Mathematical
Inequalities
& Applications
Volume 28, Number 1 (2025), 1–18 doi:10.7153/mia-2025-28-01

APPLICATIONS OF UNITARILY DIAGONALIZABLE MATRICES IN AN INDEFINITE INNER PRODUCT SPACE TO MATRIX PARTIAL ORDERS

K. KAMARAJ AND A. KARPAGAM*

(*Communicated by G. P. H. Styan*)

Abstract. Necessary and sufficient conditions for the unitary diagonalization of normal matrices in an indefinite inner product space are given. As an application of unitary diagonalization, several new characterizations of the star partial order with respect to an indefinite inner product are established. The concepts of diamond order, space pre-order, and plus order are studied in the indefinite setting. Some relations among these matrix partial orders are proved.

1. Introduction

The theory of matrix partial orderings has garnered significant attention from mathematicians over the past several decades [1], [6], [12], [17] and [20]. These orders play an important role in the study of shorted operators, which have applications in electrical networks [19]. However, in the indefinite inner product space (IIPS), the study of matrix partial orders have not received as much attention. Stanisev introduced the concept of the star partial order in an IIPS and proved some of its properties in [21]. The aim of this paper is to introduce additional matrix partial orders, such as left and right star partial orders, diamond, and plus matrix partial orders in IIPSs, and to characterize them.

The following notations are used throughout this paper. Let $\mathbb{C}^{m\times n}$ be the set of complex $m\times n$ matrices. For any matrix $A\in\mathbb{C}^{m\times n}$, A^*, $\mathscr{R}(A)$, $\mathscr{N}(A)$ and $r(A)$ denote the conjugate transpose, the range, null space and rank of A, respectively. A square matrix A is called a projection if $A=A^2$ and an orthogonal projection if A is a projection and Hermitian. The symbol I_n stands for the identity matrix of order n. The theory of generalized inverses is essential in the development of matrix partial orderings. For every matrix $A\in\mathbb{C}^{m\times n}$, the unique matrix $X\in\mathbb{C}^{n\times m}$ such that $AXA=A$, $XAX=X$, $(AX)^*=AX$ and $(XA)^*=XA$ is called the Moore-Penrose inverse of A and it is denoted by $A^\dagger$ [4].

We recall the definitions of some matrix partial orders in Euclidean settings, which will be extended to IIPS in this paper:

Mathematics subject classification (2020): 15A09, 15A23, 15A45, 46C20.

Keywords and phrases: Indefinite inner product, Moore-Penrose inverse, unitary diagonalization, matrix partial order, star partial order.

* Corresponding author.

Paper MIA-28-01

- Star order: $A \overset{*}{\leqslant} B$ if $A^*A = A^*B$ and $AA^* = BA^*$, where $A,\ B \in \mathbb{C}^{m\times n}$ [8].
- Left star order: $A* \leqslant B$ if $A^*A = A^*B$ and $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ [3].
- Right star order: $A \leqslant *B$ if $AA^* = BA^*$ and $\mathscr{R}(A^*) \subseteq \mathscr{R}(B^*)$ [3].
- Minus order: $A \overset{-}{\leqslant} B$ if $A^{(1)}A = A^{(1)}B$ and $AA^{(1)} = BA^{(1)}$ [10].
- Space order: $A \overset{S}{\leqslant} B$ if $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $\mathscr{R}(A^*) \subseteq \mathscr{R}(B^*)$ [18].
- Diamond order: $A \overset{\diamond}{\leqslant} B$ if $A \overset{S}{\leqslant} B$ and $AA^*A = AB^*A$ [2].
- Plus order: $A \overset{+}{\leqslant} B$ if $A \overset{S}{\leqslant} B$ and there are orthogonal projections $\bar{Q}$ and Q such that $A = \bar{Q}BQ$ [17].

The rest of the paper is organized as follows: In Section 2, we provide basic definitions and some known results of indefinite inner product spaces and generalized inverses of matrices in IIPSs. In Section 3, we present the notion of unitary diagonalization for N-normal and N-Hermitian matrices. As an application of unitary diagonalization, we prove some characterizations of matrix partial orderings in Section 4. Finally, we extend the concepts of space, diamond, and plus orderings to the indefinite setting.

