{(2r+1)^2}$	$\Phi_3(x)=\delta\left[\frac{r(1+B)}{r-1}x^{\frac{1}{r-1}}+B\right]\frac{u(x)}{v(x)}x^{-\frac{1}{r-1}-\frac{1}{r+1}}$
$\sigma=\frac{(r-2)(r+1)^2}{(r+2)(r-1)^2}=\frac{r^3-3r-2}{r^3-3r+2}$	$g(x)=xf(x)+2(x-1)$
$G(x)=r\ln\frac{1}{2}\left(f(x)+\sqrt{f(x)^2-4}\right)-\ln\frac{1}{2}\left(g(x)+\sqrt{g(x)^2-4}\right)$	

Table: *Functions and Constants*

First, it is very easy to get the following result.

LEMMA 2.2. *Assume $r\geqslant 3$. Then the function f is smooth and f' is positive on $[1,+\infty)$. Consequently, f is strictly increasing on $[1,+\infty)$ and the equation $f(y)=2\cosh t$ admits a unique solution $y=y(t)\geqslant 1$ for any $t\geqslant 0$. Moreover, it holds that*

$$t=\ln\frac{f(y)+\sqrt{f^2(y)-4}}{2}.$$

The key point to prove Theorem 1.4 is that the function $\frac{u}{v}+\frac{v}{u}-\varphi$ is strictly increasing on $[1,+\infty)$. We state the corresponding results in the following lemmas. First, obviously, we have

LEMMA 2.3. *Assume $r\geqslant 3$. Then functions u,v are both smooth on $[1,+\infty)$. Moreover, u' is positive and v' is negative on $[1,+\infty)$.*

The following lemma concerns some important inequalities for u and v.

LEMMA 2.4. *Assume $r\geqslant 3$. Then the following inequalities hold:*

$$\frac{u(x)}{v(x)}x^{-\frac{1}{2}(\frac{1}{r-1}+\frac{1}{r+1})}\geqslant\frac{u(1)}{v(1)},\quad\forall x>1; \tag{2.7}$$

$$\left(x^{\frac{1}{r-1}}+\theta\right)\frac{u(x)}{v(x)}x^{-\frac{1}{r-1}-\frac{1}{r+1}} > (1+\theta)\frac{u(1)}{v(1)}, \quad \forall x>1;\ 0\leqslant\theta\leqslant\theta_0\equiv\frac{1}{r}; \tag{2.8}$$

$$\left(\left(\frac{1}{v(x)}-\rho\right)u(x)\right)' > 0, \quad \forall x>1;\ 0\leqslant\rho\leqslant\rho_0\equiv\frac{4r^2+2}{(2r+1)^2}. \tag{2.9}$$

Proof. Calculating directly, we have

$$\begin{aligned}
&\left(\left(x^{\frac{1}{r-1}}+\theta\right)x^{-\frac{1}{2}\left(\frac{1}{r-1}+\frac{1}{r+1}\right)}\right)' \\
&\quad=\frac{1}{r^2-1}\left(x^{\frac{1}{r^2-1}-1}-r\theta x^{-\frac{r}{r^2-1}-1}\right)>0, \qquad \forall x>1;\ \theta\leqslant\frac{1}{r}, && (2.10)\\
&\left(-v(x)+\theta x^{-\frac{1}{2}\left(\frac{1}{r-1}+\frac{1}{r+1}\right)}\right)' \\
&\quad=\frac{r}{r^2-1}\left(\frac{r-1}{r+1}x^{-\frac{1}{r+1}-1}-\theta x^{-\frac{r}{r^2-1}-1}\right)>0, \quad \forall x>1;\ \theta\leqslant\frac{r-1}{r+1}. && (2.11)
\end{aligned}$$

Let

$$\begin{aligned}
h(x)=&\left(u(x)x^{-\frac{1}{2}\left(\frac{1}{r-1}+\frac{1}{r+1}\right)}-\frac{u(1)}{v(1)}v(x)\right)\\
=&\frac{r}{r-1}\left(x^{\frac{1}{r-1}}+\frac{2r-2}{r(2r+1)}\right)x^{-\frac{1}{2}\left(\frac{1}{r-1}+\frac{1}{r+1}\right)}\\
&+\frac{u(1)}{v(1)}\left(-v(x)+\frac{r-1}{r+1}x^{-\frac{1}{2}\left(\frac{1}{r-1}+\frac{1}{r+1}\right)}\right).
\end{aligned}$$

By (2.10) and (2.11), h is increasing strictly on $[1,+\infty)$. Consequently, $h(x)>h(1)=0$ for any $x>1$, proving (2.7).

On the other hand, by (2.7) and (2.10), we get

$$\begin{aligned}
\left(x^{\frac{1}{r-1}}+\theta\right)\frac{u(x)}{v(x)}x^{-\frac{1}{r-1}-\frac{1}{r+1}} &\geqslant \left(x^{\frac{1}{r-1}}+\theta\right)\frac{u(1)}{v(1)}x^{-\frac{1}{2}\left(\frac{1}{r-1}+\frac{1}{r+1}\right)}\\
&>(1+\theta)\frac{u(1)}{v(1)}, \qquad \forall x>1;\ 0\leqslant\theta\leqslant\theta_0.
\end{aligned}$$

That is, (2.8) holds.

