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SOME INEQUALITIES AND EQUATIONS OF g–FRAME OPERATOR MULTIPLIERS FOR FINITE GROUP REPRESENTATIONS

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Abstract. In wavelet theory, discussing the characterizing of wandering vector multipliers or frame vector multipliers for unitary representations of various groups is a very interesting problem. However, even in the case of Abelian groups, the characteristics of frame vector multipliers are still unknown. The purpose of this paper is to study g-frame operator multipliers by combining the unitary representation of finite groups with operator theory. Firstly, some new inequalities and equations that reflect the properties and characterizations of g-frame operator multipliers are discussed. With the help of group representation theory and operator theory, some necessary conditions such that a unitary operator is a g-frame operator multiplier can be found. Next, the g-frame operator multiplier for the more general case is discussed. In particular, the relationship between the g-frame operator multiplier of the direct sums of irreducible subrepresentations and the g-frame operator multiplier of the subrepresentations is obtained.

1. Introduction

Frame as a generalization of a basis for a Hilbert space was first introduced by Duffin and Schaeffer to deal with some problems concerning the nonharmonic Fourier series in 1952 [6]. The frame is defined as follows

DEFINITION 1.1. [6] A collection of vectors $F=\{f_i\}_{i=1}^k \subset H$ is a frame for a Hilbert space H if there are constants $0<A\leqslant B<\infty$ such that

$$A\|f\|^2 \leqslant \sum_{i=1}^{k} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2,$$

for any $f\in H$. Constants A and B are lower and upper frame bounds, respectively. If $A=B$, then the frame is a tight frame for H. Especially, if $A=B=1$, the frame is a Parseval frame.

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Frames and bases are similar in many ways. For example, any elements of a Hilbert space can be expanded by a frame. But because of the redundancy of frames, the expansion of elements in space is not unique. So in recent years, the frame replaced the bases and has been applied in many fields such as signal and image processing [10], quantization [1], the capacity of transmission channels [5], coding theory [13], data transmission technology [15], and so on ([7]).

To solve a variety of application problems, the frames have been generalized by many scholars. Hence some new frames such as fusion frames, g-frames, K-frames, and weaving frames are put forward. In this paper, the g-frame is mainly discussed. G-frames as a generation of traditional frames were first introduced by Sun in [20].

DEFINITION 1.2. [20] Let H and K be the Hilbert spaces, $\{K_i\}_{i\in I}$ be subspaces of K, and $\Lambda_i \in \ell(H,K_i)$ be the bounded linear operator from H to K_i. The sequence $\{\Lambda_i\}_{i\in I}$ is said to be a generalized frame, or simply a g-frame for H with respect to $\{K_i\}_{i\in I}$ if there are two positive constants A and B such that

$$A\|f\|^2 \leqslant \sum_{i\in I} \|\Lambda_i f\|^2 \leqslant B\|f\|^2,$$

for any $f \in H$. We call A and B the lower and upper g-frame bounds, respectively. In particular, if $A = B$ then the g-frame is called a tight g-frame. If $A = B = 1$, then the g-frame is called a Parseval g-frame.

It is not difficult to find that the biggest difference between the g-frame and the traditional frame is that the elements of the g-frame are operators, which combines operator theory with frame theory [18], [21]. Hence some conclusions of frames will no longer apply to g-frames, and g-frames also have some features that frames do not have [14].

In [4], they use the characterization of Fourier wavelet multipliers to prove the connectivity of Multiresolution Analysis wavelets. In fact, Fourier wavelet multipliers belong to a special class of wandering vector multipliers for the unitary system. Hence it is interesting to discuss the characterizing of wandering vector multipliers or frame vector multipliers for unitary representations of various groups which combines the theory of group representation frames and the theory of operator algebras [2,8,9,12,19]. The following is the definition of group representation.

DEFINITION 1.3. [11] Let G be a group, K be a Hilbert space, $U(K)$ be a group formed by unitary operators on K. Then a unitary representation π of G is a group homomorphism from G into the group $U(K)$.

However, little is known about operator multipliers for wavelet systems or group representations. Even in the case of Abelian groups, the characteristics of frame vector multipliers are still unknown. In [16], the authors studied the frame vector multipliers by starting with unitary representations of finite groups. Due to the structural specificity of the g-frame, we naturally want to know if the g-frame multipliers exist and if there are similar properties to frame vector multipliers. So in this paper, we generalize the frame vector multipliers to the g-frame operator multipliers. More specifically, since some conclusions of the frame will no longer apply to the g-frame, some new inequalities and equations that reflect the properties and characterizations of g-frame operator multipliers are proposed.

2. Preparation knowledge

In this section, some preparation knowledge of the g-frames and group representation are introduced, which lays the foundation for the development of the following section. First of all, the following three operators are very important in the g-frame theory.

DEFINITION 2.1. Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for H with respect to $\{K_i\}_{i\in I}$.

(I) The analysis operator of Λ is defined by

$$\Theta : H \to (\sum_{i\in I} \oplus K_i)_{\ell^2}, \Theta x = \{\Lambda_i f\}_{i\in I},$$

(II) The synthesis operator of Λ is defined by

$$\Theta^* : (\sum_{i\in I} \oplus K_i)_{\ell^2} \to H, \Theta^*\{f_i\}_{i\in I} = \sum_{i\in I} \Lambda_i^* f_i,$$

(III) The frame operator of Λ is defined by

$$S : H \to H, Sf = \Theta^*\Theta f = \sum_{i\in I} \Lambda_i^* \Lambda_i f.$$

According to Definition 1.1 and Definition 1.2, we can say that a g-frame is a Parseval g-frame if and only if for any $f \in H$, we have

$$Sf = \Theta^*\Theta f = \sum_{i\in I} \Lambda_i^* \Lambda_i f = f.$$

Similarly to a frame representation in [16], the definition of a g-frame representation can be given.

DEFINITION 2.2. Let G be a group, H and K be Hilbert spaces, π be a group homomorphism from G into the group $U(K)$. If there is a operator $\varphi : H \to K$ such that $\{\pi(g)\varphi : g \in G\}$ is a g-frame for H with respect to $\{K_i\}_{i\in I}$, then the unitary representation π for a group G is called a g-frame representation. And in this case every range of $\pi(g)\varphi$ is a subspace of K, we call it K_i. In addition, the operator $\varphi : H \to K$ is called a g-frame operator (or g-frame generator) for the representation π.

More specially, if the operator sequence $\{\pi(g)\varphi : g \in G\}$ in Definition 1.3 is a Parseval (tight) g-frame for H with respect to $\{K_i\}_{i\in I}$, then the operator $\varphi : H \to K$ is called a Parseval (tight) g-frame operator (or g-frame generator) for π. Then we introduce some lemma of π.