2. Preliminary concepts of indefinite inner product space

An indefinite inner product in $\mathbb{C}^n$ is a conjugate symmetric sesquilinear form $[x,y]$ which satisfies the regularity condition: $[x,y] = 0,\ \forall\, y \in \mathbb{C}^n$ which holds only when $x = 0$. Any indefinite inner product is associated with a unique invertible Hermitian matrix N with complex entries such that $[x,y] = \langle x, Ny\rangle$, where $\langle .,.\rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$. Such a matrix N is called a weight. A space with an indefinite inner product is called an indefinite inner product space (IIPS). The study of linear transformations over indefinite inner product space has received considerable attention over the past decades; see for instance [5], [9], [11] and the references cited therein.

We call u and v orthogonal if $[u,v] = 0$, where $u, v \in \mathbb{C}^n$. Let M and N be weights of order m and n, respectively. The *MN-adjoint* of an $m\times n$ matrix A denoted $A^{[*]}$ is defined by $A^{[*]} = N^{-1}A^*M$. We call a square matrix A is N-Hermitian, N-normal N-unitary and N-orthogonal projection if $A^{[*]} = A$, $AA^{[*]} = A^{[*]}A$, $AA^{[*]} = A^{[*]}A = I_n$ and $A = A^2 = A^{[*]}$, respectively.

The Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ in an IIPS is defined as the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four equations:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^{[*]} = AX \quad \text{and} \quad (4)\ (XA)^{[*]} = XA.$$

Such X is denoted by $A^{[\dagger]}$. A matrix X is said to be $\{1\}$-inverse of A if it satisfies the first Penrose equation and is denoted by $A^{(1)}$. The set of all $\{1\}$-inverses of A is denoted by $A\{1\}$. The sets $A\{1,2\}$, $A\{1,2,3\}$ and $A\{1,2,4\}$ are defined in similar manner. Unlike the Euclidean case, not all matrices have the Moore-Penrose inverse. The necessary and sufficient conditions for the existence of the Moore-Penrose inverse in an IIPS is $r(A) = r(AA^{[*]}) = r(A^{[*]}A)$. Also, $A^{[\dagger]}$ exists if and only if $\mathscr{R}(A)$ and $\mathscr{N}(A^{[*]})$ are orthogonal complementary subspaces [21]. More properties of Moore-Penrose inverse in an IIPS can found in [15].

The next theorem gives sufficient condition for the existence of reverse order law for Moore-Penrose inverse in an IIPS.

THEOREM 1. ([14], Theorem 3.5) *Let A and B be complex matrices of appropriate order such that $A^{[\dagger]}$ and $B^{[\dagger]}$ exist. If $\mathscr{R}(A^{[*]}AB) \subseteq \mathscr{R}(B)$ and $\mathscr{R}(BB^{[*]}A^{[*]}) \subseteq \mathscr{R}(A^{[*]})$, then $(AB)^{[\dagger]} = B^{[\dagger]}A^{[\dagger]}$.*

DEFINITION 1. A complex square matrix A is said to have the spectral property if $(A-\lambda I)^{[\dagger]}$ exists for all eigenvalues λ of A.

THEOREM 2. ([16], Theorem 3.18) *Let A be a linear operator on a real or complex finite dimensional vector space $\mathbf{V}$ and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of A. Then A is N-normal and A has a spectral property if and only if there exist N-orthogonal projections $E_1, E_2, \cdots, E_k$ on $\mathbf{V}$ such that:*

(i) $A = \lambda_1 E_1 + \cdots + \lambda_k E_k$.

(ii) $I = E_1 + \cdots + E_k$.

(iii) $E_i E_j = \mathbf{0}$, $i \neq j$.

COROLLARY 1. ([16], Corollary 3.24) *Let A be a linear operator on a finite dimensional real or complex vector space $\mathbf{V}$. Then A is N-Hermitian and has spectral property if and only if there exists distinct real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ and N-orthogonal projections $E_1, E_2, \ldots, E_k$ such that conditions (i)–(iii) of Theorem 2 hold.*

3. Unitary diagonalization

The concept of singular value decomposition (SVD) for a matrix in an IIPS was studied by Hassi [11]. Utilizing the notion of SVD, the unitary diagonalization for N-normal matrices was obtained in [13]. For completeness, we provide some basic definitions and results on unitary diagonalization in this section. More detailed properties and results can be found in [13].