Finally, it holds that

$$\frac{r+1}{r-1}\frac{\delta}{\rho_0}\frac{u(1)}{v(1)}=\frac{4r^2-1}{4r^2+2}<1.$$

Therefore $\frac{\delta}{\rho_0}\frac{u(1)}{v(1)} < \frac{r-1}{r+1}$. By (2.7) and (2.11), we have

$$\begin{aligned}
&\left(\left(\frac{1}{v(x)}-\rho\right)u(x)\right)' \\
&= \left(\frac{1}{v(x)}-\rho\right)u'(x) - \frac{u(x)}{v^2(x)}v'(x) \\
&= \frac{u'(x)}{v(x)}\left(1-\rho v(x) - \frac{u(x)}{v(x)}\cdot\frac{v'(x)}{u'(x)}\right) \\
&= \frac{u'(x)}{v(x)}\left(1-\rho v(x) + \delta\frac{u(x)}{v(x)}x^{-\frac{1}{r-1}-\frac{1}{r+1}}\right) \\
&> \frac{u'(x)}{v(x)}\left(1-\rho_0 v(x) + \delta\frac{u(1)}{v(1)}x^{-\frac{1}{2}(\frac{1}{r-1}+\frac{1}{r+1})}\right) \\
&> \frac{u'(x)}{v(x)}\left(1-\rho_0 v(1) + \delta\frac{u(1)}{v(1)}\right) = 0, \qquad \forall x>1;\ \ 0\leqslant\rho\leqslant\rho_0.
\end{aligned}$$

Therefore, (2.9) holds. This proves Lemma 2.4. □

The following lemma is crucial in proving Theorem 1.4.

LEMMA 2.5. *Assume* $r\geqslant 3$. *Then*

$$\left(\frac{u(x)}{v(x)}+\frac{v(x)}{u(x)}-\varphi(x)\right)' > 0, \qquad \forall x>1. \tag{2.12}$$

Proof. We prove the lemma in two steps.

I. Rewrite φ as

$$\begin{aligned}
\varphi(x) =& \frac{xU(x)+V(x)-2}{x-1}-2 \\
=& U(x)+\frac{U(x)-U(1)}{x-1}+\frac{V(x)-V(1)}{x-1}-2 \\
=& U(x)+\int_0^1 U'(1+(x-1)s)\,ds+\int_0^1 V'(1+(x-1)s)\,ds-2, \qquad \forall x>1.
\end{aligned}$$

We have

$$\begin{aligned}
\varphi'(x) =& U'(x)+\int_0^1 sU''(1+(x-1)s)\,ds+\int_0^1 sV''(1+(x-1)s)\,ds \\
=& U'(x)+\int_0^1 sU''(1+(x-1)s)\,ds+\frac{V''(x)}{2} \\
&-\int_0^1 \frac{s^2(x-1)}{2}V'''(1+(x-1)s)\,ds.
\end{aligned}$$

Noting that $\sigma\in(0,1)$, we have

$$V'''(x) = (1-\sigma)\frac{r}{r+1}\left(\frac{r}{r+1}-1\right)\left(\frac{r}{r+1}-2\right)x^{\frac{r}{r+1}-3} > 0, \qquad \forall x>1.$$

On the othe hand, σ is in fact chosen to satisfy $U''(1)=0$. We have

$$\begin{aligned} U''(x) =& \frac{1}{r-1}\Big(\frac{1}{r-1}-1\Big)x^{\frac{1}{r-1}-2} - \frac{1}{r+1}\Big(-\frac{1}{r+1}-1\Big)\sigma x^{-\frac{1}{r+1}-2} \\ =& -\frac{(r-2)}{(r-1)^2}\Big(x^{\frac{1}{r-1}-2} - x^{-\frac{1}{r+1}-2}\Big) < 0, \qquad \forall x > 1. \end{aligned}$$

Consequently,

$$\begin{aligned} \psi(x) \equiv& \frac{r-1}{r}u'(x) + (1+B)v'(x) - \varphi'(x) \\ =& U'(x) + \frac{V''(x)}{2} - \varphi'(x) \\ =& -\int_0^1 sU''(1+(x-1)s)\,ds + \int_0^1 \frac{s^2(x-1)}{2}V'''(1+(x-1)s)\,ds \\ >& 0 = -\frac{1}{2}U''(1) = \psi(1), \qquad \forall x > 1. \end{aligned} \tag{2.13}$$

II. The proof of (2.12) is very sensitive to the estimates (2.7)–(2.9) and (2.13). We have

$$\begin{aligned} &\Big(\frac{u(x)}{v(x)} + \frac{v(x)}{u(x)} - \varphi(x)\Big)' \\ &= \Big(\frac{1}{v(x)} - \frac{v(x)}{u^2(x)}\Big)u'(x) + \Big(\frac{1}{u(x)} - \frac{u(x)}{v^2(x)}\Big)v'(x) - \varphi'(x) \\ &= \Big(\frac{1}{v(x)} - \frac{r-1}{r} - \frac{v(x)}{u^2(x)}\Big)u'(x) \\ &\quad + \Big(\frac{1}{u(x)} - \frac{u(x)}{v^2(x)} - (1+B)\Big)v'(x) + \psi(x) \\ &= \frac{v(x)u'(x)}{u^2(x)}\Big[\frac{u^2(x)}{v(x)}\Big(\frac{1}{v(x)} - \frac{r-1}{r}\Big) - 1 \\ &\quad + \delta\Big(-\frac{u(x)}{v(x)} + \frac{u^3(x)}{v^3(x)} + (1+B)\frac{u^2(x)}{v(x)}\Big)x^{-\frac{1}{r-1}-\frac{1}{r+1}} + \frac{u^2(x)}{v(x)u'(x)}\psi(x)\Big] \\ &\equiv \frac{v(x)u'(x)}{u^2(x)}\Phi(x), \qquad x \geqslant 1. \end{aligned}$$

