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IMPROVED L^p–L^q HARDY INEQUALITIES

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Abstract. In this note, we obtain a new version of the Hardy inequality which covers the recent inequality of Frank, Laptev, and Weidl derived in [2] and improves the result of Persson and Samko established in [8]. It gives new results in one dimension. We analyse radial and non-radial multidimensional versions of the considered inequality as consequences.

1. Introduction

In 1925, G. H. Hardy described and proved the following integral inequality [4]

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1}$$

which holds for $f(x) \geqslant 0$, $p > 1$, and where f^p is integrable over $(0,\infty)$. The inequality (1) implies that the Hardy operator $\mathscr{H}f(x) = \frac{1}{x}\int_0^x f(t)dt$ is bounded in $L^p(0,\infty)$ with norm $\|\mathscr{H}\|_{L^p\to L^p} \leqslant \frac{p}{p-1}$, $p > 1$. Also, it can be shown that the norm is $\|\mathscr{H}\|_{L^p\to L^p} = \frac{p}{p-1}$ which means that the constant $\left(\frac{p}{p-1}\right)^p$ is sharp and never attained by any function in $L^p(0,\infty)$ except trivial one (see, e.g. [5]).

The discovery of the original Hardy inequality was a key for the further studies in this field. We refer to [6], where the history of the establishment of the Hardy inequality in the period 1906 – 1928 was described. Hardy's inequality (1) and its extensions have been widely used in various fields of mathematics such as functional analysis, partial differential equations, spectral theory, etc. It also has several applications in physics, particularly in quantum mechanics.

In the present paper, we are interested in the following extension from [8]

$$\left(\int_0^\infty x^\alpha \left(\int_0^x f(t)dt\right)^q dx\right)^{1/q} \leqslant C_{pq}\left(\int_0^\infty f^p(x)dx\right)^{1/p}$$

for the case $1 < p \leqslant q < \infty$ which holds for all measurable (non-negative) functions $f(t)$ on $(0,\infty)$ if and only if

$$\frac{\alpha+1}{q} = \frac{1}{p} - 1.$$

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Numerous authors have recently demonstrated a significant interest in enhancing Hardy-type inequalities. We refer to [3, 9, 12, 13] and references therein for readers seeking to explore these types of inequalities and their recent advancements in greater detail.

In this paper, we obtain the following result: Let $1<p\leqslant q<\infty$. Let $f(t)$ be any measurable function on $(0,\infty)$. Then we have the following inequality for $\frac{\alpha+1}{q}=\frac{1}{p}-1$:

$$\left(\int_0^\infty x^{\alpha+q}\sup_{0<s<\infty}\left|\min\left\{\frac{1}{x},\frac{1}{s}\right\}\int_0^s f(t)dt\right|^q dx\right)^{\frac{1}{q}} \leqslant C_{pq}\left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}}, \quad (2)$$

with the sharp Bliss constant [1]

$$C_{pq}=\left(\frac{p'}{q}\right)^{\frac{1}{p}}\left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}},$$

for the case $1<p<q<\infty$ and the constant approaches $p'=\frac{p}{p-1}$ as q approaches p. This inequality essentially provides not only the improvement of (1), but also extends the recent results from [2], [8], and [11]. Thus, we obtain an improvement of the Hardy inequality from [8] for one-dimensional case. In turn, it covers the recent improvement of the Hardy inequality from [2]. Moreover, we extend the obtained inequality to multidimensional case and establish the results in radial and non-radial setup. For this purpose, the non-increasing rearrangements technique is used as one of the main tools. These multidimensional inequalities extend the recent L^p-inequalities from [11] to L^p-L^q cases. Note that, in general, form inequalities with operators, involving suprema, were studied in [10].

The paper is organized as follows: Section 2 is devoted to some basic facts on non-increasing rearrangements and supporting lemmas. In Section 3, we prove our main results related to establishing the one-dimensional and multidimensional improvements of the L^p-L^q Hardy inequality.

2. Preliminaries

In this section, we present brief preliminaries before proceeding to main results and their consequences. We will start with an introduction to non-decreasing rearrangements. Then, we will continue with a short discussion on radial gradient operator and polar coordinate decomposition. Also, we establish a supporting result starting with the description of the weighted Hardy inequality on the half-line with non-increasing rearrangement of the function.