A linear subspace $\mathscr{W}$ of a complex indefinite inner product space $\mathbf{V}$ is non-degenerate, if the condition $x \in \mathscr{W}$, $[x,y] = 0$ for every $y \in \mathscr{W}$ implies $x = 0$. Clearly, $\mathscr{W}$ is non-degenerate if and only if $\mathscr{W}^{[\perp]}$ is a direct complement to $\mathscr{W}$, i.e., $\mathscr{W}^{[\perp]} \cap \mathscr{W} = \{0\}$. Consequently, for non-degenerate linear subspace $\mathscr{W}$, and only for them, one can construct an $[.,.]$-orthonormal basis, i.e., a basis $\{x^1, x^2, \ldots, x^s\}$, $s = \dim \mathscr{W}$ of $\mathscr{W}$ satisfying:

$$[x^i, x^j] = \begin{cases} \pm 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 1. Let E be the N-orthogonal projection onto a linear subspace $R(E)$. Then it is easy to verify that $E^{[\dagger]} = E$. Thus, $\mathbb{C}^n = \mathscr{R}(E) \oplus \mathscr{N}(E)$. Let $x \in \mathscr{R}(E)$ and suppose $[x,y] = 0$ for all $y \in \mathscr{R}(E)$. Then for all $z \in \mathbb{C}^n$, we have $[x,z] = [x, z_1 + z_2]$ $= [x,z_1] + [x,z_2] = 0$. This implies $x = 0$. Therefore, $R(E)$ is non degenerate.

DEFINITION 2. A complex square matrix A is called N-unitarily diagonalizable if and only if there exists a nonsingular matrix U and a diagonal matrix D such that $A = UDU^{-1}$ where the column vectors $u^1, u^2, \ldots, u^n$ of U satisfy $[u^i, u^j] = \pm\delta_{ij}$.

THEOREM 3. *Let* $\mathbf{V}$ *be a finite dimensional complex indefinite inner product space and let* A *be an* $n \times n$ *complex square matrix. Then the following are equivalent:*

(i) A *is* N*-normal and has spectral property.*

(ii) $\mathbf{V}$ *has an* N*-orthonormal basis consisting of eigenvectors of* A.

(iii) A *is* N*-unitarily diagonalizable.*

Proof. (i) $\Rightarrow$ (ii)

By Theorem 2, A can be written as $A = \sum_{i=1}^{k} \lambda_i E_i$, where λ_i's are the distinct eigenvalues of A and E_i's are N-orthogonal projections onto $\mathscr{R}(E_i) = \mathscr{N}(A - \lambda_i I)$. Then by Remark 1, $\mathscr{N}(A - \lambda_i I)$ is non-degenerate. Thus, we can construct a N-orthonormal basis for $\mathscr{N}(A - \lambda_i I)$ using the well-known Gram-Schmidt orthogonalization process. On this basis, every vector is an eigenvector of A. Since $\mathbf{V} = \sum_{i=1}^{k} \oplus \mathscr{R}(E_i)$, the union of all orthonormal bases of $\mathscr{N}(A - \lambda_i I)$ forms a basis for $\mathbf{V}$.

(ii) $\Rightarrow$ (iii)

Let $\{u^1, u^2, \ldots, u^n\}$ be an N-orthonormal basis for $\mathbf{V}$ consisting of eigenvectors of A and let $Au^i = \lambda_i u^i$, $i = 1, 2, \ldots, n$. Set $U = [u^1,\ u^2,\ \ldots\ , u^n]$. Then the column vectors $u^1, u^2, \ldots, u^n$ of U satisfy $[u^i, u^j] = \pm\delta_{ij}$. Moreover,

$$U^{-1}AU = U^{-1}[Au^1,\ Au^2,\ \ldots,\ Au^n] = U^{-1}[\lambda_1 u^1,\ \lambda_2 u^2,\ \ldots,\ \lambda_n u^n].$$

Let $S = \operatorname{diag}(c_1, c_2, \ldots, c_n)$ where $c_i = [u^i, u^i]$. We call S the signature matrix with respect to U. It follows that S is invertible and $S^2 = I$. Also, the i^{th} column of SU^*NU is

$$\begin{aligned}
(SU^*NU)^i &= SU^*Nu^i \\
&= S((u^i)^*NU)^* \\
&= S((u^i)^{[*]}U)^* \\
&= S\left[(u^i)^{[*]}u^1,\ (u^i)^{[*]}u^2,\ \ldots,\ (u^i)^{[*]}u^n\right]^* \\
&= S\,[0,\ 0, \cdots, c_i,\ 0 \cdots,\ 0]^* \\
&= [0,0,\ldots,1,0,\ldots,0]^*.
\end{aligned}$$