LEMMA 2.1. [16] *A unitary representation of a group G is called irreducible if $\pi(G)$ has no nontrivial invariant closed subspaces, which is equivalently to say that the commutant of $\pi(G)$ is trivial, i.e., $\pi(G)' = CI$, where I is the identity operator of H and $\pi(G)' = \{T \in B(H) : T\pi(g) = \pi(g)T, \forall g \in G\}$.*

LEMMA 2.2. [3] *Let π be a unitary representation of a finite group G on a finite dimensional Hilbert space H. Then π is unitarily equivalent to a unitary representation*

$$\pi_1^{m_1} \oplus \cdots \oplus \pi_k^{m_k},$$

where $\pi_1, \cdots, \pi_k$ are inequivalent irreducible representations, and $m_i \geqslant 1$ is the multiplicity of π.

The mainly purpose of this paper is to obtain the characterizations of g-frame operator multipliers. So in the following, we introduce the g-frame operator multiplier.

DEFINITION 2.3. Let π be a frame representations of a group G on a Hilbert space K. A unitary operator $U \in B(K)$ is called a g-frame operator multiplier if $U\varphi$ is a Parseval g-frame operator for π whenever φ is a Parseval g-frame operator.

Finally, we recall the direct sum of operators.

DEFINITION 2.4. Let φ_a and φ_b be operators from H to K, A and B be the matrices which under the orthonormal basis respectively, that is to say $\varphi_a f = Af$ and $\varphi_b f = Bf$. Then

$$(\varphi_a \oplus \varphi_b)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} Af_1 \\ Bf_2 \end{pmatrix}. \tag{1}$$

In this paper, we just write $\varphi_a \oplus \varphi_b = \varphi = (\varphi_a, \varphi_b)$, $f = (f_1, f_2)$. Hence (1) is equal to

$$\varphi f = (\varphi_a, \varphi_b) f = (\varphi_a f_1, \varphi_b f_2).$$

3. Some inequalities and equations of g-frame operator multiplier

In this section, we mainly study g-frame operator multipliers by combining the unitary representation of finite groups with operator theory. Firstly, Dongwei Li introduced some inequalities of fusion frames in [17]. In this paper, we found that those inequalities still hold in the g-frame $\{\pi(g)\varphi : g \in G\}$. Especially, due to the structural specificity of the $\{\pi(g)\varphi : g \in G\}$, we can prove the inequalities in a simpler way. Theorem 3.1 and Theorem 3.2 precisely illustrate this point.

THEOREM 3.1. *Let the unitary representation π for a group G be a g-frame representation, and the operator $\varphi : H \to K$ be a g-frame operator for the representation π. Thus $\{\pi(g)\varphi : g \in G\}$ is a g-frame for H, and S is the frame operator of $\{\pi(g)\varphi : g \in G\}$. Then for any $\lambda \in [0,2]$, we have*

$$\left(\lambda - \frac{\lambda^2}{4}\right)\langle S_J f, f\rangle + \left(1 - \frac{\lambda^2}{4}\right)\langle S_{J^c} f, f\rangle \leqslant \langle S^{-1} S_J f, S_J f\rangle + \langle S_{J^c} f, f\rangle \leqslant \langle Sf, f\rangle.$$

Proof. On the one hand, for any $J \subset G$, let's assume that $|G| = N$, and $|J| = a$. Then

$$\begin{aligned}\langle Sf, f\rangle &= \langle \sum_{g\in G}(\pi(g)\varphi)^*\pi(g)\varphi f, f\rangle = \langle \sum_{g\in G}(\varphi^*\pi(g)^*\pi(g)\varphi)f, f\rangle \\ &= \langle \sum_{g\in G}\varphi^*\varphi f, f\rangle = |G|\|\varphi f\|^2 = N\|\varphi f\|^2.\end{aligned}$$

For the same reason, we have $\langle S_J f, f\rangle = a\|\varphi f\|^2$, $\langle S_{J^c} f, f\rangle = (N-a)\|\varphi f\|^2$, and $Sf = \sum_{g\in G}(\pi(g)\varphi)^*\pi(g)\varphi f = N\varphi^*\varphi f$. Then

$$\begin{aligned}\langle S^{-1}S_J f, S_J f\rangle &= \left\langle \frac{1}{N}\varphi^{-1}(\varphi^*)^{-1}\sum_{g\in J}(\pi(g)\varphi)^*\pi(g)\varphi f, \sum_{g\in J}(\pi(g)\varphi)^*\pi(g)\varphi f\right\rangle \\ &= \left\langle \frac{a}{N}\varphi^{-1}(\varphi^*)^{-1}\varphi^*\varphi f, a\varphi^*\varphi f\right\rangle = \frac{a^2}{N}\|\varphi f\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\langle S^{-1}S_J f, S_J f\rangle + \langle S_{J^c} f, f\rangle - \langle Sf, f\rangle &= \frac{a^2}{N}\|\varphi f\|^2 + (N-a)\|\varphi f\|^2 - N\|\varphi f\|^2 \\ &= \left(\frac{a^2}{N} - a\right)\|\varphi f\|^2 \leqslant 0,\end{aligned}$$

that is to say $\langle S^{-1}S_J f, S_J f\rangle + \langle S_{J^c} f, f\rangle \leqslant \langle Sf, f\rangle$.

On the other hand, for any $\lambda \in [0,2]$, we can obtain

$$\begin{aligned}&\langle S^{-1}S_J f, S_J f\rangle + \langle S_{J^c} f, f\rangle - \left(\left(\lambda - \frac{\lambda^2}{4}\right)\langle S_J f, f\rangle + \left(1 - \frac{\lambda^2}{4}\right)\langle S_{J^c} f, f\rangle\right) \\ &= \frac{a^2}{N}\|\varphi f\|^2 + (N-a)\|\varphi f\|^2 - \left(\left(\lambda - \frac{\lambda^2}{4}\right)a\|\varphi f\|^2 + \left(1 - \frac{\lambda^2}{4}\right)(N-a)\|\varphi f\|^2\right) \\ &= \left(\frac{a^2}{N} + N - a\right)\|\varphi f\|^2 - \left(\lambda a + N - \frac{\lambda^2}{4}N - a\right)\|\varphi f\|^2 \\ &= \left(\frac{N}{4}\lambda^2 - \lambda a + \frac{a^2}{N}\right)\|\varphi f\|^2 = \frac{N}{4}\left(\lambda - \frac{2a}{N}\right)^2\|\varphi f\|^2 \geqslant 0.\end{aligned}$$