2.1. Non-increasing rearrangements

Throughout this paper we denote the non-increasing rearrangement of f by f^*. This function is non-increasing and non-negative on the interval $(0,\infty)$, satisfying the property $|\{|f| > \tau\}| = |\{f^* > \tau\}|$ for all $\tau > 0$. Here we recall L^p norm-preserving property which can be expressed as follows:

$$||f||_{L^p(0,\infty)} = ||f^*||_{L^p(0,\infty)}$$

for all $p \geqslant 1$ and for any nonnegative measurable function f in $L^p(0,\infty)$. We refer to [7, Section 15.1] for more details.

2.2. Radial gradient and polar coordinates

A function which depends only on radial part is called radially symmetric. Given that $u \in L^1(\mathbb{R}^N)$, the radial symmetric function $\tilde{u}$ can be defined as follows for any $1 < p < \infty$:

$$\tilde{u}(x) = \tilde{u}(r) := \left(\frac{1}{\omega_N}\int_{\mathbb{S}^{N-1}} |u(r\sigma)|^p \mathrm{d}\sigma\right)^{\frac{1}{p}} \text{ for any } x \in \mathbb{R}^N, \tag{3}$$

where $r = |x|$, $\sigma = \frac{x}{|x|}$, and ω_N is the surface area of the N-dimensional sphere $\mathbb{S}^{N-1}$. Given that $f(x)$ is a radial function on $[0,\infty)$, for any $x \in \mathbb{R}^N$ we have the following equality

$$f(x) = f(|x|) \quad \text{where} \quad f(x) = f(r) \text{ with } r = |x|.$$

Also, the radial gradient of a differentiable function $f(x)$ can be defined by

$$\frac{\partial f}{\partial r}(x) = \frac{x}{|x|} \cdot \nabla f(x), \tag{4}$$

where ∇ is the standard gradient on $\mathbb{R}^N$ and "$\cdot$" is the scalar product.

Given that $\mathbb{R}^N$ is the N-dimensional Euclidean space with Lebesgue measure $\mathrm{d}x$, we say that it can undergo *polar coordinate decomposition* with respect to the origin 0. For any locally integrable function f, the following equality is valid

$$\int_{\mathbb{R}^N} f(x)\mathrm{d}x = \int_0^\infty \int_{\mathbb{S}^{N-1}} f(r,\sigma)\, r^{N-1} \mathrm{d}\sigma \mathrm{d}r, \tag{5}$$

where $x = (r,\sigma) \in [0,\infty) \times \mathbb{S}^{N-1}$ with $r = |x|$. The N-dimensional unit sphere with the surface measure $\mathrm{d}\sigma$ is denoted as follows:

$$\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N \,:\, |x| = 1\}.$$

2.3. Supporting lemma

LEMMA 1. *Let $1 \leqslant N < p \leqslant q < \infty$ and α is some real number satisfying $\frac{\alpha+N}{q} = \frac{N}{p}$. Then for any measurable function f the following weighted inequality holds:*

$$\left(\int_0^\infty r^{\alpha+N-1} \sup_{0<s<\infty}\left|\min\left\{\frac{1}{r},\frac{1}{s}\right\}\int_0^s f(t)\mathrm{d}t\right|^q \mathrm{d}r\right)^{\frac{1}{q}} \leqslant C_{Npq}\left(\int_0^\infty r^{N-1}\,|f^*(r)|^p \mathrm{d}r\right)^{\frac{1}{p}}, \tag{6}$$

where f^* is the non-increasing rearrangement of f. The expression of the constant can be given by

$$C_{Npq} = \left(\frac{p-1}{p-N}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}}, \tag{7}$$

for the case $N<p<q<\infty$ where Γ is the Gamma function. As q approaches p, the constant approaches $\frac{p}{p-N}$.