Thus, $SU^*NU = I$, since $c_i^2 = 1$, or equivalently, $U^{-1} = SU^*N$. The i^{th} column of $U^{-1}AU$ is

$$\begin{aligned}
\left(U^{-1}AU\right)^i &= (SU^*NAU)^i \\
&= \left(SU^*N[\lambda_1 u^1,\ \lambda_2 u^2,\ \ldots,\ \lambda_n u^n]\right)^i \\
&= \lambda_i SU^*Nu^i \\
&= \lambda_i S\left((u^i)^{[*]}U\right)^* \\
&= \lambda_i S\left[(u^i)^{[*]}u^1,\ (u^i)^{[*]}u^2,\ \ldots,\ (u^i)^{[*]}u^n\right]^* \\
&= \lambda_i S[0,0,\cdots,c_i,0,\cdots,0]^* \\
&= [0,0,\ldots,\lambda_i,0,\ldots,0]^*.
\end{aligned}$$

Let $D = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$. Then $U^{-1}AU = D$. Thus A is N-unitarily diagonalizable.

(iii) $\Rightarrow$ (i)

Let $A = UDU^{-1}$. Then the column vectors $\{u^1, u^2, \ldots, u^n\}$ of U are eigenvectors of A such that $[u^i, u^j] = \pm\delta_{ij}$. Let $S = \operatorname{diag}(c_1, c_2, \ldots, c_n)$, where $c_i = [u^i, u^i]$. Then S is invertible and $S^2 = I$. It is easy to check that $U^{-1} = SU^*N$ as above. Since $SD = DS$ then we have,

$$AA^{[*]} = AN^{-1}A^*N = UDSD^*U^*N = USDD^*U^*N$$

and

$$A^{[*]}A = N^{-1}A^*NA = USD^*SDSU^*N = USDD^*U^*N.$$

Thus A is N-normal. Moreover,

$$\begin{aligned}
(A-\lambda I)(A-\lambda I)^{[*]} &= U(D-\lambda I)U^{-1}(U^{-1})^{[*]}(D-\lambda I)^{[*]}U^{[*]} \\
&= U(D-\lambda I)SN(D-\lambda I)^{[*]}U^{[*]} \\
&= US(D-\lambda I)(D-\lambda I)^*U^*N.
\end{aligned}$$

Thus

$$\begin{aligned}
\operatorname{rank}(A-\lambda I) &= \operatorname{rank}(D-\lambda I) \\
&= \operatorname{rank}\left((D-\lambda I)(D-\lambda I)^*\right) \\
&= \operatorname{rank}\left((A-\lambda I)(A-\lambda I)^{[*]}\right).
\end{aligned}$$

Similarly, $\operatorname{rank}(A-\lambda I) = \operatorname{rank}\left((A-\lambda I)^{[*]}(A-\lambda I)\right)$. Thus A has spectral property. This completes the proof. $\square$

COROLLARY 2. *Let A be a linear operator on a finite-dimensional real or complex vector space $\mathbf{V}$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. Then A is N-normal and has spectral property if and only if there exist N-orthogonal projectors $P_1, P_2, \cdots, P_n$ on $\mathbf{V}$ such that:*

(i) $A = \lambda_1 P_1 + \cdots + \lambda_n P_n$.

(ii) $I = P_1 + \cdots + P_n$.

(iii) $P_i P_j = \mathbf{0},\ i \neq j$.

Proof. Suppose A is N-normal and has spectral property. Then by Theorem 3, there exists an N-orthonormal basis $\{u^1, u^2, \ldots, u^n\}$ consisting of eigenvectors of A. Let $P_i = c_i u^i (u^i)^{[*]}$, where $c_i = [u^i, u^i] = \pm 1,\ i = 1, 2, \ldots, n$. Then it is easy to check that P_i satisfies the conditions (i)–(iii), above.

Conversely, if conditions (i)–(iii) hold and $P_i^{[*]} = P_i$ for all $i = 1, 2, \ldots, n$, then $AA^{[*]} = \sum_{i=1}^{n} |\lambda_i|^2 P_i = A^{[*]}A$. Thus A is N-normal. By (i) and (ii),

$$A - \lambda_j I = \sum_{\substack{i=1 \\ i \neq j}}^{n} (\lambda_i - \lambda_j) P_i,\ \forall j = 1, 2, \ldots, n.$$

So by (iii)

$$(A - \lambda_j I)(A - \lambda_j I)^{[*]} = \sum_{\substack{i=1 \\ i \neq j}}^{n} |\lambda_i - \lambda_j|^2 P_i,\ \forall j.$$