We have

$$\begin{aligned} \Phi(x) =& \frac{u^2(x)}{v(x)u'(x)}\psi(x) - 1 \\ &+ \frac{u(x)}{v(x)}\Big(\Big(\frac{1}{v(x)} - \frac{r-1}{r}\Big)u(x) + \delta\frac{u^2(x)}{v^2(x)}x^{-\frac{1}{r-1}-\frac{1}{r+1}}\Big) \\ &+ \delta\Big(\frac{r(1+B)}{r-1}x^{\frac{1}{r-1}} + B\Big)\frac{u(x)}{v(x)}x^{-\frac{1}{r-1}-\frac{1}{r+1}} \\ \equiv& \Phi_1(x) + \Phi_2(x) + \Phi_3(x), \qquad x \geqslant 1. \end{aligned}$$

By (2.13),

$$\Phi_1(x) \geqslant -1 = \Phi_1(1), \qquad x > 1. \tag{2.14}$$

We have

$$\frac{r}{r-1}\rho_0 = \frac{r}{r-1}\frac{4r^2+2}{(2r+1)^2} = \frac{4r^3+2r}{4r^3-3r-1} > 1.$$

Thus, $\frac{r-1}{r} < \rho_0$. By (2.9) and (2.7) of Lemma 2.4,

$$\begin{aligned}
&\Big(\frac{1}{v(x)} - \frac{r-1}{r}\Big)u(x) + \delta\frac{u^2(x)}{v^2(x)}x^{-\frac{1}{r-1}-\frac{1}{r+1}} \\
&> \Big(\frac{1}{v(1)} - \frac{r-1}{r}\Big)u(1) + \delta\frac{u^2(1)}{v^2(1)} \\
&= \Big(\frac{r+1}{r-1} - \frac{2r+1}{r} + \frac{2r-1}{2r+1}\Big)\frac{2r-1}{2r+1} \\
&= \Big(\frac{2}{r-1} - \frac{1}{r} - \frac{2}{2r+1}\Big)\frac{2r-1}{2r+1} > 0, \qquad \forall x > 1.
\end{aligned}$$

This implies

$$\begin{aligned}
\Phi_2(x) >& \frac{u(x)}{v(x)}\Big(\Big(\frac{1}{v(1)} - \frac{r-1}{r}\Big)u(1) + \delta\frac{u^2(1)}{v^2(1)}\Big) \\
>& \frac{u(1)}{v(1)}\Big(\Big(\frac{1}{v(1)} - \frac{r-1}{r}\Big)u(1) + \delta\frac{u^2(1)}{v^2(1)}\Big) = \Phi_2(1), \qquad \forall x > 1.
\end{aligned} \tag{2.15}$$

Now, consider the last term Φ_3. We have

$$\begin{aligned}
\frac{1}{\theta_0}\frac{r-1}{r}\frac{B}{1+B} &= r\frac{r-1}{r}\frac{r^2-2r-1}{r^3-r^2-r-1} \\
&= \frac{r^3-3r^2+r+1}{r^3-r^2-r-1} < 1.
\end{aligned}$$

Therefore, $\frac{(r-1)B}{r(1+B)} < \theta_0$. Then, it follows from (2.8) of Lemma 2.4 that

$$\Phi_3(x) > \Phi_3(1), \qquad \forall x > 1. \tag{2.16}$$

Combining (2.14)–(2.16) and noting that

$$\begin{aligned}
\varphi'(1) &= \frac{1}{2}\big((x-1)\varphi(x)\big)''\big|_{x=1} = \frac{1}{2}\big(x^{\frac{r}{r-1}} + x^{\frac{r}{r+1}} - 2x\big)''\big|_{x=1} \\
&= \frac{2r^2}{(r+1)^2(r-1)^2},
\end{aligned}$$

we finally get

$$\begin{aligned}\Phi(x) >\Phi(1) &= \frac{u^2(x)}{v(x)u'(x)}\Big(\frac{u(x)}{v(x)}+\frac{v(x)}{u(x)}-\varphi(x)\Big)'\Big|_{x=1}\\ &=\frac{u^2(1)}{v(1)u'(1)}\Big(\frac{2r(2r^2+1)}{(2r+1)^2(r-1)^2}-\frac{2r(2r^2+1)}{(2r-1)^2(r+1)^2}-\frac{2r^2}{(r+1)^2(r-1)^2}\Big)\\ &=\frac{u^2(1)}{v(1)u'(1)}\Big(\frac{8r^2(2r^2-1)(2r^2+1)}{(2r-1)^2(2r+1)^2(r-1)^2(r+1)^2}-\frac{2r^2}{(r+1)^2(r-1)^2}\Big)\\ &=\frac{u^2(1)}{v(1)u'(1)}\cdot\frac{2r^2(8r^2-5)}{(2r-1)^2(2r+1)^2(r-1)^2(r+1)^2}>0, \qquad \forall x>1.\end{aligned}$$

And (2.12) follows. $\square$

3. Proof of Theorem 1.3

For clarity, we denote $X \propto Y$ if $\operatorname{sgn}(X)=\operatorname{sgn}(Y)$ in the following.
First, we have

LEMMA 3.1. *Theorem 1.3 holds for $r>0$ if and only if it holds for $r \geqslant 1$.*

Proof. By the formulation of Theorem 1.3 with hyperbolic cosine, assume inequality (2.4) to hold for $r=r_0$ with $r_0>1$. Then we have

$$\Big(\frac{\cosh r_0 t}{\cosh t}\Big)^{\frac{1}{r_0-1}}+\Big(\frac{\cosh t}{\cosh r_0 t}\Big)^{\frac{1}{r_0+1}}>2\cosh t, \qquad \forall t>0.$$

This is equivalent to

$$\Big(\frac{\cosh t}{\cosh \frac{t}{r_0}}\Big)^{\frac{1}{r_0-1}}+\Big(\frac{\cosh \frac{t}{r_0}}{\cosh t}\Big)^{\frac{1}{r_0+1}}>2\cosh \frac{t}{r_0}, \qquad \forall t>0.$$