Thus $\langle S^{-1}S_J f, S_J f\rangle + \langle S_{J^c} f, f\rangle \geqslant (\lambda - \frac{\lambda^2}{4})\langle S_J f, f\rangle + (1 - \frac{\lambda^2}{4})\langle S_{J^c} f, f\rangle$. □

THEOREM 3.2. *Let the unitary representation π for a group G be a g-frame representation, and the operator $\varphi : H \to K$ be a g-frame operator for the representation π. Thus $\{\pi(g)\varphi : g \in G\}$ is a g-frame for H, and S is the frame operator of $\{\pi(g)\varphi : g \in G\}$. Then for any $\lambda \in [1,2]$, we have*

$$\begin{aligned}\left(2\lambda - \frac{\lambda^2}{2} - 1\right)\langle S_J f, f\rangle + \left(1 - \frac{\lambda^2}{2}\right)\langle S_{J^c} f, f\rangle &\leqslant \langle S^{-1}S_J f, S_J f\rangle + \langle S^{-1}S_{J^c} f, S_{J^c} f\rangle \\ &\leqslant \langle Sf, f\rangle.\end{aligned}$$

Proof. One the one hand, for any $\lambda \in [1,2]$ and $J \subset G$, let's assume that $|G| = N$, and $|J| = a$. Then

$$\begin{aligned}\langle S^{-1}S_J f, S_J f\rangle + \langle S^{-1}S_{J^c} f, S_{J^c} f\rangle &= \frac{a^2}{N}\|\varphi f\|^2 + \frac{(N-a)^2}{N}\|\varphi f\|^2 \\ &= \frac{N^2 - 2a(N-a)}{N}\|\varphi f\|^2 \leqslant N\|\varphi f\|^2 \\ &\leqslant \lambda N\|\varphi f\|^2 = \lambda\langle Sf, f\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\langle S^{-1}S_J f, S_J f\rangle + \langle S^{-1}S_{J^c} f, S_{J^c} f\rangle - \left(\left(2\lambda - \frac{\lambda^2}{2} - 1\right)\langle S_J f, f\rangle + \left(1 - \frac{\lambda^2}{2}\right)\langle S_{J^c} f, f\rangle\right) \\ &= \frac{N^2 - 2a(N-a)}{N}\|\varphi f\|^2 - \left(\left(2\lambda - \frac{\lambda^2}{2} - 1\right)a + \left(1 - \frac{\lambda^2}{2}\right)(N-a)\right)\|\varphi f\|^2 \\ &= \left(\frac{N}{2}\lambda^2 - 2\lambda a + \frac{2a^2}{N}\right)\|\varphi f\|^2 = \frac{N}{2}\left(\lambda - \frac{2a}{N}\right)^2\|\varphi f\|^2 \geqslant 0\end{aligned}$$

Thus $\langle S^{-1}S_J f, S_J f\rangle + \langle S^{-1}S_{J^c} f, S_{J^c} f\rangle \geqslant (2\lambda - \frac{\lambda^2}{2} - 1)\langle S_J f, f\rangle + (1 - \frac{\lambda^2}{2})\langle S_{J^c} f, f\rangle$. □

Then some equations that reflect the properties of a Parseval g-frame operator for the representation π can be obtained.

THEOREM 3.3. *Let G be a finite group, K and H be Hilbert spaces, π be a g-frame representation of G on K. Then we have*

(i) If φ is a Parseval g-frame operator for the representation π, then $t\varphi$ is a Parseval g-frame operator for the representation π for any unimodular scalars t.

(ii) If φ is a Parseval g-frame operator for the representation π, then $\|\varphi\|^2 = \frac{1}{|G|}$.

(iii) If $\varphi \in B(H,K)$ such that $\varphi^*\varphi = cI$, then φ is a tight g-frame operator for the representation π. Moreover, if $\|\varphi\|^2 = \frac{1}{|G|}$, then φ is a Parseval g-frame operator for the representation π.

Proof. (i) For the operator sequence $\{\pi(g)t\varphi\}_{g\in G}$, since φ is a Parseval g-frame operator for the representation π, we have

$$\begin{aligned}\sum_{g\in G}(\pi(g)t\varphi)^*\pi(g)t\varphi f &= \sum_{g\in G}\|t\|^2(\varphi^*\pi(g)^*\pi(g)\varphi)f \\ &= \sum_{g\in G}\varphi^*\pi(g)^*\pi(g)\varphi f \\ &= \sum_{g\in G}(\pi(g)\varphi)^*(\pi(g)\varphi)f = f.\end{aligned}$$

Thus $\{\pi(g)t\varphi\}_{g\in G}$ is a Parseval g-frame operator for the representation π, and $t\varphi$ is a Parseval g-frame operator for the representation π.

(ii) Since φ is a Parseval g-frame operator for π, then

$$\begin{aligned}\sum_{g\in G}\|\pi(g)\varphi f\|^2 &= \sum_{g\in G}\langle \pi(g)\varphi f, \pi(g)\varphi f\rangle \\ &= \sum_{g\in G}\langle (\pi(g)\varphi)^*\pi(g)\varphi f, f\rangle \\ &= \langle \sum_{g\in G}(\pi(g)\varphi)^*\pi(g)\varphi f, f\rangle = \langle f, f\rangle = \|f\|^2.\end{aligned}$$

And,

$$\sum_{g\in G}\|\pi(g)\varphi f\|^2 = |G|\|\varphi f\|^2.$$

Hence $|G|\|\varphi f\|^2 = \|f\|^2$, thus $\|\varphi\|^2 = \frac{1}{|G|}$.

(iii) Since $\varphi^*\varphi = cI$, it is easy to check that,

$$\sum_{g\in G}(\pi(g)\varphi)^*(\pi(g)\varphi)f = \sum_{g\in G}\varphi^*\varphi f = c|G|f.$$

Hence φ is a tight g-frame operator for π.

Moreover, we obtain that

$$\|\pi(g)\varphi f\|^2 = \langle \pi(g)\varphi f, \pi(g)\varphi f\rangle = \langle \varphi^*\varphi f, f\rangle = c\|f\|^2.$$

That is to say $\|\varphi\|^2 = c = \frac{1}{|G|}$. Furthermore,

$$\sum_{g\in G}(\pi(g)\varphi)^*(\pi(g)\varphi)f = c|G|f = f.$$

And so φ is a Parseval g-frame operator for π. □

Next the more general case is considered. Let $K = K_1\oplus K_2\oplus\cdots\oplus K_m$, $\pi = \pi_1\oplus\pi_2\oplus\cdots\oplus\pi_m$ and $H' = H\oplus H\oplus\cdots\oplus H$. Then the relationship between the Parseval g-frame operator for π with the Parseval g-frame operator for π_i can be obtained from Theorem 3.4.