Proof. By using [11, Lemma 3.1] with $g(r)=r^{\alpha+N-1}$ and $h(r)=r$ for $r\in(0,\infty)$, we have

$$\left(\int_0^\infty r^{\alpha+N-1} \sup_{0<s<\infty}\left|\min\left\{\frac{1}{r},\frac{1}{s}\right\}\int_0^s f(t)\mathrm{d}t\right|^q \mathrm{d}r\right)^{\frac{1}{q}} \leqslant \left(\int_0^\infty r^{\alpha+N-q-1}\left(\int_0^r f^*(t)\mathrm{d}t\right)^q \mathrm{d}r\right)^{\frac{1}{q}}.$$

Now exploiting [8, Theorem 2.7 (a)] in the right-hand side of the above we obtain the following inequality

$$\left(\int_0^\infty r^{\alpha+N-q-1}\left(\int_0^r f^*(t)\mathrm{d}t\right)^q \mathrm{d}r\right)^{\frac{1}{q}} \leqslant C_{Npq}\left(\int_0^\infty r^{N-1}|f^*(r)|^p \mathrm{d}r\right)^{\frac{1}{p}}.$$

Applying this and formulas (3.1)–(3.2) from [8], we obtain (6). □

3. Improvement of the L^p-L^q Hardy inequality

In this section, we establish a novel Hardy inequality which improves the result of [8], which essentially covers the result obtained in [2] when $q=p$. In addition, we extend this version of the one-dimensional inequality to the multidimensional setting.

3.1. Main results

First, we present a new version of the L^p-L^q Hardy inequality in the integral form which gives the improvement of the inequality obtained in [8, Theorem 2.7] and covers the result obtained in [2, Theorem 4] when $p=q$.

THEOREM 1. *Let $1<p\leqslant q<\infty$. Let f be any measurable function in $L_p(0,\infty)$. Then we have the following inequality for $\frac{\alpha+1}{q}=\frac{1}{p}-1$:*

$$\left(\int_0^\infty x^{\alpha+q} \sup_{0<s<\infty}\left|\min\left\{\frac{1}{x},\frac{1}{s}\right\}\int_0^s f(t)\mathrm{d}t\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \leqslant C_{pq}\left(\int_0^\infty |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}, \tag{8}$$

where the constant is sharp constant and denoted by

$$C_{pq} = \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}},$$

for the case $1 < p < q < \infty$ *and as* q *approaches* p*, this constant approaches* $\frac{p}{p-1}$.

Proof. Applying [11, Lemma 3.1], for $h(x) = x$ and $g(x) = x^{\alpha+q}$, we have the following inequality:

$$\left(\int_0^\infty x^{\alpha+q} \sup_{0<s<\infty} \left|\min\left\{\frac{1}{x},\frac{1}{s}\right\}\int_0^s f(t)\mathrm{d}t\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \leqslant \left(\int_0^\infty x^\alpha \left(\int_0^x f^*(t)\mathrm{d}t\right)^q \mathrm{d}x\right)^{\frac{1}{q}}.$$

Now we apply [8, Theorem 2.1] to get

$$\left(\int_0^\infty x^\alpha \left(\int_0^x f^*(t)dt\right)^q \mathrm{d}x\right)^{\frac{1}{q}} \leqslant C_{pq}\left(\int_0^\infty (f^*(x))^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Finally, using the fact that the L^p norm of its rearrangement is equal to the L^p norm of the function, i.e.,

$$\left(\int_0^\infty (f^*(x))^p \mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_0^\infty |f^*(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_0^\infty |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

for $1 \leqslant p < \infty$, we complete the proof. Also, by observing the fact that

$$\begin{aligned}&\left(\int_0^\infty x^{\alpha+q} \sup_{0<s<\infty} \left|\min\left\{\frac{1}{x},\frac{1}{s}\right\}\int_0^s f(t)\mathrm{d}t\right|^q \mathrm{d}x\right)^{\frac{1}{q}}\\ &\geqslant \left(\int_0^\infty x^\alpha \left|\int_0^s f(t)\mathrm{d}t\right|^q \mathrm{d}x\right)^{\frac{1}{q}}\\ &\geqslant \left(\int_0^\infty x^\alpha \left(\int_0^s f(t)\mathrm{d}t\right)^q \mathrm{d}x\right)^{\frac{1}{q}},\end{aligned}$$

and the best constant obtained in [8, Theorem 2.1] we establish the optimality here. □

REMARK 1. Let $1 < p \leqslant q < \infty$ and $\frac{\alpha+1}{q} = \frac{1}{p} - 1$. Set $u(x) = \int_0^x f(t)dt$ and $u'(x) = f(x)$ in (8). Then, for any locally absolutely continuous function u on $(0,\infty)$ with $\liminf_{x\to 0}|u(x)| = 0$, we have

$$\left(\int_0^\infty x^{\alpha+q} \sup_{0<s<\infty} \left|\min\left\{\frac{1}{x},\frac{1}{s}\right\}u(s)\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \leqslant C_{pq}\left(\int_0^\infty |u'(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}, \tag{9}$$

which is equivalent to (8) and covers [2, Theorem 1] when $p=q$.

