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NECESSARY AND SUFFICIENT CONDITIONS FOR A MIXED BIVARIATE MEAN OF THREE PARAMETERS TO BE SCHUR m–POWER CONVEX

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This paper is dedicated to Professor Bai-Ni Guo for her retirement in August 2024

(*Communicated by I. Perić*)

Abstract. In the work, by virtue of some techniques in the theory of majorization, the authors find necessary and sufficient conditions for a mixed bivariate mean of three parameters to be Schur m-power convex.

1. Preliminaries

We recall two fundamental definitions in the theory of majorization.

DEFINITION 1. ([5, 13]) Let $\boldsymbol{x}=(x_1,x_2,\dots,x_n)$ and $\boldsymbol{y}=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$.

1. The n-tuple $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$, in symbols $\boldsymbol{x}\prec\boldsymbol{y}$, if

$$\sum_{i=1}^{k}x_{[i]}\leqslant\sum_{i=1}^{k}y_{[i]},\quad 1\leqslant k\leqslant n-1\quad\text{and}\quad\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}y_i,$$

where $x_{[1]}\geqslant x_{[2]}\geqslant\cdots\geqslant x_{[n]}$ and $y_{[1]}\geqslant y_{[2]}\geqslant\cdots\geqslant y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in descending order.

2. A set $\Omega\subseteq\mathbb{R}^n$ is called to be convex if

$$(\alpha x_1+\beta y_1,\alpha x_2+\beta y_2,\dots,\alpha x_n+\beta y_n)\in\Omega$$

for any $\boldsymbol{x}$ and $\boldsymbol{y}\in\Omega$, where $\alpha,\beta\in[0,1]$ such that $\alpha+\beta=1$.

3. A function $\varphi:\Omega\to\mathbb{R}$ is said to be Schur-convex (or Schur-concave, respectively) if the majorizing relation $\boldsymbol{x}\prec\boldsymbol{y}$ on Ω implies the inequality $\varphi(\boldsymbol{x})\lesseqgtr\varphi(\boldsymbol{y})$.

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REMARK 1. In the papers [3, 4, 6, 8, 9, 11, 12, 16, 18, 25], for example, there have been many results on investigations of the Schur-convexity and related ones.

DEFINITION 2. ([21, 22, 23]) Let $f:\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}\dfrac{x^m-1}{m}, & m\ne 0;\\ \ln x, & m=0.\end{cases} \tag{1}$$

A function $\varphi:\Omega\subseteq\mathbb{R}_+^n\to\mathbb{R}$ is said to be Schur m-power convex (or Schur m-power concave, respectively) on Ω if the majorizing relation

$$f(\boldsymbol{x})=(f(x_1),f(x_2),\dots,f(x_n))\prec f(\boldsymbol{y})=(f(y_1),f(y_2),\dots,f(y_n))$$

on Ω implies the inequality $\varphi(\boldsymbol{x})\lesseqgtr\varphi(\boldsymbol{y})$.

REMARK 2. When taking $m=1,\ 0,\ -1$, or say, when putting $f(x)=x$, $\ln x$, $\frac{1}{x}$, in Definition 2, we can derive definitions of the Schur-convexity (see [5, 13], for example), the Schur-geometric convexity (see [2, 27], for example), and the Schur-harmonic convexity (see [1, 19, 20], for example), respectively.

REMARK 3. In the papers [15, 17, 24, 26], for example, there have been many results on investigations of the Schur m-convexity and related ones.

REMARK 4. The function $f(x)$ defined by (1) has an integral representation

$$f(x)=\int_1^x u^{m-1}\,\mathrm{d}u.$$

This integral representation is a special case of the function

$$h_{a,b}(x)=\int_a^b u^{x-1}\,\mathrm{d}u=\begin{cases}\dfrac{b^x-a^x}{x}, & x\ne 0\\ \ln b-\ln a, & x=0\end{cases}$$

for $b>a>0$ and $x\in\mathbb{R}$, that is, $h_{1,x}(m)=f(x)$. The function $h_{a,b}(x)$ was investigated and applied in over one hundred of works by Qi and his coauthors since 1997; see the last paragraph in [7, p. 13], the paper [10], and [17, Remark 1], for example. Definition 2 is a special case of the definition of the so-called Schur f-convex functions introduced in [21, 22, 23] by Yang.