Then

$$\begin{aligned}&\Big[\Big(\frac{\cosh \frac{t}{r_0}}{\cosh t}\Big)^{\frac{1}{\frac{1}{r_0}-1}}+\Big(\frac{\cosh t}{\cosh \frac{t}{r_0}}\Big)^{\frac{1}{\frac{1}{r_0}+1}}-2\cosh t\Big]\\ &=\Big[\Big(\frac{\cosh \frac{t}{r_0}}{\cosh t}\Big)^{-1-\frac{1}{r_0-1}}+\Big(\frac{\cosh t}{\cosh \frac{t}{r_0}}\Big)^{1-\frac{1}{r_0+1}}-2\cosh t\Big]\\ &=\frac{\cosh t}{\cosh \frac{t}{r_0}}\Big[\Big(\frac{\cosh t}{\cosh \frac{t}{r_0}}\Big)^{\frac{1}{r_0-1}}+\Big(\frac{\cosh \frac{t}{r_0}}{\cosh t}\Big)^{\frac{1}{r_0+1}}-2\cosh \frac{t}{r_0}\Big]>0, \qquad \forall t>0.\end{aligned}$$

This shows (2.4) holds for $r=\frac{1}{r_0}$. This proves Lemma 3.1. $\square$

Proof of Theorem 1.3. By Lemma 3.1, we can suppose that $r \geqslant 1$ without loss of generality. Define

$$\begin{aligned} F_0(t) &= \left(\frac{\cosh rt}{\cosh t}\right)^{\frac{1}{r+1}} \left[\left(\frac{\cosh rt}{\cosh t}\right)^{\frac{1}{r-1}} + \left(\frac{\cosh t}{\cosh rt}\right)^{\frac{1}{r+1}} - 2\cosh t\right] \\ &= \frac{\cosh^{\frac{2r}{r^2-1}} rt}{\cosh^{\frac{2r}{r^2-1}} t} + 1 - 2\cosh^{\frac{1}{r+1}} rt \cosh^{\frac{r}{r+1}} t, \qquad t \geqslant 0. \end{aligned}$$

Noting that $X - Y \propto \ln \frac{X}{Y}$ for positive numbers X and Y, we have

$$\begin{aligned} F_0'(t) &= \frac{2r}{r^2-1} \frac{\cosh^{\frac{2r}{r^2-1}} rt}{\cosh^{\frac{2r}{r^2-1}} t} \big(r\tanh rt - \tanh t\big) \\ &\quad - \frac{2r}{r+1} \cosh^{\frac{1}{r+1}} rt \cosh^{\frac{r}{r+1}} t \big(\tanh rt + \tanh t\big) \\ &\propto F(t), \qquad t > 0, \end{aligned}$$

where

$$F(t) = \ln\left[\frac{1}{r-1} \frac{\cosh^{\frac{1}{r-1}} rt}{\cosh^{\frac{r}{r-1}} t} \frac{\big(r\tanh rt - \tanh t\big)}{\big(\tanh rt + \tanh t\big)}\right], \qquad t > 0.$$

We have

$$\begin{aligned} F'(t) &= \frac{r}{r-1}(\tanh rt - \tanh t) - \frac{\frac{r}{\cosh^2 rt} + \frac{1}{\cosh^2 t}}{\tanh rt + \tanh t} + \frac{\frac{r^2}{\cosh^2 rt} - \frac{1}{\cosh^2 t}}{r\tanh rt - \tanh t} \\ &= \frac{1}{r-1} \frac{\frac{1}{\cosh^2 t} - \frac{r^2}{\cosh^2 rt}}{\tanh rt + \tanh t} + \frac{\frac{r^2}{\cosh^2 rt} - \frac{1}{\cosh^2 t}}{r\tanh rt - \tanh t} \\ &= \frac{(r\tanh t - \tanh rt)\left(\frac{r^2}{\cosh^2 rt} - \frac{1}{\cosh^2 t}\right)}{(r-1)(r\tanh rt - \tanh t)(\tanh rt + \tanh t)}, \qquad t > 0. \end{aligned} \tag{3.1}$$

It is lucky to have (3.1) which by Lemma 2.1(ii) implies that

$$F'(t) \propto \frac{r\cosh t - \cosh rt}{r-1}, \qquad t > 0.$$

Then, by Lemma 2.1(iv), there exists a $t_r > 0$ such that $F'(t)$ is positive in $(0, t_r)$ and negative in $(t_r, +\infty)$. Combining this fact with $F(0^+) = F(+\infty) = 0$, we get that $F(t)$ is positive in $(0, +\infty)$. Consequently $F_0'(t)$ is positive in $(0, +\infty)$. Then (2.4) follows from $F_0(0) = 0$. Therefore, Theorem 1.3 holds. □

REMARK 1. In the above proof, r can be equal to 1. That is, we need only to look at the corresponding limits of the functions that appeared in the above proof. More precisely, for $r = 1$, it holds that

$$F_0(t) = e^{t\tanh t} + 1 - 2\cosh t, \qquad \forall t \geqslant 0.$$

$$F_0'(t) = e^{t\tanh t}\left(\tanh t + t\cosh^{-2} t\right) - 2\sinh t, \qquad \forall t > 0,$$

$$F(t) = \ln\left(e^{t\tanh t}\left(\tanh t + t\cosh^{-2} t\right)\right) - \ln\left(2\sinh t\right), \qquad \forall t > 0.$$

$$F'(t) = \frac{2(1 - t\tanh t)(\sinh 2t - 2t)}{\sinh 2t(\sinh 2t + 2t)} \propto \cosh t - t\sinh t, \qquad \forall t > 0.$$

4. Proof of Theorem 1.4

We turn to prove Theorem 1.4. If we use the approach used to prove Theorem 1.3, it will be very complex since we do not have luck to establish an equality like (3.1) in this case. Thus, we seek for another approach.