Inequality (9) is an improvement of

$$\left(\int_0^\infty \left|\frac{u(x)}{x^{\frac{1}{q}-\frac{1}{p}+1}}\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \leqslant C_{pq}\left(\int_0^\infty |u'(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}, \quad 1<p\leqslant q<\infty, \tag{10}$$

since

$$\begin{aligned}\frac{|u(x)|^q}{x^q} &\leqslant \max\left\{\sup_{0<s\leqslant x}\frac{|u(s)|^q}{x^q}, \sup_{x\leqslant s<\infty}\frac{|u(s)|^q}{s^q}\right\}\\ &= \sup_{0<s<\infty}\left|\min\left\{\frac{1}{x},\frac{1}{s}\right\}u(s)\right|^q\end{aligned}$$

by (13).

3.2. Radial version

First, we present the inequality for the compactly supported smooth radial function space denoted as $C^\infty_{c,rad}(\mathbb{R}^N\setminus\{0\})$.

THEOREM 2. *Let $N<p\leqslant q<\infty$ and $\frac{\alpha+N}{q}=\frac{N}{p}$. Then we have*

$$\begin{aligned}&\left(\int_{\mathbb{R}^N}|x|^\alpha \max\left\{\sup_{\bar{B}(0;|x|)\setminus\{0\}}\frac{|u(y)|^q}{|x|^q}, \sup_{B^c(0;|x|)}\frac{|u(y)|^q}{|y|^q}\right\}\mathrm{d}x\right)^{\frac{1}{q}}\\ &\leqslant C_{Npq}\sigma_N^{\frac{1}{q}-\frac{1}{p}}\left(\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}},\end{aligned} \tag{11}$$

for all $u\in C^\infty_{c,rad}(\mathbb{R}^N\setminus\{0\})$. Here σ_N is the surface area of the unit sphere in $\mathbb{R}^N$ and C_{Npq} is defined in (7).

Proof. For $u\in C^\infty_{c,rad}(\mathbb{R}^N\setminus\{0\})$ we use the notation $u(y)=u(s)$ for $s=|y|$. Recall the polar coordinate decomposition $x=(r,\sigma)$ where $r=|x|\in(0,\infty)$ and $\sigma=\frac{x}{|x|}\in\mathbb{S}^{N-1}$. Then we deduce

$$\begin{aligned}&\int_{\mathbb{R}^N}|x|^\alpha \max\left\{\sup_{\bar{B}(0;|x|)\setminus\{0\}}\frac{|u(y)|^q}{|x|^q}, \sup_{B^c(0;|x|)}\frac{|u(y)|^q}{|y|^q}\right\}\mathrm{d}x\\ &=\int_0^\infty\int_{\mathbb{S}^{N-1}} r^{\alpha+N-1}\max\left\{\sup_{0<s\leqslant r}\frac{|u(s)|^q}{r^q}, \sup_{r\leqslant s<\infty}\frac{|u(s)|^q}{s^q}\right\}\mathrm{d}\sigma\mathrm{d}r.\end{aligned} \tag{12}$$

Before going further let us mention the following identities

$$\begin{aligned}
&\sup_{0<s<\infty}\left|\min\left\{\frac{1}{r},\frac{1}{s}\right\}u(s)\right|^q\\
&=\sup_{0<s<\infty}\min\left\{\frac{1}{r^q},\frac{1}{s^q}\right\}|u(s)|^q\\
&=\max\left\{\sup_{0<s\leqslant r}\min\left\{\frac{1}{r^q},\frac{1}{s^q}\right\}|u(s)|^q,\ \sup_{r\leqslant s<\infty}\min\left\{\frac{1}{r^q},\frac{1}{s^q}\right\}|u(s)|^q\right\}\\
&=\max\left\{\sup_{0<s\leqslant r}\frac{|u(s)|^q}{r^q},\ \sup_{r\leqslant s<\infty}\frac{|u(s)|^q}{s^q}\right\}.
\end{aligned}\tag{13}$$