2. Motivations and main results

In the paper [14], Wang, Shi, and Fu defined the mixed mean

$$W_n(\boldsymbol{x};\omega_1,\omega_2;p)=\begin{cases}\left[\dfrac{\omega_1H_n(\boldsymbol{x}^p)+\omega_2G_n(\boldsymbol{x}^p)}{\omega_1+\omega_2}\right]^{1/p}, & 0\leqslant\omega_1,\omega_2<\infty\\ [H_n(\boldsymbol{x}^p)]^{1/p}, & \omega_1=\infty\\ G_n(\boldsymbol{x}), & \omega_2=\infty\quad\text{or}\quad p=0\end{cases} \tag{2}$$

for $n \geqslant 2$, $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$, $p \in \mathbb{R}$, and $\omega_1, \omega_2 \geqslant 0$ such that $\omega_1 + \omega_2 \neq 0$, where

$$H_n(\boldsymbol{x}^p) = \frac{n}{\sum_{i=1}^n \frac{1}{x_i^p}} \quad \text{and} \quad G_n(\boldsymbol{x}^p) = \prod_{i=1}^n x_i^{p/n} = G_n^p(\boldsymbol{x})$$

are the harmonic mean and geometric mean of $\boldsymbol{x}^p = (x_1^p, x_2^p, \dots, x_n^p)$, respectively. Hereafter, three authors studied the Schur convexity, the Schur-geometric convexity, and the Schur-harmonic convexity of the mixed mean $W_n(\boldsymbol{x}; \omega_1, \omega_2; p)$, which are recited as follows.

THEOREM 1. ([14, Theorem 1.2]) *Let $n \geqslant 2$, $\boldsymbol{x} \in \mathbb{R}_+^n$, $p \in \mathbb{R}$, and $\omega_1, \omega_2 \geqslant 0$ such that $\omega_1 + \omega_2 \neq 0$.*

1. *If $p \geqslant -1$, the mixed mean $W_n(\boldsymbol{x}; \omega_1, \omega_2; p)$ is Schur-concave in $\boldsymbol{x} \in \mathbb{R}_+^n$.*
2. *If $p \geqslant 0$, the mixed mean $W_n(\boldsymbol{x}; \omega_1, \omega_2; p)$ is Schur-geomertrically concave in $\boldsymbol{x} \in \mathbb{R}_+^n$.*
3. *If $p < 0$, the mixed mean $W_n(\boldsymbol{x}; \omega_1, \omega_2; p)$ is Schur-geomertrically convex in $\boldsymbol{x} \in \mathbb{R}_+^n$.*
4. *If $p \leqslant 1$, the mixed mean $W_n(\boldsymbol{x}; \omega_1, \omega_2; p)$ is Schur-harmonically convex in $\boldsymbol{x} \in \mathbb{R}_+^n$.*

THEOREM 2. ([14, Theorem 1.4]) *Let $n \geqslant 2$, $\boldsymbol{x} \in \mathbb{R}_+^n$, $p \in \mathbb{R}$, and $\omega_1, \omega_2, \omega_2^* \geqslant 0$ such that $\omega_2 \geqslant \omega_2^*$ and $\omega_1 + \omega_2^* \neq 0$.*

1. *If $p \leqslant -1$, then the ratio $\frac{W_n(\boldsymbol{x};\omega_1,\omega_2;p)}{W_n(\boldsymbol{x};\omega_1,\omega_2^*;p)}$ is Schur-concave in $\boldsymbol{x} \in \mathbb{R}_+^n$.*
2. *If $p \leqslant 0$, then the ratio $\frac{W_n(\boldsymbol{x};\omega_1,\omega_2;p)}{W_n(\boldsymbol{x};\omega_1,\omega_2^*;p)}$ is Schur-geomertrically concave in $\boldsymbol{x} \in \mathbb{R}_+^n$.*
3. *If $p \geqslant 0$, then the ratio $\frac{W_n(\boldsymbol{x};\omega_1,\omega_2;p)}{W_n(\boldsymbol{x};\omega_1,\omega_2^*;p)}$ is Schur-geomertrically convex in $\boldsymbol{x} \in \mathbb{R}_+^n$.*
4. *If $p \geqslant 1$, then the ratio $\frac{W_n(\boldsymbol{x};\omega_1,\omega_2;p)}{W_n(\boldsymbol{x};\omega_1,\omega_2^*;p)}$ is Schur-harmonically convex in $\boldsymbol{x} \in \mathbb{R}_+^n$.*

In order to study the Schur m-power convexity of the mixed mean in (2) for $n = 2$, we reformulate the definition of the mixed mean of two variables and three parameters as

$$W_2(a,b;\omega_1,\omega_2;p) = \begin{cases} \left[\dfrac{\omega_1 H(a^p,b^p) + \omega_2 G(a^p,b^p)}{\omega_1 + \omega_2}\right]^{1/p}, & p \neq 0 \\ G(a,b), & p = 0 \end{cases} \tag{3}$$

for $(a,b) \in \mathbb{R}_+^2$, $p \in \mathbb{R}$, and $\omega_1, \omega_2 \in \mathbb{R}_0 = [0,\infty)$ such that $\omega_1 + \omega_2 \neq 0$.