THEOREM 3.4. *Let G be a finite group, $\pi = \pi_1\oplus\pi_2\oplus\cdots\oplus\pi_m$ be a g-frame representation of G on $K = K_1\oplus K_2\oplus\cdots\oplus K_m$, where π_1, $\cdots$, π_m be inequivalent irreducible representations of G, $H' = H\oplus H\oplus\cdots\oplus H$. Then we have:*

(i) If φ_i is a Parseval g-frame operator of H on K_i, then $\varphi = (\varphi_1,\cdots,\varphi_m)$ is a Parseval g-frame operator of H' on K, and $\|\varphi\|^2 = \frac{1}{|G|}$.

(ii) If $\varphi = (\varphi_1,\cdots,\varphi_m)$ is a Parseval g-frame operator of H' on K, then $\varphi_i^*\varphi_i = \frac{1}{|G|}I_H$. Thus φ_i is a Parseval g-frame operator of H on K_i, and $\|\varphi_i\|^2 = \frac{1}{|G|}$.

Proof. (i) First for any $f = (f_1,\cdots,f_m)\in H'$, according to the definition of φ we have $\varphi f = (\varphi_1 f_1,\cdots,\varphi_m f_m)$, where $f_i\in H$.

Since φ_i is a Parseval g-frame operator of H on K_i, then

$$\begin{aligned}\sum_{g\in G}(\pi(g)\varphi)^*\pi(g)\varphi f &= \sum_{g\in G}(\varphi^*\pi(g)^*\pi(g)\varphi)f = \sum_{g\in G}(\varphi^*\varphi)f\\ &= \sum_{g\in G}(\varphi_1^*\varphi_1 f_1,\cdots,\varphi_m^*\varphi_m f_m)\\ &= (\sum_{g\in G}\varphi_1^*\pi_1(g)^*\pi_1(g)\varphi_1 f_1,\cdots,\sum_{g\in G}\varphi_m^*\pi_m(g)^*\pi_m(g)\varphi_m f_m)\\ &= (f_1,\cdots,f_m) = f,\end{aligned}$$

hence φ is a Parseval g-frame operator of H' on K. And from Theorem 3.3 we know that $\|\varphi\|^2 = \frac{1}{|G|}$.

(ii) Since π_i is irreducible and φ is a Parseval g-frame operator of H on K, then for any $f \in H'$

$$\begin{aligned}\sum_{g\in G}(\pi(g)\varphi)^*\pi(g)\varphi f &= \sum_{g\in G}(\varphi^*\pi(g)^*\pi(g)\varphi)f = \sum_{g\in G}(\varphi^*\varphi)f\\ &= \sum_{g\in G}(\varphi_1^*\varphi_1 f_1,\cdots,\varphi_m^*\varphi_m f_m)\\ &= |G|(\varphi_1^*\varphi_1 f_1,\cdots,\varphi_m^*\varphi_m f_m) = f = (f_1,\cdots,f_m).\end{aligned}$$

Hence $\varphi_i^*\varphi_i = \frac{1}{|G|}I_H$ for all $i = 1,2,\cdots,m$.

Moreover we obtain that

$$\sum_{g\in G}\varphi_i^*\pi_i(g)^*\pi_i(g)\varphi_i f_i = |G|\frac{1}{|G|}f_i = f_i.$$

Thus φ_i is a Parseval g-frame operator of H on K_i. In addition, from Theorem 3.3 we know that $\|\varphi_i\|^2 = \frac{1}{|G|}$. □

In the next, the conditions such that a unitary operator $U = (u_{ij})_{m\times m}$ to be a g-frame operator multiplier can be obtained.

THEOREM 3.5. *Let G be a finite group, $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_m$ be a g-frame representation of G on $K = K_1 \oplus K_2 \oplus \cdots \oplus K_m$, where π_1, $\cdots$, π_m be inequivalent irreducible representations of G, $H' = H \oplus H \oplus \cdots \oplus H$. Let $U = (u_{ij})_{m\times m}$ be a unitary operator, where each u_{ij} be an operator from K_j to K_i. If $U = (u_{ij})_{m\times m}$ satisfies the following conditions:*

(i) $u_{ij}^*u_{ik} = 0$ for any $j \neq k$,

(ii) $u_{ij}^*u_{ij} = \lambda_{ij}I_{K_j}$,

(iii) $\sum_{i=1}^m \lambda_{ij} = 1$,

then $U = (u_{ij})_{m\times m}$ is a g-frame operator multiplier.

Proof. Assume that a unitary operator $U = (u_{ij})_{m\times m}$ satisfies the conditions (i)–(iii). Let a operator $\varphi = (\varphi_1,\cdots,\varphi_m)$ be a Parseval g-frame operator of H' on

K, where φ_i is a operator from H to K_i. Then the question is equal to proof that $U\varphi$ is a Parseval g-frame operator of H' on K.

Firstly, $U\varphi = (\sum_{j=i}^{m} u_{ij}\varphi_j)_{i=1}^{m}$ is a operator from H' to K, where $\sum_{j=i}^{m} u_{ij}\varphi_j$ is a operator from H to K_i. And for any $f \in H'$,

$$\sum_{g\in G}(\pi(g)U\varphi)^*\pi(g)U\varphi f = \sum_{g\in G}(\varphi^*U^*\pi(g)^*\pi(g)U\varphi)f = \sum_{g\in G}(\varphi^*U^*U\varphi)f. \quad (2)$$

Since $U = (u_{ij})_{m\times m}$ satisfies the conditions (i)–(iii), and $\varphi = (\varphi_1, \cdots, \varphi_m)$ is a Parseval g-frame operator,

$$\sum_{g\in G}(\pi(g)U\varphi)^*\pi(g)U\varphi f = \sum_{g\in G}(\varphi^*\varphi)f = \sum_{g\in G}(\varphi^*\pi(g)^*\pi(g)\varphi)f = f. \quad (3)$$

Hence $U\varphi$ is a Parseval g-frame operator of H' on K. And $U = (u_{ij})_{m\times m}$ is a g-frame operator multiplier. □

Due to the structural specificity of the g-frame, some conclusions about frame will no longer hold true in the g-frame. For example, in [16], the reverse of the Theorem 3.5 is also true. But in the g-frame theory it is not true. Hence we discuss new equations which reflect characterizations of the g-frame operator multiplier $U = (u_{ij})_{m\times m}$.