By using these, we compute

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N}|x|^\alpha\max\left\{\sup_{\bar{B}(0\,;|x|)\setminus\{0\}}\frac{|u(y)|^q}{|x|^q},\ \sup_{B^c(0\,;|x|)}\frac{|u(y)|^q}{|y|^q}\right\}\mathrm{d}x\right)^{\frac{1}{q}}\\
&=\left(\int_{\mathbb{S}^{N-1}}\int_0^\infty r^{\alpha+N-1}\sup_{0<s<\infty}\left|\min\left\{\frac{1}{r},\frac{1}{s}\right\}u(s)\right|^q\mathrm{d}r\mathrm{d}\sigma\right)^{\frac{1}{q}}\\
&=\left(\int_{\mathbb{S}^{N-1}}\int_0^\infty r^{\alpha+N-1}\sup_{0<s<\infty}\left|\min\left\{\frac{1}{r},\frac{1}{s}\right\}\int_0^s\frac{\partial u}{\partial t}(t)\mathrm{d}t\right|^q\mathrm{d}r\mathrm{d}\sigma\right)^{\frac{1}{q}}\\
&\overset{\text{Lemma 1}}{\leqslant} C_{Npq}|\sigma_N|^{\frac{1}{q}-\frac{1}{p}}\left(\int_{\mathbb{S}^{N-1}}\int_0^\infty r^{N-1}\left|\left(\frac{\partial u}{\partial r}\right)^*(r)\right|^p\mathrm{d}r\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
&=C_{Npq}\sigma_N^{\frac{1}{q}-\frac{1}{p}}\left(\int_{\mathbb{R}^N}\left|\left(\frac{\partial u}{\partial|x|}\right)^*(x)\right|^p\mathrm{d}x\right)^{\frac{1}{p}}\\
&=C_{Npq}\sigma_N^{\frac{1}{q}-\frac{1}{p}}\left(\int_{\mathbb{R}^N}\left|\frac{\partial u}{\partial|x|}(x)\right|^p\mathrm{d}x\right)^{\frac{1}{p}}\\
&=C_{Npq}\sigma_N^{\frac{1}{q}-\frac{1}{p}}\left(\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^p\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

In the middle, we have used Lemma 1 for $f(t)=\frac{\partial u}{\partial t}(t)$ and the norm preserving property for the function $\frac{\partial u}{\partial r}$. □

3.3. Non-radial setting of the results

In this subsection, we state the non-radial version of Theorem 2. Note that radialisation technique is commonly used in order to establish a non-radial inequality from radial one.

THEOREM 3. *Let $N < p \leqslant q < \infty$ and $\frac{\alpha+N}{q} = \frac{N}{p}$. Then, for all $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, we have*

$$\left(\max\left\{ \int_0^\infty r^{N-1+\alpha} \sup_{0<s\leqslant r} \int_{\mathbb{S}^{N-1}} \frac{|u(s\sigma)|^q}{r^q} \mathrm{d}\sigma \mathrm{d}r, \int_0^\infty r^{N-1+\alpha} \sup_{r\leqslant s<\infty} \int_{\mathbb{S}^{N-1}} \frac{|u(s\sigma)|^q}{s^q} \mathrm{d}\sigma \mathrm{d}r \right\} \right)^{\frac{1}{q}} \leqslant C_{Npq} \sigma_N^{\frac{1}{q}-\frac{1}{p}} \left(\int_{\mathbb{R}^N} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^p \mathrm{d}x \right)^{\frac{1}{p}}. \tag{14}$$

Here σ_N is the surface area of the unit sphere in $\mathbb{R}^N$ and C_{Npq} is defined in (7).