Post- multiplying by P_s, where $s \neq j$ and $1 \leqslant s \leqslant n$, we get

$$(A - \lambda_j I)(A - \lambda_j I)^{[*]} P_s = |\lambda_s - \lambda_j|^2 P_s,\ \forall j.$$

Since $|\lambda_s - \lambda_j| \neq 0$ then

$$P_s = \frac{1}{|\lambda_s - \lambda_j|^2} (A - \lambda_j I)(A - \lambda_j I)^{[*]} P_s,\ \forall j.$$

Using the above formula for P_i in equation (i), we get

$$\begin{aligned} A - \lambda_j I &= \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{(\lambda_i - \lambda_j)}{|\lambda_i - \lambda_j|^2} (A - \lambda_j I)(A - \lambda_j I)^{[*]} P_i \\ &= (A - \lambda_j I)(A - \lambda_j I)^{[*]} \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\bar{\lambda_i} - \bar{\lambda_j}\right)^{-1} P_i,\ \forall j. \end{aligned}$$

Thus, $\operatorname{rank}(A - \lambda_j I) \leqslant \operatorname{rank}\left((A - \lambda_j I)(A - \lambda_j I)^{[*]}\right) \leqslant \operatorname{rank}(A - \lambda_j I)\ \ \forall j$. Therefore, $\operatorname{rank}(A - \lambda_j I) = \operatorname{rank}\left((A - \lambda_j I)(A - \lambda_j I)^{[*]}\right)$. Since $(A - \lambda_j I)$ is N-normal, then $(A - \lambda_j I)^{[\dagger]}$ exists for all $j = 1, 2, \ldots, n$. This completes the proof. $\square$

COROLLARY 3. *Let P be a $n \times n$ complex square matrix. If P is projection then the following are equivalent:*

(i) P is an N-orthogonal projection.

(ii) P is an N-unitarily diagonalizable.

Proof. Suppose, P is an N-orthogonal projection. Then $I-P$ is also an N-orthogonal projection. Clearly, $P^{[\dagger]}$ and $(I-P)^{[\dagger]}$ are exist. Thus, P has spectral property. Therefore, by Theorem 3, P is N-unitarily diagonalizable.

Conversely, let us assume that P is an N-unitarily diagonalizable matrix. Then by Theorem 3, there exists an invertible matrix U such that $P = U\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}U^{-1}$. Moreover, $U^{-1} = SU^*N$ where $S = \operatorname{diag}(c_1, c_2, \ldots, c_n)$, $c_i = \pm 1$. It is easy to check that $P^2 = P$. Since, S is a diagonal matrix, thus

$$\begin{aligned} P^{[*]} &= N^{-1}NUS\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}U^*N = US\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}U^*N \\ &= U\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}SU^*N = U\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}U^{-1} = P. \end{aligned}$$

This completes the proof. □

4. Star partial order, left and right star partial orders

As a natural extension, Stanisev defined the star partial order in an IIPS as follows: $A \overset{[*]}{\leqslant} B$ if $AA^{[*]} = BA^{[*]}$ and $A^{[*]}A = A^{[*]}B$ [21]. In this paper, we utilize the definitions of matrix partial orders based on the works referenced in [6], [7] and [20]. The primary aim of using these definitions is to explore the potential extension of matrix partial orders to infinite dimensional Kerin spaces.

DEFINITION 3. Let A and B be any two complex matrices of the same order. We say A is below B with respect to

- Star order: $A \overset{[*]}{\leqslant} B$ if there exist M-orthogonal projection P and N-orthogonal projection Q such that $A = PA = PB$, $A = AQ = BQ$, $\mathscr{R}(P) = \mathscr{R}(A) = \mathscr{R}(AA^{[*]})$ and $\mathscr{N}(Q) = \mathscr{N}(A) = \mathscr{N}(A^{[*]}A)$.

- Left star order: $A \,{}^{[*]}\!\!\leqslant B$ if and only if there exist M-orthogonal projection P and projection Q such that $A = PA = PB$, $A = AQ = BQ$, $\mathscr{R}(P) = \mathscr{R}(A) = \mathscr{R}(AA^{[*]})$, $\mathscr{N}(A) = \mathscr{N}(Q)$.

- Right star order: $A \leqslant^{[*]} B$ if and only if there exist projection P and N-orthogonal projection Q such that $A = AP = BP$, $A = AQ = BQ$, $\mathscr{R}(P) = \mathscr{R}(A)$, $\mathscr{N}(A) = \mathscr{N}(Q) = \mathscr{N}(A^{[*]}A)$.