THEOREM 3.6. *Let G be a finite group, $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_m$ be a g-frame representation of G on $K = K_1 \oplus K_2 \oplus \cdots \oplus K_m$, where $\pi_1, \cdots, \pi_m$ be inequivalent irreducible representations of G, $H' = H \oplus H \oplus \cdots \oplus H$. Let $U = (u_{ij})_{m\times m}$ be a unitary operator, where each u_{ij} be an operator from K_j to K_i, φ_i be a Parseval g-frame operator of H on K_i. If $U = (u_{ij})_{m\times m}$ is a g-frame operator multiplier, then*

(i) $\varphi_j^* u_{ij}^* u_{ik}\varphi_k = 0$ for any $j \neq k$,

(ii) $u_{ij}^* u_{ij} = \lambda_{ij} I_{k_j}$,

(iii) $\sum_{j=1}^{m} \lambda_{ij} = 1$.

Proof. Let $U = (u_{ij})_{m\times m}$ be a g-frame operator multiplier, $\varphi = (\varphi_1, \cdots, \varphi_m)$ be a Parseval g-frame operator of H' on K, then $U\varphi$ be a Parseval g-frame operator of H' on K.

(i) According to Theorem 3.4, we know that φ_i is a Parseval g-frame operator of H on K_i, and $\|\varphi_i\|^2 = \frac{1}{|G|}$. In addition, from Theorem 3.3 we have $t_i\varphi_i$ is a Parseval g-frame operator of H on K_i, and $\varphi = (t_1\varphi_1, \cdots, t_m\varphi_m)$ be a Parseval g-frame operator of H' on K for any unimodular scalars t_i.

Since $U\varphi$ is a Parseval g-frame operator of H' on K, according to (ii) in Theorem 3.4 we get that

$$(\sum_{j=1}^{m} u_{ij}\varphi_j)^* \sum_{j=1}^{m} u_{ij}\varphi_j = \frac{1}{|G|}I,$$

and $\sum_{j=1}^{m} u_{ij}\varphi_j$ is a Parseval g-frame operator of H on K_i. Moreover

$$\|\sum_{j=1}^{m} u_{ij}\varphi_j\|^2 = \frac{1}{|G|}. \quad (4)$$

Then for any $f \in H$, (4) is equal to

$$\sum_{j=1}^{m} \|u_{ij}\varphi_j f\|^2 + 2Re \sum_{1\leqslant j<k\leqslant m} \langle u_{ij}\varphi_j f, u_{ik}\varphi_k f\rangle = \frac{1}{|G|}\|f\|^2. \tag{5}$$

Replace φ_1 in (5) with $-\varphi_1$, we get that

$$\sum_{j=1}^{m} \|u_{ij}\varphi_j f\|^2 - 2Re \sum_{2\leqslant k\leqslant m} \langle u_{i1}\varphi_1 f, u\varphi_k f\rangle + 2Re \sum_{2\leqslant j<k\leqslant m} \langle u_{ij}\varphi_j f, u\varphi_k f\rangle = \frac{1}{|G|}\|f\|^2. \tag{6}$$

According to (5)+(6), we have that

$$\sum_{j=1}^{m} \|u_{ij}\varphi_j f\|^2 + 2Re \sum_{2\leqslant j<k\leqslant m} \langle u_{ij}\varphi_j f, u_{ik}\varphi_k f\rangle = \frac{1}{|G|}\|f\|^2. \tag{7}$$

Repeat the above steps, and use $-\varphi_l$ to replace φ_l, then we obtain that

$$\sum_{j=1}^{m} \|u_{ij}\varphi_j f\|^2 + 2Re \sum_{l\leqslant j<k\leqslant m} \langle u_{ij}\varphi_j f, u_{ik}\varphi_k f\rangle = \frac{1}{|G|}\|f\|^2, \tag{8}$$

where $l = 2,3,\cdots,m$.

That is to say $\sum_{j=1}^{m} \|u_{ij}\varphi_j f\|^2 = \frac{1}{|G|}\|f\|^2$, and $\langle u_{ij}\varphi_j f, u_{ik}\varphi_k f\rangle = 0$. Hence $\varphi_j^* u_{ij}^* u_{ik}\varphi_k = 0$, and so (i) hold.

(ii) For any $f \in H$, Let

$$\|\lambda_{ij} f\|^2 = \frac{\|\varphi_i\|^2\|f\|^2 - \sum_{k\neq j}\|u_{ik}\varphi_k f\|^2}{\|\varphi_j\|^2}. \tag{9}$$

Because $\sum_{j=1}^{m} u_{ij}\varphi_j$ is a Parseval g-frame operator of H on K_i, we have that

$$\|\sum_{j=1}^{m} u_{ij}\varphi_j f\|^2 = \frac{1}{|G|}\|f\|^2. \tag{10}$$

According to (i), we obtain that

$$\|\lambda_{ij} f\|^2 = \frac{\|u_{ij}\varphi_j f\|^2}{\|\varphi_j\|^2}.$$

Thus for any operator φ,

$$\|\lambda_{ij}\varphi f\|^2 = \|u_{ij}\varphi f\|^2.$$

Hence

$$u_{ij}^* u_{ij} = \lambda_{ij} I_{k_j},$$

and so (ii) hold.

(iii) Moreover, since $u_{ij}^* u_{ij} = \lambda_{ij} I_{k_j}$, $u_{ij}^* u_{ik} = 0$ for $j \neq k$, and φ_i is a Parseval g-frame operator of H on K_i, we obtain that,

$$\|\sum_{j=1}^{m} u_{ij}\varphi_j\|^2 = \max_{f\in H} \frac{\|\sum_{j=1}^{m} u_{ij}\varphi_j f\|^2}{\|f\|^2}$$
$$= \sum_{j=1}^{m} \max_{f\in H} \lambda_{ij} \frac{\|\varphi_j f\|^2}{\|f\|^2} = \frac{1}{|G|}\sum_{j=1}^{m}\lambda_{ij} = \frac{1}{|G|}.$$

That is to say,

$$\sum_{j=1}^{m}\lambda_{ij} = 1,$$

and so (iii) hold. □

So far we get the properties of the g-frame operator multiplier. Furthermore, if $\dim(K_1) = \dim(K_2) = \cdots = \dim(K_m)$, then the g-frame operator multiplier U has a more special representation.

