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MONOTONICITY OF RATIOS OF BERNOULLI POLYNOMIALS

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Abstract. Monotonicity patterns of the ratios B_k/B_{k+2} of Bernoulli polynomials are determined.

The Bernoulli polynomials $B_k(x)$ may be defined by the formula

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!} \tag{1}$$

for real x and $|t|<2\pi$; see e.g. the first displayed formula in [3].

These polynomials arise, along with the Bernoulli numbers, in studies of various special functions and, in particular, with the Riemann zeta function and the Hurwitz zeta function, as well as in the Euler–Maclaurin formula; see e.g. Section D.2 in [1].

Take any nonnegative integer k and any positive integer n. It is known – see e.g. [3, p. 125] that

$$B_{2n+1}(x)\neq 0 \quad\text{for}\quad x\in(0,1/2) \tag{2}$$

and

$$\text{there exists a unique } x_{2n}\in(0,1/2) \text{ such that } B_{2n}(x_{2n})=0. \tag{3}$$

Consider the ratio

$$r_k:=\frac{B_k}{B_{k+2}}. \tag{4}$$

It follows from (2) and (3) that this ratio is defined on the entire interval $(0,1/2)$ if k is odd and on the union $(0,x_{k+2})\cup(x_{k+2},1/2)$ of two intervals if k is even.

Now we can state the main result of this note:

THEOREM 1. *Take any nonnegative integer k.*

$$\begin{aligned}&\textit{If } k \textit{ is odd, then } r_k \textit{ is increasing on } (0,1/2).\\&\textit{If } k \textit{ is even, then } r_k \textit{ is increasing on } (0,x_{k+2}) \textit{ and on } (x_{k+2},1/2).\end{aligned} \tag{†}$$

Recall the identity $B_k(1-x)=(-1)^kB_k(x)$ for real x, which easily follows from, say, definition (1). We now see that $r_k(1-x)=r_k(x)$ for $x\in(0,1)$. So, we have

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COROLLARY 2. *Take any nonnegative integer k.*

If k is odd, then r_k is decreasing on $(1/2,1)$.
If k is even, then r_k is decreasing on $(1/2,1-x_{k+2})$ and on $(1-x_{k+2},1)$.

Theorem 1 and Corollary 2 are illustrated below by the graphs $\{(x,r_k(x))\colon 0<x<1\}$ for $k=3$ (left) and $k=4$ (right, with only part of the graph shown, because of the infinite discontinuities of r_4 at $x_{k+2}=x_6=0.247\ldots$ and $1-x_{k+2}=1-x_6$):

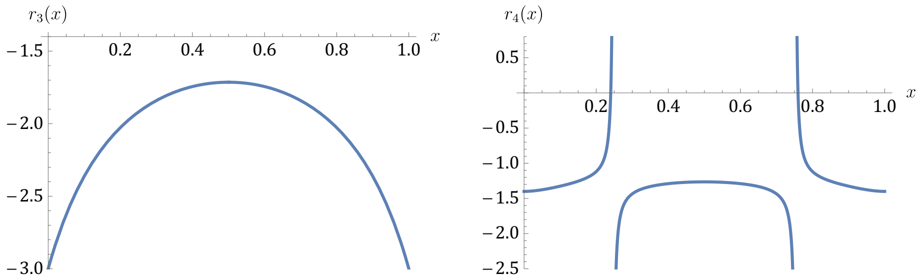

Figure 1: *Graphs of r_3 (left) and r_4 (right).*

Proof of Theorem 1. First, let us recall some additional facts regarding the Bernoulli polynomials $B_k(x)$ and the Bernoulli numbers (which will be denoted here by b_k):

$$B_k(x)=\sum_{j=0}^{k}\binom{k}{j}b_{k-j}x^j \tag{5}$$

– see e.g. Exercise 44(c) in Ch. 1 of [1]; for $n\geqslant 1$,

$$b_{2n+1}=0 \quad \text{and} \quad b_{2n}=\frac{(-1)^{n+1}2(2n)!}{(2\pi)^{2n}}\zeta(2n) \tag{6}$$

– see e.g. [2, formula (12.38)] or [1, Theorem 1.2.4], so that

$$b_{2n}\overset{\text{sign}}{=}(-1)^{n+1}; \tag{7}$$

$$B_k'=kB_{k-1} \tag{8}$$

for $k\geqslant 1$ – see e.g. [3, formulas (1.1) and (1.5)]. Everywhere here, by default k and n are any nonnegative integers, unless specified otherwise; and $\overset{\text{sign}}{=}$ denotes the equality in sign.

Moreover – see e.g. [3, p. 125], for $n\geqslant 1$,

$$B_{2n+1}(0)=B_{2n+1}(1/2)=0. \tag{9}$$

It follows from (8), (5), and (7) that for $n \geqslant 1$ we have $B'_{2n+1}(0) = (2n+1)B_{2n}(0) = (2n+1)b_{2n} \overset{\text{sign}}{=} (-1)^{n+1}$ and hence, by (2) and (9),

$$B_{2n+1}(x) \overset{\text{sign}}{=} (-1)^{n+1} \text{ for } x \in (0,1/2); \tag{10}$$

this holds for for $n = 0$ as well, since $B_1(x) = x - 1/2$.