Similar to Lemma 3.1, we have

LEMMA 4.1. *Theorem 1.4 holds for $r > 0$ if and only if it holds for $r \geqslant 1$.*

Proof. By the formulation of Theorem 1.4 with hyperbolic cosine, assume inequality (2.5) to hold for $r = r_0$ with $r_0 > 1$. Then

$$\left(\frac{\cosh r_0 t}{\cosh t}\right)^{\frac{2}{r_0-1}} + \left(\frac{\cosh t}{\cosh r_0 t}\right)^{\frac{2}{r_0+1}} < 2\cosh 2t, \qquad \forall t > 0.$$

This is equivalent to

$$\left(\frac{\cosh t}{\cosh \frac{t}{r_0}}\right)^{\frac{2}{r_0-1}} + \left(\frac{\cosh \frac{t}{r_0}}{\cosh t}\right)^{\frac{2}{r_0+1}} < 2\cosh \frac{2t}{r_0}, \qquad \forall t > 0.$$

By Lemma 2.1(iii), $\frac{\cosh^2 t}{\cosh 2t}$ is decreasing strictly on $[0, +\infty)$. Thus,

$$\frac{\cosh^2 \frac{t}{r_0} \cosh 2t}{\cosh \frac{2t}{r_0} \cosh^2 t} > 1, \qquad \forall t > 1.$$

Consequently,

$$\begin{aligned}
&\left[\left(\frac{\cosh \frac{t}{r_0}}{\cosh t}\right)^{\frac{2}{\frac{1}{r_0}-1}} + \left(\frac{\cosh t}{\cosh \frac{t}{r_0}}\right)^{\frac{2}{\frac{1}{r_0}+1}} - 2\cosh 2t\right] \\
&= \left[\left(\frac{\cosh \frac{t}{r_0}}{\cosh t}\right)^{-2-\frac{2}{r_0-1}} + \left(\frac{\cosh t}{\cosh \frac{t}{r_0}}\right)^{2-\frac{2}{r_0+1}} - 2\cosh 2t\right] \\
&= \frac{\cosh^2 t}{\cosh^2 \frac{t}{r_0}}\left[\left(\frac{\cosh t}{\cosh \frac{t}{r_0}}\right)^{\frac{2}{r_0-1}} + \left(\frac{\cosh \frac{t}{r_0}}{\cosh t}\right)^{\frac{2}{r_0+1}} - \frac{\cosh^2 \frac{t}{r_0}\cosh 2t}{\cosh \frac{2t}{r_0}\cosh^2 t}\cdot 2\cosh \frac{2t}{r_0}\right] \\
&< \frac{\cosh^2 t}{\cosh^2 \frac{t}{r_0}}\left[\left(\frac{\cosh t}{\cosh \frac{t}{r_0}}\right)^{\frac{2}{r_0-1}} + \left(\frac{\cosh \frac{t}{r_0}}{\cosh t}\right)^{\frac{2}{r_0+1}} - 2\cosh \frac{2t}{r_0}\right] < 0, \qquad \forall t > 0.
\end{aligned}$$

This shows (2.5) holds for $r = \frac{1}{r_0}$. This proves Lemma 4.1. $\square$

Proof of Theorem 1.4. Obviously, the theorem holds for $r = 0$. Then, to prove the theorem, by Lemma 4.1, it suffices to prove it for the cases of $r \geqslant 1$. We mention that the proof for Case 1 below was given in [5]. For the cases of $3 < r \leqslant 7$, Theorem 1.4 was also established there.

Case 1. $1 \leqslant r \leqslant 3$.
We have

$$\begin{aligned} C_3^2(a,b) + C_{-1}^2(a,b) &= \frac{a^3+b^3}{a+b} + \left(\frac{a^{-1}+b^{-1}}{a+b}\right)^{-1} \\ &= a^2 - ab + b^2 + ab = a^2 + b^2, \qquad a,b > 0. \end{aligned}$$

Thus, by Proposition 2.2(iii),

$$C_r^2(a,b) + C_{-r}^2(a,b) < C_3^2(a,b) + C_{-1}^2(a,b) = a^2 + b^2, \qquad b > a > 0.$$

This means that Theorem 1.4 holds in this case.

Case 2. $r \in (3, +\infty)$.
We have shown that Theorem 1.4 is equivalent to (2.6), that is

$$f\left(\frac{\cosh rt + 1}{\cosh t + 1}\right) < f(y), \qquad \forall t > 0, \tag{4.1}$$

where $y = y(t)$ is the implicit function of

$$f(y) = 2\cosh t, \qquad t \geqslant 0.$$

By Lemma 2.2, f is strictly increasing. Therefore, (4.1) is equivalent to

$$\frac{\cosh rt + 1}{\cosh t + 1} < y, \qquad \forall t > 0.$$

That is,

$$2\cosh rt < y(2\cosh t + 2) - 2 = y(f(y) + 2) - 2 = g(y), \qquad \forall t > 0. \tag{4.2}$$

Recalling from Lemma 2.2 that

$$t = \ln \frac{f(y) + \sqrt{f^2(y) - 4}}{2}, \qquad t \geqslant 0,$$

we see that (4.2) is equivalent to

$$r \ln \frac{f(y) + \sqrt{f^2(y) - 4}}{2} < \ln \frac{g(y) + \sqrt{g^2(y) - 4}}{2}, \qquad \forall t > 0.$$