COROLLARY 3.1. *Let G be a finite group, $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_m$ be a frame representation of G on $K = K_1 \oplus K_2 \oplus \cdots \oplus K_m$, where $\pi_1, \cdots, \pi_m$ be inequivalent irreducible representations of G, $H' = H \oplus H \oplus \cdots \oplus H$. Let $U = (u_{ij})_{m\times m}$ be a g-frame operator multiplier. If $\dim(K_1) = \dim(K_2) = \cdots = \dim(K_m)$ and there is a Parseval g-frame operator $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_m)$ where φ_i is a reversible operator, then there is a permutation τ of $\{1, \cdots, m\}$ such that $u_{i\tau(i)}$ is unitary and $u_{ij} = 0$ whenever $j \neq \tau(i)$ for all $i = 1, \cdots, m$.*

Proof. Since $\dim(K_i) = \dim(K_j)$, then there is a $u_{ij} \neq 0$, and it is a reversible operator.

Hence for any $k \neq j$, we have

$$(u_{ij}^*)^{-1}(\varphi_j^*)^{-1}\varphi_j^* u_{ij}^* u_{ik}\varphi_k = ((u_{ij}^*)^{-1}(\varphi_j^*)^{-1})\varphi_j^* u_{ij}^* u_{ik}\varphi_k = u_{ik}\varphi_k.$$

According to theorem 3.6, we know that for any $k \neq j$, $\varphi_j^* u_{ij}^* u_{ik}\varphi_k = 0$. So we obtain that

$$u_{ik}\varphi_k = 0.$$

Furthermore, we have that

$$u_{ik}\varphi_k\varphi_k^{-1} = u_{ik} = 0.$$

That is to say there is exactly one nonzero entry in each row of U. And U is unitary, so we also obtain that there is exactly one nonzero entry in each column as well. Hence there is a permutation τ of $\{1, \cdots, m\}$ such that $u_{i\tau(i)}$ is unitary and $u_{ij} = 0$ whenever $j \neq \tau(i)$ for all $i = 1, \cdots, m$. □

Moreover, if $\dim(K_1) < \dim(K_2) < \cdots < \dim(K_m)$, then the g-frame operator multiplier is $U = diag(u_{11}, \cdots, u_{mm})$.

COROLLARY 3.2. *Let G be a finite group, $\pi=\pi_1\oplus\pi_2\oplus\cdots\oplus\pi_m$ be a frame representation of G on $K=K_1\oplus K_2\oplus\cdots\oplus K_m$, where $\pi_1,\ \cdots,\ \pi_m$ be inequivalent irreducible representations of G, $H'=H\oplus H\oplus\cdots\oplus H$. Let $U=(u_{ij})_{m\times m}$ be a g-frame operator multiplier. If* $\dim(K_1)<\dim(K_2)<\cdots<\dim(K_m)$, *then $U=diag(u_{11},\cdots,u_{mm})$.*

The proof of the corollary3.2 is similar to corollary2.8 which in [16].

Next we find that for a g-frame operator multiplier $U=(u_{ij})_{m\times m}$, if there is a Parseval g-frame operator such that U satisfies the conditions of Theorem 3.7, then U has properties different from Therorem3.6. Before introducing Theorem 3.7, we need to introduce a Lemma.

LEMMA 3.1. [16] *Let A be a linear operator on a Hilbert space H. If $\langle Ax,y\rangle=0$ for all $x,y\in H$ with $x\perp y$ and $\|x\|=\|y\|$, then $A=\lambda I$.*

Now we use Lemma 3.1 to proof Theorem 3.7.

THEOREM 3.7. *Let G be a finite group, $\pi=\pi_1\oplus\pi_2\oplus\cdots\oplus\pi_m$ be a frame representation of G on $K=K_1\oplus K_2\oplus\cdots\oplus K_m$, where $\pi_1,\ \cdots,\ \pi_m$ be inequivalent irreducible representations of G, $H'=H\oplus H\oplus\cdots\oplus H$. Let $U=(u_{ij})_{m\times m}$ be a g-frame vector multiplier, which each u_{ij} be an operator from K_j to K_i. If there are Parseval g-frame operators φ_i of H on K_i such that $\varphi_i^*\varphi_j=0$ and $(\sum_{j=1}^m u_{ij}\varphi_j)^*\sum_{j=1}^m u_{i'j}\varphi_j=0$, then $u_{ij}^*u_{ij}$ is a scalar multiple of I whenever $j\neq j'$ or $i=i'$.*

Proof. (i) Assume that $i=i'$.

If $j=j'$, then according to Theorem 3.6, we have $u_{ij}^*u_{ij}=\lambda_{ij}I$. If $j\neq j'$, we know $\varphi_j^*u_{ij}^*u_{ik}\varphi_k=0$.

Thus for any $f\in H$, we obtain that

$$\langle\varphi_j^*u_{ij}^*u_{ik}\varphi_k f,f\rangle=0=\langle u_{ik}\varphi_k f,u_{ij}\varphi_j f\rangle.$$

Since φ_j is a Parseval g-frame operator of H on K_j such $\varphi_k^*\varphi_j=0$, then

$$\varphi_k f\perp\varphi_j f.$$

And

$$\|\varphi_k f\|^2=\frac{1}{|G|}\|f\|^2=\|\varphi_i f\|^2.$$

Hence according to Lemma 3.1, we get that $u_{ik}^*u_{ij}=\lambda I$.

(ii) Assume that $i\neq i'$.

Without loss of generality, we discuss $u_{21}^*u_{12}$. Since

$$(\sum_{j=1}^m u_{ij}\varphi_j)^*\sum_{j=1}^m u_{i'j}\varphi_j=0, \tag{11}$$

then (11) is equal to

$$(u_{21}\varphi_1+u_{22}\varphi_2+z_2)^*(u_{11}\varphi_1+u_{12}\varphi_2+z_1)=0. \tag{12}$$

First use $-\varphi_1$ to replace φ_1 in 12, and subtract the new equation from the above equation we get

$$\varphi_1^* u_{21}^* u_{12}\varphi_2 + \varphi_1^* u_{21}^* z_1 + \varphi_2^* u_{22}^* u_{11}\varphi_1 + z_2^* u_{11}^* \varphi_1 = 0. \tag{13}$$

Then use $-\varphi_2$ to replace φ_1 in 13, and subtract the new equation from the above equation we get

$$\varphi_1^* u_{21}^* u_{12}\varphi_2 + \varphi_2^* u_{22}^* u_{11}\varphi_1 = 0. \tag{14}$$

Now use $i\varphi_1$ to replace φ_1 in 14, and subtract the new equation from the above equation we get

$$\varphi_1^* u_{21}^* u_{12}\varphi_2 = 0. \tag{15}$$

Hence we obtain that for any $f \in H$, $\langle u_{12}\varphi_2 f, u_{21}\varphi_1 f\rangle = 0$.