Proof. Let $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ and $\tilde{u}$ be the radial symmetric function associated to it. Using (3), for any $1 < p < \infty$, we define the radial symmetric function $\tilde{u}$ as follows:

$$\tilde{u}(x) = \tilde{u}(r) := \left(\frac{1}{\sigma_N} \int_{\mathbb{S}^{N-1}} |u(r\sigma)|^p \, \mathrm{d}\sigma \right)^{\frac{1}{p}} \quad \text{for any } x \in \mathbb{R}^N.$$

Then exploiting [11, Lemma 4.2] with $f(x) = |x|^\alpha$ and then substituting the result into Theorem 2, we deduce

$$\begin{aligned} &\left(\max\left\{ \int_0^\infty r^{N-1+\alpha} \sup_{0<s\leqslant r} \int_{\mathbb{S}^{N-1}} \frac{|u(s\sigma)|^q}{r^q} \mathrm{d}\sigma \mathrm{d}r, \int_0^\infty r^{N-1+\alpha} \sup_{r\leqslant s<\infty} \int_{\mathbb{S}^{N-1}} \frac{|u(s\sigma)|^q}{s^q} \mathrm{d}\sigma \mathrm{d}r \right\} \right)^{\frac{1}{q}} \\ &\leqslant \left(\int_{\mathbb{R}^N} |x|^\alpha \max\left\{ \sup_{\bar{B}(0\,;\,|x|)\setminus\{0\}} \frac{|\tilde{u}(y)|^q}{|x|^q}, \sup_{B^c(0\,;\,|x|)} \frac{|\tilde{u}(y)|^q}{|y|^q} \right\} \mathrm{d}x \right)^{\frac{1}{q}} \\ &\leqslant C_{Npq} \sigma_N^{\frac{1}{q}-\frac{1}{p}} \left(\int_{\mathbb{R}^N} \left| \frac{x}{|x|} \cdot \nabla \tilde{u}(x) \right|^p \mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned}$$

Next, using [11, Lemma 4.1] with $f(x) = 1$, we have

$$\left(\int_{\mathbb{R}^N} \left| \frac{x}{|x|} \cdot \nabla \tilde{u}(x) \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \leqslant \left(\int_{\mathbb{R}^N} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^p \mathrm{d}x \right)^{\frac{1}{p}},$$

Finally, combining the above two estimates we obtain

$$\left(\max\left\{ \int_0^\infty r^{N-1+\alpha} \sup_{0<s\leqslant r} \int_{\mathbb{S}^{N-1}} \frac{|u(s\sigma)|^q}{r^q} \mathrm{d}\sigma \mathrm{d}r, \int_0^\infty r^{N-1+\alpha} \sup_{r\leqslant s<\infty} \int_{\mathbb{S}^{N-1}} \frac{|u(s\sigma)|^q}{s^q} \mathrm{d}\sigma \mathrm{d}r \right\} \right)^{\frac{1}{q}} \leqslant C_{Npq} \sigma_N^{\frac{1}{q}-\frac{1}{p}} \left(\int_{\mathbb{R}^N} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^p \mathrm{d}x \right)^{\frac{1}{p}},$$

which is the desired result (14). $\square$

REMARK 2. The above result gives the following L^p-L^q Hardy inequality. Let $N < p \leqslant q < \infty$ and $\frac{\alpha+N}{q} = \frac{N}{p}$. Then, for all $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |x|^\alpha \frac{|u(x)|^q}{|x|^q} \mathrm{d}x\right)^{\frac{1}{q}} \\
&\leqslant \left(\max\left\{\int_0^\infty r^{N-1+\alpha} \sup_{0<s\leqslant r} \int_{\mathbb{S}^{N-1}} \frac{|u(s\sigma)|^q}{r^q} \mathrm{d}\sigma \mathrm{d}r,\right.\right. \\
&\left.\left.\int_0^\infty r^{N-1+\alpha} \sup_{r\leqslant s<\infty} \int_{\mathbb{S}^{N-1}} \frac{|u(s\sigma)|^q}{s^q} \mathrm{d}\sigma \mathrm{d}r\right\}\right)^{\frac{1}{q}} \\
&\leqslant C_{Npq} \sigma_N^{\frac{1}{q}-\frac{1}{p}} \left(\int_{\mathbb{R}^N} \left|\frac{x}{|x|} \cdot \nabla u(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\
&\leqslant C_{Npq} \sigma_N^{\frac{1}{q}-\frac{1}{p}} \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

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