Since $\{y(t)\big|t>0\}=(1,+\infty)$, what we need to prove is that $G(x)<0$ for any $x>1$ with

$$G(x)\equiv G(x;r)=r\ln\frac{f(x)+\sqrt{f^2(x)-4}}{2}-\ln\frac{g(x)+\sqrt{g^2(x)-4}}{2},\quad x\geqslant 1.$$

For positive numbers X,X_1,Y,Y_1, we have $\frac{Y}{X}-\frac{Y_1}{X_1}\propto\frac{X_1^2}{X^2}-\frac{Y_1^2}{Y^2}$ and $X-Y\propto\frac{1}{Y}-\frac{1}{X}$. Thus, noting that $f',g'>0$ and

$$\frac{g^2(x)-4}{x^2\big(f^2(x)-4\big)}=\frac{f(x)+2-4x^{-1}}{f(x)-2}=1+\frac{4}{\varphi(x)},$$

we have

$$\begin{aligned}
G'(x)=&\frac{rf'(x)}{\sqrt{f^2(x)-4}}-\frac{g'(x)}{\sqrt{g^2(x)-4}}\\
\propto&\frac{g^2(x)-4}{x^2\big(f^2(x)-4\big)}-\frac{\big(g'(x)\big)^2}{\big(rxf'(x)\big)^2}\\
=&1+\frac{4}{\varphi(x)}-\frac{\big(u(x)+v(x)\big)^2}{\big(u(x)-v(x)\big)^2}\\
=&4\left(\frac{1}{\varphi(x)}-\frac{u(x)v(x)}{\big(u(x)-v(x)\big)^2}\right)\\
\propto&\frac{\big(u(x)-v(x)\big)^2}{u(x)v(x)}-\varphi(x)\\
=&\frac{u(x)}{v(x)}+\frac{v(x)}{u(x)}-\varphi(x)-2.
\end{aligned}$$

By Lemma 2.5,

$$\Big(\frac{u(x)}{v(x)}+\frac{v(x)}{u(x)}-\varphi(x)-2\Big)'>0,\qquad\forall x>1.$$

While

$$\begin{aligned}
&\Big(\frac{u(x)}{v(x)}+\frac{v(x)}{u(x)}-\varphi(x)-2\Big)\Big|_{x=1}\\
&=\frac{\big(u(1)-v(1)\big)^2}{u(1)v(1)}-\varphi(1)\\
&=\frac{4r^2}{(4r^2-1)(r^2-1)}-\frac{2}{r^2-1}\\
&=-\frac{2(2r^2-1)}{(4r^2-1)(r^2-1)}<0.
\end{aligned}$$

On the other hand, it is easy to see that

$$\lim_{x\to+\infty} x^{-\frac{1}{r-1}}\left(\frac{u(x)}{v(x)}+\frac{v(x)}{u(x)}-\big(\varphi(x)+2\big)\right)=\frac{r}{r-1}+0-1=\frac{1}{r-1}.$$

Thus

$$\lim_{x\to+\infty}\left(\frac{u(x)}{v(x)}+\frac{v(x)}{u(x)}-\varphi(x)-2\right)=+\infty.$$

Therefore, there is an x_r in $(1,+\infty)$, such that $\frac{u}{v}+\frac{v}{u}-\varphi(x)-2$ is negative in $(1,x_r)$ and positive in $(x_r,+\infty)$. Consequently, G' is negative in $(1,x_r)$ and positive in $(x_r,+\infty)$. Since $G(1)=0$ and

$$\begin{aligned}\lim_{x\to+\infty} G(x) &= \lim_{x\to+\infty}\left(r\ln\frac{1}{2}\left(f(x)+\sqrt{f(x)^2-4}\right)-\ln\frac{1}{2}\left(g(x)+\sqrt{g(x)^2-4}\right)\right)\\ &= \lim_{x\to+\infty}\left(r\ln x^{\frac{1}{r-1}}-\ln x^{\frac{r}{r-1}}\right)=0,\end{aligned}$$

we get that G is negative in $(1,+\infty)$. Therefore, Theorem 1.4 holds in this case.

We get the proof. $\square$

5. Proofs of Theorems 1.1 and 1.2

It was proved in [5] that Theorem 1.3 implies Theorem 1.1, while Theorem 1.4 implies Theorem 1.2. More precisely, for fixed r, there is a $\beta_r>1$ such that

$$L_r(1,b)+L_{-r}(1,b)>2L_0(1,b), \qquad \forall b\in(1,\beta_r);\;\; r\in(0,+\infty], \tag{5.1}$$

$$L_r(1,b)+L_{-r}(1,b)<b+1, \qquad \forall b\in(1,\beta_r);\;\; r\in[0,+\infty). \tag{5.2}$$

Then, using (1.2), theorems 1.3 and 1.4, one can easily extend the inequalities (5.1) and (5.2) valid for all $b\in(1,\beta_r)$ to those valid for all $b\in(1,\beta_r^2)$. By induction, we can get (1.4) and (1.5), getting theorems 1.1 and 1.2.

It is easy to verify that

$$\lim_{b\to1}\frac{L_r(1,b)+L_{-r}(1,b)-2L_0(1,b)}{(b-1)^4}=\frac{r^2}{960}, \tag{5.3}$$

$$\lim_{b\to1}\frac{L_r(1,b)+L_{-r}(1,b)-1-b}{(b-1)^2}=-\frac{1}{6}. \tag{5.4}$$

Then, (5.1) and (5.2) hold. Nevertheless, it is a little complex to get (5.3). In order to avoid using (5.3), we will give a proof of Theorem 1.1 different from that in [5].