- Minus order: $A \overset{-}{\leqslant} B$ if and only if there exist projections P and Q such that $A = AP = BP$, $A = AQ = BQ$, $\mathscr{R}(P) = \mathscr{R}(A)$, $\mathscr{N}(A) = \mathscr{N}(Q)$.

REMARK 2. From Definition 3, it is easy to observe that $A \overset{[*]}{\leqslant} B \Rightarrow A \overset{[*]}{\leqslant} B \Rightarrow A \overset{-}{\leqslant} B$. Additionally in view of Baksalary [1, p. 164], it is observed that $A \overset{-}{\leqslant} B$ if and only if $A = AB^{(1)}A = BB^{(1)}A = AB^{(1)}B$ for some $B^{(1)} \in B\{1\}$. Thus, it is clear that the definition for minus order given in Definition 3 is equivalent to $AA^{(1)} = BA^{(1)}$ and $A^{(1)}A = A^{(1)}B$, for some $A^{(1)}$.

THEOREM 4. *Let A and B have any two complex $m \times n$ matrices. Then the following are equivalent:*

(i) $A \overset{[*]}{\leqslant} B$.

(ii) $A^{[\dagger]}$ *exists,* $\mathscr{R}(A)[\perp]\mathscr{R}(B-A)$ *and* $\mathscr{R}(A^{[*]})[\perp]\mathscr{R}(B^{[*]} - A^{[*]})$.

Proof. Suppose $A \overset{[*]}{\leqslant} B$. Then by Definition 3, there exist M-orthogonal projection P and N-orthogonal projection Q such that $A = PA = PB$, $A = AQ = BQ$. Thus, $B - A = B - PB = (I-P)B$ and $(I-P)A = 0$. Clearly, $[Ax, (B-A)y] = [Ax, (I-P)By] = [(I-P)Ax, By] = 0$ for all x and y. Therefore, $\mathscr{R}(A)[\perp]\mathscr{R}(B-A)$. Similarly, we can prove the other one using the fact that $A^{[*]} = A^{[*]}Q = B^{[*]}Q$. The existence of Moore-Penrose inverse of A directly follows from the definition.

Conversely, if $A^{[\dagger]}$ exists then, $\mathscr{R}(A) \oplus \mathscr{N}(A^{[*]}) = \mathbb{C}^m$ and $\mathscr{R}(A^{[*]}) \oplus \mathscr{N}(A) = \mathbb{C}^n$. Thus, there exists an N-orthogonal projection P such that $\mathscr{R}(P) = \mathscr{R}(A)$ and $\mathscr{N}(P) = \mathscr{R}(A)^{[\perp]} = \mathscr{N}(A^{[*]})$. Therefore, $\mathscr{R}(B-A) \subseteq \mathscr{R}(A)^{[\perp]} \subseteq \mathscr{N}(A^{[*]}) = \mathscr{N}(P)$. This shows that $PB = PA = A$. Similarly, we can prove the other one. This completes the proof. $\square$

LEMMA 1. *Let A and B be two complex matrices of the same order such that $A^{[\dagger]}$ and $B^{[\dagger]}$ exist. If $AA^{[\dagger]} = BA^{[\dagger]}$ and $A^{[\dagger]}A = A^{[\dagger]}B$, then*

1. $(BA^{[\dagger]})^{[\dagger]} = AB^{[\dagger]}$.
2. $(A^{[\dagger]}B)^{[\dagger]} = B^{[\dagger]}A$.

Proof. To prove (i), by Theorem 1, it is enough to show $\mathscr{R}(B^{[*]}BA^{[\dagger]}) \subseteq \mathscr{R}(A^{[\dagger]})$ and $\mathscr{R}(A^{[\dagger]}(A^{[\dagger]})^{[*]}B^{[*]}) \subseteq \mathscr{R}(B^{[*]})$. Thus,

$$\mathscr{R}(B^{[*]}BA^{[\dagger]}) = \mathscr{R}(B^{[*]}AA^{[\dagger]}) = \mathscr{R}((AA^{[\dagger]}B)^{[*]}) = \mathscr{R}((AA^{[\dagger]}A)^{[*]}) = \mathscr{R}(A^{[*]}) \subseteq \mathscr{R}(A^{[\dagger]}).$$

Also,

$$\mathscr{R}(A^{[\dagger]}(A^{[\dagger]})^{[*]}B^{[*]}) = \mathscr{R}((A^{[\dagger]}A)^{[*]}A^{[\dagger]}(A^{[\dagger]})^{[*]}B^{[*]}) = \mathscr{R}((A^{[\dagger]}B)^{[*]}A^{[\dagger]}(A^{[\dagger]})^{[*]}B^{[*]}))$$
$$= \mathscr{R}(B^{[*]}(A^{[\dagger]})^{[*]}A^{[\dagger]}(A^{[\dagger]})^{[*]}B^{[*]}) \subseteq \mathscr{R}(B^{[*]}).$$

(ii) can be proved in similar manner. $\square$

The following theorem provides main characterization of star partial order.