The main aim of this paper is to find necessary and sufficient conditions for the mixed mean $W_2(a,b;\omega_1,\omega_2;p)$ to be Schur m-power convex with respect to $(a,b) \in \mathbb{R}_+^2$ for $p \in \mathbb{R}$ and $\omega_1, \omega_2 \in \mathbb{R}_0$ such that $\omega_1 + \omega_2 \neq 0$.

Our main results are stated in the following three theorems.

THEOREM 3. *For fixed $p\in\mathbb{R}$, $m=0$, and $\omega_1,\omega_2\in\mathbb{R}_0$ such that $\omega_1+\omega_2\neq 0$, the mixed mean $W_2(a,b;\omega_1,\omega_2;p)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $p\leqslant 0$ (or $p\geqslant 0$, respectively) for $\omega_1,\omega_2\in\mathbb{R}_0$ such that $\omega_1+\omega_2\neq 0$.*

THEOREM 4. *For fixed $p\in\mathbb{R}$, $m>0$, and $\omega_1,\omega_2\in\mathbb{R}_0$ such that $\omega_1+\omega_2\neq 0$, the mixed mean $W_2(a,b;\omega_1,\omega_2;p)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $(p;\omega_1,\omega_2)\in A_1$ (or $(p;\omega_1,\omega_2)\in A_2$, respectively), where*

$$A_1=\{(p;\omega_1,\omega_2):m+p\leqslant 0,\omega_1\in\mathbb{R}_0,\omega_2=0\}$$

and

$$A_2=\{(p;\omega_1,\omega_2):p\in\mathbb{R},\omega_1,\omega_2\in\mathbb{R}_0,(m+p)\omega_1+m\omega_2\geqslant 0\}.$$

THEOREM 5. *For fixed $p\in\mathbb{R}$, $m<0$, and $\omega_1,\omega_2\in\mathbb{R}_0$ such that $\omega_1+\omega_2\neq 0$, the mixed mean $W_2(a,b;\omega_1,\omega_2;p)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $(p;\omega_1,\omega_2)\in B_1$ (or $(p;\omega_1,\omega_2)\in B_2$, respectively), where*

$$B_1=\{(p;\omega_1,\omega_2):p\in\mathbb{R},\omega_1,\omega_2\in\mathbb{R}_0,(m+p)\omega_1+m\omega_2\geqslant 0\}$$

and

$$B_2=\{(p;\omega_1,\omega_2):m+p\geqslant 0,\omega_1\in\mathbb{R}_0,\omega_2=0\}.$$

In this paper, we will use Yang's method in [21, 22, 23] to prove the above Theorems 3 to 5.

3. Lemmas

For proving the above three theorems, we need the following lemmas.

LEMMA 1. ([21, 22, 23]) *Let $\Omega\subset\mathbb{R}_+^n$ be a symmetric set such that $\Omega^\circ\neq\emptyset$ and let the function $\varphi:\Omega\to\mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex (or Schur m-power concave, respectively) on Ω if and only if φ is symmetric on Ω and the inequalities*

$$\frac{x_1^m-x_2^m}{m}\left[x_1^{1-m}\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1}-x_2^{1-m}\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right]\gtreqless 0,\quad m\neq 0$$

and

$$(\ln x_1-\ln x_2)\left[x_1\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1}-x_2\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right]\gtreqless 0,\quad m=0$$

are valid for $\boldsymbol{x}\in\Omega^\circ$.

REMARK 5. If letting $m = 1, 0, -1$ in Lemma 1 respectively, we can deduce criteria theorems for the Schur-convexity (see [5, 13]), for the Schur-geometric convexity (see [2, 27]), and for the Schur-harmonic convexity (see [1, 19, 20]), respectively.