Next, by (8) and (10), for $n \geqslant 1$ we have $B'_{2n} \overset{\text{sign}}{=} B_{2n-1} \overset{\text{sign}}{=} (-1)^n$ (on $(0,1/2)$). Also, $B_{2n}B'_{2n} > 0$ on $(x_{2n},1)$ and $B_{2n}B'_{2n} < 0$ on $(0,x_{2n})$. So,

$$B_{2n} \overset{\text{sign}}{=} \begin{cases} (-1)^{n+1} & \text{on } (0,x_{2n}), \\ (-1)^n & \text{on } (x_{2n},1). \end{cases} \tag{11}$$

Moreover, it was shown in [4] that $x_{2n} < x_{2n+2}$ for all $n \geqslant 1$. It follows that

$$r_{2n} \begin{cases} > 0 & \text{on } (x_{2n},x_{2n+2}) \neq \emptyset, \\ < 0 & \text{on } (x_{2n+2},1) \end{cases}$$

and hence

$$r_{2n} \to \begin{cases} \infty & \text{as } x \uparrow x_{2n+2}, \\ -\infty & \text{as } x \downarrow x_{2n+2}. \end{cases} \tag{12}$$

After these preliminaries, the proof of (†) will be done by induction on k. The induction base, for $k = 0$ and $k = 1$, is checked easily, since $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = 1/6 - x + x^2$, and $B_3(x) = x/2 - 3x^2/2 + x^3$.

Suppose now that the statement (†) of Theorem 1 holds for some integer $k \geqslant 1$. We then have to show that (†) holds with $k+1$ in place of k.

This will be done using so-called l'Hospital-type rules for monotonicity [5], as follows. Consider the "derivative ratio" for r_{k+1} (cf. (4)): by (8),

$$\rho_{k+1} := \frac{B'_{k+1}}{B'_{k+3}} = \frac{k+1}{k+3}\frac{B_k}{B_{k+2}} = \frac{k+1}{k+3} r_k,$$

so that the monotonicity pattern of ρ_{k+1} is the same as that of r_k:

$$\begin{aligned} &\text{If } k \text{ is odd, then } \rho_{k+1} \text{ is increasing on } (0,1/2). \\ &\text{If } k \text{ is even, then } \rho_{k+1} \text{ is increasing on } (0,x_{k+2}) \text{ and on } (x_{k+2},1/2). \end{aligned} \tag{13}$$

Consider first the case when $k = 2n$, even. Note that the condition $k \geqslant 1$ implies in this case that $n \geqslant 1$. Then, by [5, Proposition 4.1] – used with $f = B_{k+1} = B_{2n+1}$ and $g = B_{k+3} = B_{2n+3}$, (13), and (9), we see that (†) implies that $r'_{k+1} > 0$ on $(0,x_{k+2})$ and on $(x_{k+2},1/2)$, so that the induction step is completed in this case.

Consider now the case when $k = 2n-1 \geqslant 1$, odd. Then it is not true that $B_{k+1}(0) = 0$ or $B_{k+1}(1/2) = 0$. So, in this case, we have to use so-called general l'Hospital-type rules for monotonicity. More specifically, we will use [5, Table 1.1], with $f = B_{k+1} = B_{2n}$ and $g = B_{k+3} = B_{2n+2}$, so that $r_{k+1} = f/g$ and, by (8) and (10), $g' \overset{\text{sign}}{=}$

$B_{2n+1} \overset{\text{sign}}{=} (-1)^{n+1}$ on the entire interval $(0,1/2)$. So, by (3), $gg' \overset{\text{sign}}{=} B_{2n+2}B_{2n+1} \overset{\text{sign}}{=} B_{2n+2}(-1)^{n+1}$, which, by (11), is <0 on the interval $(0,x_{2n+2}) = (0,x_{k+3})$ and >0 on the interval $(x_{2n+2},1/2) = (x_{k+3},1/2)$. Hence, by lines 3 and 1 of mentioned [5, Table 1.1] and (13), we get that $r_{k+1} = r_{2n}$ is

- up-down on $(0,x_{k+3})$ (that is, for some $c_1 \in [0,x_{k+3}]$ the function $r_{k+1} = r_{2n}$ is increasing on $(0,c_1)$ and decreasing on (c_1,x_{k+3});
- down-up on $(x_{k+3},1/2)$ (that is, for some $c_2 \in [x_{k+3},1/2]$ the function $r_{k+1} = r_{2n}$ is decreasing on (x_{k+3},c_2) and increasing on $(c_2,1/2)$.

However, in view of (12), r_{2n} cannot be decreasing in any left neighborhood of $x_{k+3} = x_{2n+2}$, and r_{2n} cannot be decreasing in any right neighborhood of $x_{k+3} = x_{2n+2}$. So, in the above "up-down" and "down-up" items, $c_1 = x_{k+3} = c_2$; that is, $r_{k+1} = r_{2n}$ is increasing on $(0,x_{k+3})$ and on $(x_{k+3},1/2)$.

Thus, whether k is even or odd, (†) holds with $k+1$ in place of k, which completes the induction step. □

REMARK 3. The above proof of Theorem 1 is an edited version of the answer [6] given by the author on 4 January 2024 on MathOverflow, a question-and-answer site for professional mathematicians. After this paper was submitted on 8 February 2024, the author learned on 25 March 2024 from the answer [7] by MathOverflow user qifeng618 about Proposition 1 in the accepted paper [9], which considered the case (†) in Theorem 1 for odd k. Further, from the comment [8], again by user qifeng618, on 10 May 2024, the author learned of further related results [10] dealing also with ratios $B_{2n}(t)/B_{2n+1}(t)$, $B_{2m}(t)/B_{2n}(t)$, and $B_{2n}(t)/B_{2n-1}(t)$ for natural m and n.

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