THEOREM 5. *Let A and B have any two $m\times n$ complex matrices of the same order. Then the following are equivalent:*

(i) $A \overset{[*]}{\leqslant} B$.

(ii) There exists invertible matrices U and V such that

$$A = U\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}V^{-1} \quad \text{and} \quad B = U\begin{pmatrix} A_1 & 0 \\ 0 & B_4 \end{pmatrix}V^{-1},$$

for some matrix B_4 and invertible A_1. Moreover, $U^{-1} = SU^*M$ and $V^{-1} = RV^*N$, where S and R are signature matrices with respect to U and V respectively.

(iii) $A^{[\dagger]}$ *exists,* $AA^{[*]} = BA^{[*]}$ *and* $A^{[*]}A = A^{[*]}B$.

(iv) $AA^{[\dagger]} = BA^{[\dagger]}$ *and* $A^{[\dagger]}A = A^{[\dagger]}B$.

(v) $AA^{[\dagger]}B = A = BA^{[\dagger]}A$.

Moreover if $B^{[\dagger]}$ exists, then the following statement is equivalent to (i)–(v)

(vi) $AA^{[\dagger]} = AB^{[\dagger]}$ *and* $A^{[\dagger]}A = B^{[\dagger]}A$.

Proof. (i) $\Rightarrow$ (ii) Since $A \overset{[*]}{\leqslant} B$, there exist M-orthogonal projection P and N-orthogonal projection Q such that $A = PA = PB$, $A = AQ = BQ$, $\mathscr{R}(P) = \mathscr{R}(A) = \mathscr{R}(AA^{[*]})$ and $\mathscr{N}(Q) = \mathscr{N}(A) = \mathscr{N}(A^{[*]}A)$. Clearly, by Corollary 3, there exist invertible matrices U and V such that $P = U\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}U^{-1}$ and $Q = V\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}V^{-1}$, where r is equal to rank of A. Moreover, $U^{-1} = SU^*M$ and $V^{-1} = RV^*N$ where $S = \operatorname{diag}(c_1, c_2, \ldots, c_m)$, $c_i = \pm 1$ and $R = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, $d_i = \pm 1$.

The matrices A and B can be written, with suitable block matrices, as

$$A = U\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}V^{-1} \quad \text{and} \quad B = U\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}V^{-1}.$$

Now, using the fact that $A = PA$ and $AQ = A$, we get $A_2 = 0$, $A_3 = 0$ and $A_4 = 0$. Similarly, using the fact that $PB = A$ and $BQ = A$, we get $B_1 = A_1$, $B_2 = 0$ and $B_3 = 0$. Also, $\operatorname{rank}(A_1) = \operatorname{rank}(A) = \operatorname{rank}(P) = r$. Thus A_1 is invertible.

(ii) $\Rightarrow$ (iii)

Suppose A and B have the decomposition mentioned in (ii). Then

$$\begin{aligned} A^{[*]} &= N^{-1}(V^{-1})^*\begin{pmatrix} A_1^* & 0 \\ 0 & 0 \end{pmatrix}U^*M \\ &= N^{-1}(V^{-1})^*\begin{pmatrix} A_1^* & 0 \\ 0 & 0 \end{pmatrix}SU^{-1} \\ &= N^{-1}(V^{-1})^*\begin{pmatrix} A_1^*S_1 & 0 \\ 0 & 0 \end{pmatrix}U^{-1}. \end{aligned}$$

Where $S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$ and $R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$. Since, $U^{-1} = SU^*M$ and $V^{-1} = RV^*N$

It is easy to verify that,

$$A^{[*]}A = N^{-1}(V^*)^{-1}\begin{pmatrix} A_1^*S_1A_1 & 0 \\ 0 & 0 \end{pmatrix}V^{-1} = A^{[*]}B$$

and

$$AA^{[*]} = U\begin{pmatrix} A_1R_1A_1^* & 0 \\ 0 & 0 \end{pmatrix}V^*M = BA^{[*]}.$$

Also, $\operatorname{rank}(A^{[*]}A) = \operatorname{rank}(A_1^*S_1A_1) = \operatorname{rank}(A_1) = \operatorname{rank}(A)$. Similarly, $\operatorname{rank}(AA^{[*]})$ $= \operatorname{rank}(A_1R_1A_1^*) = \operatorname{rank}(A_1) = \operatorname{rank}(A)$. Thus, $A^{[\dagger]}$ exists.