LEMMA 2. *For fixed $p, m \in \mathbb{R}$ with $m \neq 0$, the mean $W_2(a,b;\omega_1,\omega_2;p)$ given by (3) is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $\varphi(t) \geqslant 0$ (or $\varphi(t) \leqslant 0$, respectively) for all $t > 0$, where*

$$\begin{aligned}\varphi(t) = &-8\omega_1 \sinh[(m+p)t] \\ &-2\omega_2\{\sinh[(m+2p)t] + 2\sinh(mt) + \sinh[(m-2p)t]\}.\end{aligned} \tag{4}$$

Proof. Without loss of generality, we assume $a > b$.

If $p \neq 0$, by the equation (3), we obtain the partial derivatives

$$\frac{\partial W_2(a,b;\omega_1,\omega_2;p)}{\partial a} = \frac{\Lambda}{a}\left[4\omega_1 a^{p/2} b^{3p/2} + \omega_2(a^p + b^p)^2\right] \tag{5}$$

and

$$\frac{\partial W_2(a,b;\omega_1,\omega_2;p)}{\partial b} = \frac{\Lambda}{b}\left[4\omega_1 a^{3p/2} b^{p/2} + \omega_2(a^p + b^p)^2\right], \tag{6}$$

where

$$\Lambda = \frac{(ab)^{p/2}[W_2(a,b;\omega_1,\omega_2;p)]^{1-p}}{2(\omega_1+\omega_2)(a^p+b^p)^2}.$$

Employing Lemma 1 and utilizing the expressions (5) and (6), we acquire

$$\begin{aligned}\Delta_{p,m} &= \frac{a^m - b^m}{m}\left[a^{1-m}\frac{\partial W_2(a,b;\omega_1,\omega_2;p)}{\partial a} - b^{1-m}\frac{\partial W_2(a,b;\omega_1,\omega_2;p)}{\partial b}\right] \\ &= -\frac{(a^m-b^m)\Lambda}{m(ab)^m}\left[4\omega_1(ab)^{p/2}(a^{p+m} - b^{p+m}) + \omega_2(a^p+b^p)^2(a^m-b^m)\right] \\ &= -\frac{(a^m-b^m)\Lambda}{m(ab)^{m/2-p}}\left\{4\omega_1\left[\left(\frac{a}{b}\right)^{(m+p)/2} - \left(\frac{a}{b}\right)^{-(m+p)/2}\right]\right. \\ &\quad \left. + \omega_2\left[\left(\frac{a}{b}\right)^{p/2} + \left(\frac{a}{b}\right)^{-p/2}\right]^2\left[\left(\frac{a}{b}\right)^{m/2} - \left(\frac{a}{b}\right)^{-m/2}\right]\right\}.\end{aligned}$$

Putting $\ln\sqrt{\frac{a}{b}} = t$ and considering the definitions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$, we deduce

$$\Delta_{p,m} = \frac{(a^m - b^m)\Lambda}{m(ab)^{m/2-p}}\varphi(t). \tag{7}$$

It is clear that $W_2(a,b;\omega_1,\omega_2;0)=G(a,b)$ for $(a,b)\in\mathbb{R}_+^2$. By virtue of Lemma 1, we have

$$\begin{aligned}\Delta_{0,m}&=\frac{a^m-b^m}{m}\left[a^{1-m}\frac{\partial W_2(a,b;\omega_1,\omega_2;0)}{\partial a}-b^{1-m}\frac{\partial W_2(a,b;\omega_1,\omega_2;0)}{\partial b}\right]\\&=\frac{(a^m-b^m)\sqrt{ab}}{2m}\left(\frac{1}{a^m}-\frac{1}{b^m}\right)\\&=-\frac{(a^m-b^m)(ab)^{(1-m)/2}}{m}\sinh(mt).\end{aligned}$$

It is easy to verify that $\lim_{p\to0}\Delta_{p,m}=\Delta_{0,m}$. Consequently, the formula (7) holds for all $p\in\mathbb{R}$.