Since $\varphi_1 f \perp \varphi_2 f$ and $\|\varphi_k f\|^2 = \|\varphi_i f\|^2$, so from lemma 3.1, we get $u_{21}^* u_{12} = \lambda I$. $\square$

4. A g-frame operator multiplier for the general case

In this section, the g-frame operator multiplier for the general case is discussed. More specifically, let $\pi = \pi_1^{m_1} \oplus \pi_2^{m_2} \oplus \cdots \oplus \pi_t^{m_t}$ and $K = K_1^{m_1} \oplus K_2^{m_2} \oplus \cdots \oplus K_t^{m_t}$, $n_j = m_1 + \cdots + m_j (1 \leqslant j \leqslant k)$ and $\Lambda_i = \{n_{j-1}+1, \cdots, n_j\}$. Then we discuss the properties of the Parseval g-frame operator and the g-frame operator multiplier of H' on K in this case.

First of all, similar to Theorem 3.4, we discuss the relationship between the Parseval g-frame operator of H^{m_i} on $K_i^{m_i}$ and the Parseval g-frame operator of H' on K.

THEOREM 4.1. *Let G be a finite group, $\pi = \pi_1^{m_1} \oplus \pi_2^{m_2} \oplus \cdots \oplus \pi_t^{m_t}$ be a frame representation of G on $K = K_1^{m_1} \oplus K_2^{m_2} \oplus \cdots \oplus K_t^{m_t}$, where π_1, $\cdots$, π_t be inequivalent irreducible representations of G, $H' = H^{m_1} \oplus H^{m_2} \oplus \cdots \oplus H^{m_t}$.*

Let $\varphi = (\varphi_{11}, \cdots, \varphi_{1m_1}, \cdots, \varphi_{i1}, \cdots, \varphi_{im_i}, \cdots, \varphi_{t1}, \cdots, \varphi_{tm_t})$. Then the following are equivalent:

(i) For all $i = 1, 2, \cdots, t$, $(\varphi_{i1}, \cdots, \varphi_{im_i})$ is a Parseval g-frame operator of H^{m_i} on $K_i^{m_i}$,

(ii) $\varphi = (\varphi_{11}, \cdots, \varphi_{1m_1}, \cdots, \varphi_{i1}, \cdots, \varphi_{im_i}, \cdots, \varphi_{t1}, \cdots, \varphi_{tm_t})$ is a Parseval g-frame operator of H' on K.

Proof. (i) $\Rightarrow$ (ii) According to theorem 3.4, it is easy to check that if $(\varphi_{i1}, \cdots, \varphi_{im_i})$ is a Parseval g-frame operator of H^{m_i} on $K_i^{m_i}$, then for any $f_i = (f_{i1}, f_{i2}, \cdots, f_{im_i}) \in H^{m_i}$ we have

$$\sum_{g \in G} (\varphi_{i1}^* \varphi_{i1} f_{i1}, \cdots, \varphi_{im_i}^* \varphi_{imi} f_{im_i}) = f_i.$$

Thus for any $f' = (f_1, f_2, \cdots, f_t) \in H$,

$$\sum_{g\in G}(\varphi_{11}^*\varphi_{11}f_{11}, \cdots, \varphi_{1m_1}^*\varphi_{1m1}f_{1m_1}, \cdots, \varphi_{t1}^*\varphi_{t1}f_{t1}, \cdots, \varphi_{tm_t}^*\varphi_{tmt}f_{tm_t})$$
$$= (f_1, f_2, \cdots, f_t) = f',$$

which is equal to φ is a Parseval g-frame operator of H' on K.

(ii) $\Rightarrow$(i) Since φ is a Parseval g-frame operator of H' on K, then for any $f' = (f_1, f_2, \cdots, f_t) \in H$, we have

$$\sum_{g\in G}(\varphi_{11}^*\varphi_{11}f_{11}, \cdots, \varphi_{1m_1}^*\varphi_{1m1}f_{1m_1}, \cdots, \varphi_{t1}^*\varphi_{t1}f_{t1}, \cdots, \varphi_{tm_t}^*\varphi_{tmt}f_{tm_t})$$
$$= (f_1, f_2, \cdots, f_t) = f'.$$

Thus for any $f_i = (f_{i1}, f_{i2}, \cdots, f_{im_i}) \in H^{m_i}$

$$\sum_{g\in G}(\varphi_{i1}^*\varphi_{i1}f_{i1}, \cdots, \varphi_{im_i}^*\varphi_{imi}f_{im_i}) = f_i.$$

Hence $(\varphi_{i1}, \cdots, \varphi_{im_i})$ is a Parseval g-frame operator of H^{m_i} on $K_i^{m_i}$. □

Then similar to Theorem 3.6, we obtain the equations that reflect properties of g-frame operator multiplier $U = (u_{ij})_{m\times m}$ of H' on K.

THEOREM 4.2. *Let G be a finite group, $\pi = \pi_1^{m_1} \oplus \pi_2^{m_2} \oplus \cdots \oplus \pi_t^{m_t}$ be a frame representation of G on $K = K_1^{m_1} \oplus K_2^{m_2} \oplus \cdots \oplus K_t^{m_t}$, where π_1, $\cdots$, π_t be inequivalent irreducible representations of G, $H' = H^{m_1} \oplus H^{m_2} \oplus \cdots \oplus H^{m_t}$. Let $U = (u_{ij})_{m\times m}$ be a unitary operator, where each u_{ij} is an operator from K_j to K_i, $i \in \Lambda_i$ and $j \in \Lambda_j$, $\varphi_j (j \in \Lambda_i)$ be a Parseval g-frame operator of H on K_i. If $U = (u_{ij})_{m\times m}$ is a g-frame operator multiplier, then*

(i) $\varphi_j^* u_{ij}^* u_{ik}\varphi_k = 0$ whenever $j \in \Lambda_j, k \in \Lambda_k$ or $j, k \in \Lambda_i$,
(ii) $u_{ij}^* u_{ij} = \lambda_{ij} I_{k_j}$,
(iii) $\sum_{j=1}^m \lambda_{ij} = 1$.

The proof of the theorem 4.2 is similar to Theorem 3.6, and will not be repeated here.

In the next Theorem, the relationship between a g-frame operator multiplier for H^{m_i} on K_i and a g-frame operator multiplier for H' on K is discussed.