First, for real number $r\neq0$, we have

$$\frac{b^r-1}{r(b-1)}=1+\frac{r-1}{2}(b-1)+\frac{(r-1)(r-2)}{6}(b-1)^2+o\big((b-1)^2\big), \qquad b\to1.$$

Thus, for real number $r \neq 0, 1$, we have

$$\begin{aligned} L_r(1,b) =& 1 + \frac{1}{r-1}\Big(\frac{r-1}{2}(b-1) + \frac{(r-1)(r-2)}{6}(b-1)^2\Big) \\ &+ \frac{1}{2}\frac{1}{r-1}\Big(\frac{1}{r-1} - 1\Big)\Big(\frac{r-1}{2}(b-1)\Big)^2 + o\big((b-1)^2\big) \\ =& 1 + \frac{b-1}{2} + \frac{r-2}{24}(b-1)^2 + o\big((b-1)^2\big), \qquad b \to 1. \end{aligned} \tag{5.5}$$

Actually, it is easy to verify that (5.5) holds also for $r = 0$ and 1.

By (1.2), it is easy to get that

$$\begin{aligned} L_r(a,b) &= \lim_{n\to+\infty}\Big(L_r\big(a^{2^{-n}}, b^{2^{-n}}\big)\prod_{k=1}^{n} C_r\big(a^{2^{-k}}, b^{2^{-k}}\big)\Big) \\ &= \prod_{k=1}^{\infty} C_r\big(a^{2^{-k}}, b^{2^{-k}}\big), \qquad \forall a,b > 0;\ -\infty \leqslant r \leqslant +\infty. \end{aligned} \tag{5.6}$$

Proof of Theorem 1.1. Let $r \in (0, +\infty]$. We suppose that $b > a = 1$ without loss of generality.

By (2.1), (2.2), and Theorem 1.3,

$$L_r(1,b)\big(C_r(1,b) - C_0(1,b)\big) > L_{-r}(1,b)\big(C_0(1,b) - C_{-r}(1,b)\big), \qquad \forall b > 1.$$

Therefore, by (1.2), we have that,

$$\begin{aligned} &L_r(1,b) + L_{-r}(1,b) \\ &\quad = L_r(1,\sqrt{b})C_r(1,\sqrt{b}) + L_{-r}(1,\sqrt{b})C_{-r}(1,\sqrt{b}) \\ &\quad > \big(L_r(1,\sqrt{b}) + L_{-r}(1,\sqrt{b})\big)C_0(1,\sqrt{b}) > \cdots \\ &\quad > \big(L_r(1,b^{2^{-n}}) + L_{-r}(1,b^{2^{-n}})\big)\prod_{k=1}^{n} C_0\big(1,b^{2^{-k}}\big), \qquad \forall n \geqslant 1. \end{aligned}$$

Passing to the limit as $n \to +\infty$, and using (5.6), we get that

$$L_r(1,b) + L_{-r}(1,b) > 2\prod_{k=1}^{\infty} C_0\big(1,b^{2^{-k}}\big) = 2L_0(1,b).$$

Completing the proof. $\square$

Proof of Theorem 1.2. Let $r \in [0, +\infty)$. By (5.5), we have (5.4). Thus there exists a $\beta_r > 1$ such that,

$$L_r(1,b) + L_{-r}(1,b) < b + 1, \qquad \forall b \in (1, \beta_r). \tag{5.7}$$

Thus, by (2.1) and (2.2),

$$\big(b - L_r(1,b)\big)C_r(1,b) > \big(L_{-r}(1,b) - 1\big)C_{-r}(1,b), \qquad \forall b \in (1, \beta_r).$$

Consequently, by (1.2), Theorem 1.4, and the inequality $st \leqslant \frac{1}{2}\left(s^2+t^2\right)$ for reals s,t in the penultimate line, we have

$$\begin{aligned}
&L_r(1,b^2)+L_{-r}(1,b^2)\\
&\quad = L_r(1,b)C_r(1,b)+L_{-r}(1,b)C_{-r}(1,b)\\
&\quad < bC_r(1,b)+C_{-r}(1,b)\\
&\quad \leqslant \frac{1}{2}\left(b^2+C_r^2(1,b)+1+C_{-r}^2(1,b)\right)\\
&\quad < b^2+1, \qquad \forall b \in (1,\beta_r).
\end{aligned}$$

Thus,

$$L_r(1,b)+L_{-r}(1,b) < b+1, \qquad \forall b \in (1,\beta_r^2).$$

Therefore, we have extended the inequality (5.7) valid for all $b \in (1,\beta_r)$ to one valid for all $b \in (1,\beta_r^2)$. By induction, we can get (1.5), completing the proof. $\square$

6. Further results

In this section, we will give some corollaries. From observations on the first two pages of the present paper we have

$$\begin{aligned}
ab =& C_{+\infty}(a,b)C_{-\infty}(a,b) = C_{-1}^2(a,b)\\
=& L_{+\infty}(a,b)L_{-\infty}(a,b) = L_{-1}^2(a,b).
\end{aligned}$$

Similar to Proposition 1.1, we have

THEOREM 6.1. *Let* $0<s<r<+\infty$, $b>a>0$. *Then*

$$ab < C_r(a,b)C_{-r}(a,b) < C_s(a,b)C_{-s}(a,b) < \left(\frac{a+b}{2}\right)^2. \tag{6.1}$$

Proof. We can suppose that $b>a=1$. By (2.3),

$$\begin{aligned}
&\frac{\partial}{\partial r}\left[\ln\left(C_r(1,b)C_{-r}(1,b)\right)\right]\\
&\quad = \int_0^1 \left(\frac{b^{1+t(r-1)}}{(b^{1+t(r-1)}+1)^2} - \frac{b^{1-t(r+1)}}{(b^{1-t(r+1)}+1)^2}\right) t\ln^2 b\,dt\\
&\quad = \int_0^1 \left(\frac{1}{(b^{\frac{1+t(r-1)}{2}}+b^{-\frac{1+t(r-1)}{2}})^2} - \frac{1}{(b^{\frac{1-t(r+1)}{2}}+b^{-\frac{1-t(r+1)}{2}})^2}\right) t\ln^2 b\,dt\\
&\quad < 0, \qquad \forall r>0.
\end{aligned}$$

Therefore, $\ln\left(C_r(1,b)C_{-r}(1,b)\right)$ is strictly decreasing on $r \in [0,+\infty)$. This implies (6.1). $\square$

Proposition 1.1 can be generalized as the following theorem.