Since $a>b$, the positivity $\frac{(a^m-b^m)\Lambda}{m(ab)^{m/2-p}}>0$ is valid. By Lemma 1, we see that the mixed mean $W_2(a,b;\omega_1,\omega_2;p)$ is Schur m-power convex (or Schur m-power concave, respectively) with respect to $(a,b)\in\mathbb{R}_+^2$ if and only if $\varphi(t)\gtreqless 0$ for all $t>0$. The proof of Lemma 2 is thus complete. □

LEMMA 3. *Let $\varphi(t)$ be given by* (4). *Then*

$$\lim_{t\to0^+}\frac{\varphi(t)}{t}=\lim_{t\to0^+}\varphi'(t)=-8[\omega_1(m+p)+\omega_2m].$$

Proof. Since $\varphi(0)=0$, applying the L'Hospital rule yields

$$\begin{aligned}\lim_{t\to0^+}\frac{\varphi(t)}{t}&=\lim_{t\to0^+}\varphi'(t)\\&=-8\omega_1(m+p)-2\omega_2(m+2p+2m+m-2p)\\&=-8[\omega_1(m+p)+\omega_2m].\end{aligned}$$

The proof of Lemma 3 is thus complete. □

LEMMA 4. *Let $m,p\in\mathbb{R}$ with $m\ne0$, $\beta=\max\{|m+2p|,|m-2p|\}$, and $\varphi(t)$ be given by* (4).

1. *When $m>0$,*
   (a) $\lim_{t\to\infty}\frac{\varphi(t)}{e^{\beta t}}\geqslant0$ *if and only if $\omega_1\in\mathbb{R}_+$, $\omega_2=0$, and $p\ne0$;*
   (b) $\lim_{t\to\infty}\frac{\varphi(t)}{e^{\beta t}}\leqslant0$ *if and only if $\omega_1,\omega_2\in\mathbb{R}_0$ and $p\in\mathbb{R}$.*
2. *When $m<0$,*
   (a) $\lim_{t\to\infty}\frac{\varphi(t)}{e^{\beta t}}\geqslant0$ *if and only if $\omega_1,\omega_2\in\mathbb{R}_0$ and $p\in\mathbb{R}$;*
   (b) $\lim_{t\to\infty}\frac{\varphi(t)}{e^{\beta t}}\leqslant0$ *if and only if $\omega_1\in\mathbb{R}_+$, $\omega_2=0$, and $p\ne0$.*

Proof. Using the expression (4) and the L'Hospital rule, we obtain

$$\begin{aligned}\lim_{t\to\infty}\frac{2\beta\varphi(t)}{e^{\beta t}}&=\lim_{t\to\infty}\frac{2\varphi'(t)}{e^{\beta t}}\\&=-4\lim_{t\to\infty}\frac{1}{e^{\beta t}}\{4\omega_1(m+p)\cosh[(p+m)t]\\&\quad+\omega_2[(m+2p)\cosh((m+2p)t)\\&\quad+2m\cosh(mt)+(m-2p)\cosh((m-2p)t)]\}\\&=\begin{cases}-8(\omega_1+\omega_2)m, & p=0,m\neq 0;\\-2\omega_2(m+2p), & mp>0;\\-2\omega_2(m-2p), & mp<0.\end{cases}\end{aligned}$$

When $m>0$, we obtain

$$\lim_{t\to\infty}\frac{2\beta\varphi(t)}{e^{\beta t}}=\begin{cases}-8(\omega_1+\omega_2)m<0, & p=0;\\-2\omega_2(m+2|p|)\leqslant 0, & p\neq 0.\end{cases}$$

Thus, we verified the necessary and sufficient condition that $\lim_{t\to\infty}\frac{\beta\varphi(t)}{e^{\beta t}}\geqslant 0$ if and only if $\omega_1\in\mathbb{R}_+$, $\omega_2=0$, and $p\neq 0$.

When $m<0$, we derive

$$\lim_{t\to\infty}\frac{2\beta\varphi(t)}{e^{\beta t}}=\begin{cases}-8(\omega_1+\omega_2)m>0, & p=0;\\2\omega_2(|m|+2|p|)\geqslant 0, & p\neq 0.\end{cases}$$

Hence, we confirmed the necessary and sufficient condition that $\lim_{t\to\infty}\frac{\beta\varphi(t)}{e^{\beta t}}\leqslant 0$ if and only if $\omega_1\in\mathbb{R}_+$, $\omega_2=0$, and $p\neq 0$. The proof of Lemma 4 is complete. □

4. Proofs of main results

Now we start out to prove our main results.

Proof of Theorem 3. For $m=0$, by Lemma 2, we need to prove the necessary and sufficient conditions that $\varphi(t)\gtreqless 0$ for all $t>0$ if and only if $p\lesseqgtr 0$ for $\omega_1,\omega_2\in\mathbb{R}_0$ with $\omega_1+\omega_2\neq 0$.