THEOREM 4.3. *Let G be a finite group, $\pi = \pi_1^{m_1} \oplus \pi_2^{m_2} \oplus \cdots \oplus \pi_t^{m_t}$ be a frame representation of G on $K = K_1^{m_1} \oplus K_2^{m_2} \oplus \cdots \oplus K_t^{m_t}$, where π_1, $\cdots$, π_t be inequivalent irreducible representations of G, $H' = H^{m_1} \oplus H^{m_2} \oplus \cdots \oplus H^{m_t}$. Let $U = (u_{ij})_{m\times m}$ be a unitary operator, where each u_{ij} be an operator from K_j to $K_i (i \in \Lambda_i$ and $j \in \Lambda_j)$, $\varphi_j (j \in \Lambda_i)$ be a Parseval g-frame operator of H on K_i. Let $U = (u_{ij})_{m\times m}$ be a g-frame operator multiplier for H' on K, if for any $j \in \Lambda_i$, $\sum_{i\in\Lambda_i} \lambda_{ij} = 1$, then $U' = (u_{ij})_{i,j\in\Lambda_i}$ is a g-frame operator multiplier for H^{m_i} on K_i.*

Proof. Without loss of generality, we discuss whether $U' = (u_{ij})_{i,j\in\{1,2,\cdots m_1\}}$ is a g-frame operator multiplier for H^{m_1} on K_1, and other cases can be proved in the same way.

First of all, let $\varphi = (\varphi_{11},\cdots,\varphi_{1m_1},\cdots,\varphi_{i1},\cdots,\varphi_{im_i},\cdots,\varphi_{t1},\cdots,\varphi_{tm_t})$ be a Parseval g-frame operator of H' on K. Then from Theorem 4.1, we have that $\varphi_1 = (\varphi_{11},\cdots,\varphi_{1m_1})$ is a Parseval g-frame operator of H on K_1. Next we consider $U'\varphi_1$.

Since $\varphi_j^* u_{ij}^* u_{ik}\varphi_k = 0$ whenever $j\in\Lambda_j, k\in\Lambda_k$ or $j,k\in\Lambda_i$, and $u_{ij}^*u_{ij} = \lambda_{ij}I_{k_j}$, thus for any $f = (f_1,f_2,\cdots,f_{m_1})\in H^{m_1}$, we can obtain that

$$\begin{aligned}
&\sum_{g\in G}(\pi_1(g)^{m_1}U'\varphi_1)^*\pi_1(g)^{m_1}U'\varphi_1 f\\
&= |G|(U'\varphi_1)^*U'\varphi_1 f = |G|((\sum_{i=1}^{m_1}\lambda_{i1})\varphi_{11}^*\varphi_{11}f_1,\cdots,(\sum_{i=1}^{m_1}\lambda_{im_1})\varphi_{1m_1}^*\varphi_{1m_1}f_{m_1})\\
&= |G|(\varphi_{11}^*\varphi_{11}f_1,\cdots,\varphi_{1m_1}^*\varphi_{1m_1}f_{m_1}) = (f_1,f_2,\cdots,f_{m_1}) = f.
\end{aligned}$$

That is to say $U'\varphi_1$ is a Parseval g-frame operator of H on K_1. Hence $U' = (u_{ij})_{i,j\in\Lambda_i}$ is a g-frame operator multiplier for H^{m_i} on K_i. □

Finally we find that if $\dim(K_1) = \dim(K_2) = \cdots = \dim(K_t)$, then $U = diag(U_i)_{i=1}^m$ is a g-frame operator multiplier of H' on K if and any if U_i is a g-frame operator multiplier for H^{m_i} on K_i.

THEOREM 4.4. *Let G be a finite group, $\pi = \pi_1^{m_1}\oplus\pi_2^{m_2}\oplus\cdots\oplus\pi_t^{m_t}$ be a frame representation of G on $K = K_1^{m_1}\oplus K_2^{m_2}\oplus\cdots\oplus K_t^{m_t}$, where π_1, $\cdots$, π_t be inequivalent irreducible representations of G and $\dim(K_1) = \dim(K_2) = \cdots = \dim(K_t)$, $H' = H^{m_1}\oplus H^{m_2}\oplus\cdots\oplus H^{m_t}$. Then $U = diag(U_i)_{i=1}^m$ is a g-frame operator multiplier of H' on K if and only if $U_i = (u_{ij})_{j\in\Lambda_i}$ is a g-frame operator multiplier for H^{m_i} on K_i, where each u_{ij} is an operator from K_j to K_i($i\in\Lambda_i$ and $j\in\Lambda_j$).*

Proof. On the one hand, similar to coroally3.2, if $\dim(K_1) = \dim(K_2) = \cdots = \dim(K_t)$, then the g-frame operator multiplier U of H' on K satisfy that $U = diag(U_i)_{i=1}^m$.

According to Theorem 4.2, we know that $u_{ij}^*u_{ij} = \lambda_{ij}I_{k_j}$ and $\sum_{j=1}^m\lambda_{ij} = 1$. Since $u_{ij}(i\in\Lambda_i, j\in\Lambda_j) = 0$, $\lambda_{ij}(i\in\Lambda_i, j\in\Lambda_j) = 0$, then

$$\sum_{i,j\in\Lambda_i}\lambda_{ij} = 1.$$

Then from Theorem 3.5 we obtain that U_i is a g-frame operator multiplier for H^{m_i} on K_i.

On the other hand, if U_i is a g-frame operator multiplier for H^{m_i} on K_i, we consider $U = diag(U_i)_{i=1}^m$.

First of all, let $\varphi_i = (\varphi_{i1},\cdots,\varphi_{im_i})$ is a Parseval g-frame operator of H^{m_i} on $K_i^{m_i}$, $f_i = (f_{i1}, f_{i2},\cdots,f_{im_i})\in H^{m_i}$, then

$$\begin{aligned}
\sum_{g\in G}(\pi(g)U\varphi)^*\pi(g)U\varphi f &= |G|(U\varphi)^*U\varphi f = (|G|(U_i\varphi_i)^*U_i\varphi_i f_i)_{i=1}^t\\
&= (\sum_{g\in G}(\pi_i(g)U_i\varphi_i)^*\pi_i(g)U_i\varphi_i f_i)_{i=1}^t.
\end{aligned}$$

Since U_i is a g-frame operator multiplier for H^{m_i} on K_i and $\varphi_i = (\varphi_{i1}, \cdots, \varphi_{im_i})$ is a Parseval g-frame operator of H^{m_i} on $K_i^{m_i}$, we have that

$$(\sum_{g\in G}(\pi_i(g)U_i\varphi_i)^*\pi_i(g)U_i\varphi_i f_i)_{i=1}^t = (f_i)_{i=1}^t = f.$$

That is imply that $U\varphi$ is a Parseval g-frame operator of H' on K. Hence U is a g-frame operator multiplier of H' on K. □

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