THEOREM 6.2. *Let* $0<s<r<+\infty$, $b>a>0$. *Then*

$$ab<L_r(a,b)L_{-r}(a,b)<L_s(a,b)L_{-s}(a,b)<L_0^2(a,b). \tag{6.2}$$

Proof. We need only to prove the middle inequality. Let $0<s<r<+\infty$, $b>a=1$. By (5.6) and (6.1), we have

$$\begin{aligned} L_r(1,b)L_{-r}(1,b) &= \prod_{k=1}^{\infty}\Big(C_r\big(1,b^{2^{-k}}\big)C_{-r}\big(1,b^{2^{-k}}\big)\Big) \\ &< \prod_{k=1}^{\infty}\Big(C_s\big(1,b^{2^{-k}}\big)C_{-s}\big(1,b^{2^{-k}}\big)\Big)=L_s(1,b)L_{-s}(1,b). \end{aligned}$$

Completing the proof. □

Concerning $L_r^{\alpha}+L_{-r}^{\alpha}$ and $C_r^{\alpha}+C_{-r}^{\alpha}$ for other cases of α, we have

THEOREM 6.3. *Let* $0<r<+\infty$, $0<\alpha\leqslant 1\leqslant\beta<+\infty$ *and* $b>a>0$. *Then*

$$a^{\alpha}+b^{\alpha}<C_r^{\alpha}(a,b)+C_{-r}^{\alpha}(a,b)<2\Big(\frac{3a^2+2ab+3b^2}{8}\Big)^{\frac{\alpha}{2}}, \tag{6.3}$$

$$2C_0^{\beta}(a,b)<C_r^{\beta}(a,b)+C_{-r}^{\beta}(a,b), \tag{6.4}$$

$$C_r^{2\beta}(a,b)+C_{-r}^{2\beta}(a,b)<a^{2\beta}+b^{2\beta}, \tag{6.5}$$

$$2L_0^{\beta}(a,b)<L_r^{\beta}(a,b)+L_{-r}^{\beta}(a,b)<a^{\beta}+b^{\beta}. \tag{6.6}$$

Proof. By properties of convex/concave functions, we have

$$a^{\alpha}+(x+y-a)^{\alpha}\leqslant x^{\alpha}+y^{\alpha}\leqslant 2\big(\frac{x+y}{2}\big)^{\alpha},\qquad \forall a\leqslant x\leqslant y, \tag{6.7}$$

$$a^{\beta}+(x+y-a)^{\beta}\geqslant x^{\beta}+y^{\beta}\geqslant 2\big(\frac{x+y}{2}\big)^{\beta},\qquad \forall a\leqslant x\leqslant y. \tag{6.8}$$

Thus, using (1.6), (1.7), (6.1), (6.7) and (6.8), we get

$$\begin{aligned} a^{\alpha}+b^{\alpha} &< a^{\alpha}+(C_r(a,b)+C_{-r}(a,b)-a)^{\alpha} && \text{(by (1.6))} \\ &\leqslant C_r^{\alpha}(a,b)+C_{-r}^{\alpha}(a,b) && \text{(by (6.7))} \\ &\leqslant 2\Big(\frac{C_r(a,b)+C_{-r}(a,b)}{2}\Big)^{\alpha} && \text{(by (6.7))} \\ &= 2\Big(\frac{C_r^2(a,b)+C_{-r}^2(a,b)+2C_r(a,b)C_{-r}(a,b)}{4}\Big)^{\frac{\alpha}{2}} \\ &< 2\Big(\frac{a^2+b^2+2\big(\frac{a+b}{2}\big)^2}{4}\Big)^{\frac{\alpha}{2}}, && \text{(by (1.7), (6.1))} \\ &= 2\Big(\frac{3a^2+2ab+3b^2}{8}\Big)^{\frac{\alpha}{2}}, \end{aligned}$$

$$
\begin{aligned}
2C_0^\beta(a,b) &< 2\left(\frac{C_r(a,b)+C_{-r}(a,b)}{2}\right)^\beta && \text{(by (1.6))}\\
&\leqslant C_r^\beta(a,b)+C_{-r}^\beta(a,b) && \text{(by (6.8))}
\end{aligned}
$$

and

$$
\begin{aligned}
&C_r^{2\beta}(a,b)+C_{-r}^{2\beta}(a,b)\\
&\quad< a^{2\beta}+\left(C_r^2(a,b)+C_{-r}^2(a,b)-a^2\right)^\beta && \text{(by (6.8))}\\
&\quad\leqslant a^{2\beta}+b^{2\beta}. && \text{(by (1.7))}
\end{aligned}
$$

Using (1.4), (1.5) and (6.8), we have

$$
\begin{aligned}
2L_0^\beta(a,b) &< 2\left(\frac{L_r(a,b)+L_{-r}(a,b)}{2}\right)^\beta && \text{(by (1.4))}\\
&\leqslant L_r^\beta(a,b)+L_{-r}^\beta(a,b) && \text{(by (6.8))}\\
&< a^\beta+\left(L_r(a,b)+L_{-r}(a,b)-a\right)^\beta && \text{(by (6.8))}\\
&< a^\beta+b^\beta. && \text{(by (1.5))}
\end{aligned}
$$

The proof is completed. □

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