When $m=0$, we see immediately that $\varphi(t)=-8\omega_1\sinh(pt)\gtreqless 0$ for all $t>0$ if and only if $p\lesseqgtr 0$ for $\omega_1,\omega_2\in\mathbb{R}_0$ with $\omega_1+\omega_2\neq 0$. The proof of Theorem 3 is thus complete. □

Proof of Theorem 4. For $m>0$, according to Lemma 2, we need to prove the necessary and sufficient conditions that $\varphi(t)\gtreqless 0$ for all $t>0$ if and only if $(p;\omega_1,\omega_2)\in A_1$ (or $(p;\omega_1,\omega_2)\in A_2$, respectively). We now prove the necessary and sufficient conditions by splitting into two cases.

1. Using Lemma 2, we see that the positivity $\varphi(t) \geqslant 0$ for all $t > 0$ is valid if and only if
$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} \geqslant 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\beta\varphi(t)}{e^{\beta t}} \geqslant 0. \tag{8}$$
Further, by Lemma 3 and the first item in Lemma 4, the inequalities in (8) are equivalent to
$$\begin{cases} -8\left[\omega_1(m+p)+\omega_2 m\right] \geqslant 0; \\ \omega_2 = 0, \omega_1 \in \mathbb{R}_+, p \neq 0, \end{cases}$$
that is, they are equivalent to the conditions $\omega_2 = 0$, $\omega_1 \in \mathbb{R}_+$, and $m+p \leqslant 0$. Therefore, the inequalities in (8) are equivalent to
$$(p;\omega_1,\omega_2) \in A_1 = \{(p;\omega_1,\omega_2) : p \in \mathbb{R}, \omega_1 \in \mathbb{R}_0, m+p \leqslant 0, \omega_2 = 0\}.$$

2. From Lemma 2, we see that the negativity $\varphi(t) \leqslant 0$ for all $t > 0$ is valid if and only if
$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} \leqslant 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\beta\varphi(t)}{e^{\beta t}} \leqslant 0. \tag{9}$$
By Lemma 3 and the first item in Lemma 4, we see that the inequalities in (9) are equivalent to
$$\begin{cases} -8\left[\omega_1(m+p)+\omega_2 m\right] \leqslant 0; \\ \omega_1, \omega_2 \in \mathbb{R}_0, \quad p \in \mathbb{R}, \end{cases}$$
that is, the positivity $\omega_1(m+p)+\omega_2 m \geqslant 0$ is valid. Therefore, the inequalities in (9) are equivalent to
$$(p;\omega_1,\omega_2) \in A_2 = \{(p;\omega_1,\omega_2) : p \in \mathbb{R}, \omega_1,\omega_2 \in \mathbb{R}_0, \omega_1(m+p)+\omega_2 m \geqslant 0\}.$$

The proof of Theorem 4 is thus finished. □

Similar to the proof of Theorem 4, the proof of Theorem 5 is given as follows.

Proof of Theorem 5. For $m < 0$, using Lemma 2, we need to prove the necessary and sufficient conditions that $\varphi(t) \gtreqless 0$ for all $t > 0$ if and only if $(p;\omega_1,\omega_2) \in B_1$ (or $(p;\omega_1,\omega_2) \in B_2$, respectively). We now prove the necessary and sufficient conditions by dividing into the following two steps.

1. Using Lemma 2 to the second item in Lemma 4, we see that the positivity $\varphi(t) \geqslant 0$ is valid for all $t > 0$ if and only if
$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} \geqslant 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\beta\varphi(t)}{e^{\beta t}} \geqslant 0. \tag{10}$$
The inequalities in (10) are equivalent to
$$\begin{cases} -8\left[\omega_1(m+p)+\omega_2 m\right] \geqslant 0; \\ \omega_1, \omega_2 \in \mathbb{R}_0, \quad p \in \mathbb{R}, \end{cases}$$
simply speaking, $\omega_1(m+p)+\omega_2 m \leqslant 0$.

2. From Lemma 2 to the second item in Lemma 4, we see that the positivity $\varphi(t) \leqslant 0$ is valid for all $t > 0$ if and only if

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} \leqslant 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\beta\varphi(t)}{e^{\beta t}} \leqslant 0. \tag{11}$$

The inequalities (11) are equivalent to

$$\begin{cases} -8\left[\omega_1(m+p)+\omega_2 m\right] \leqslant 0; \\ \omega_2 = 0, \quad \omega_1 \in \mathbb{R}_+, \quad p \neq 0, \end{cases}$$

in other words, they are equivalent to $\omega_1 \in \mathbb{R}_+$, $\omega_2 = 0$, and $m+p \geqslant 0$.

The proof of Theorem 5 is complete. □

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