(iii) $\Rightarrow$ (iv)

Since, $A^{[\dagger]}$ exists then, $\left(AA^{[*]}\right)^{[\dagger]}$ and $\left(A^{[*]}A\right)^{[\dagger]}$ are exist and $A^{[\dagger]} = A^{[*]}\left(AA^{[*]}\right)^{[\dagger]}$ $= \left(A^{[*]}A\right)^{[\dagger]}A^{[*]}$. Post-multiplying by $\left(AA^{[*]}\right)^{[\dagger]}$ in $AA^{[*]} = BA^{[*]}$, we get $AA^{[\dagger]} =$ $BA^{[\dagger]}$. Similarly, pre-multiplying by $\left(A^{[*]}A\right)^{[\dagger]}$ in $A^{[*]}A = A^{[*]}B$, we get $A^{[\dagger]}A = A^{[\dagger]}B$.

(iv) $\Rightarrow$ (v)

Post- and pre-multiplying by A, we get $AA^{[\dagger]}B = A = BA^{[\dagger]}A$.

(iv) $\Rightarrow$ (vi)

By Lemma 1, we get $AA^{[\dagger]} = AB^{[\dagger]}$ and $A^{[\dagger]}A = B^{[\dagger]}A$.

(v) $\Rightarrow$ (i)

Since, $AA^{[\dagger]}B = A$ and $BA^{[\dagger]}A = A$. Choose $P = AA^{[\dagger]}$ and $Q = A^{[\dagger]}A$. Clearly, $A = AA^{[\dagger]}A = PA$ and $A = BA^{[\dagger]}A = BQ$. Thus $A = PA = BQ$. Since $A^{[\dagger]}$ exists, thus $\mathscr{R}(A) = \mathscr{R}(AA^{[*]}) = \mathscr{R}(AA^{[\dagger]}) = \mathscr{R}(P)$ and $\mathscr{N}(A) = \mathscr{N}(A^{[*]}A) = \mathscr{N}(A^{[\dagger]}A) = \mathscr{N}(Q)$. It shows that $A \overset{[*]}{\leqslant} B$.

(v) $\Rightarrow$ (vi)

From the assumptions, it is easy to observe that $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $\mathscr{R}(A^{[*]}) \subseteq \mathscr{R}(B^{[*]})$ equivalently, $BB^{[\dagger]}A = A = AB^{[\dagger]}B$. Post-multiplying $AA^{[\dagger]}B = A$ by $B^{[\dagger]}$ we get $AB^{[\dagger]} = AA^{[\dagger]}BB^{[\dagger]} = (BB^{[\dagger]}AA^{[\dagger]})^{[*]} = AA^{[\dagger]}$. Other one is similar.

(vi) $\Rightarrow$ (iii)

Let $AA^{[\dagger]} = AB^{[\dagger]}$ and $A^{[\dagger]}A = B^{[\dagger]}A$, then by Lemma 1, $AA^{[\dagger]} = (AA^{[\dagger]})^{[\dagger]} = (AB^{[\dagger]})^{[\dagger]} = BA^{[\dagger]}$. Post-multiplying by $AA^{[*]}$, we get $AA^{[\dagger]}AA^{[*]} = BA^{[\dagger]}AA^{[*]}$. This gives $AA^{[*]} = BA^{[*]}$. Similarly, we can be prove $A^{[*]}A = A^{[*]}B$.

This completes the theorem. □

THEOREM 6. *The relation $\overset{[*]}{\leqslant}$ is a partial order on $\mathbb{C}^{m\times n}$.*

Proof. From Definition 3, it is easy to observe that $A \overset{[*]}{\leqslant} A$. It ensures that $\overset{[*]}{\leqslant}$ reflexive. Suppose $A \overset{[*]}{\leqslant} B$ and $B \overset{[*]}{\leqslant} A$, then there exist M-orthogonal projections P_1 and P_2 and N-orthogonal projections Q_1 and Q_2 such that $A = P_1A = P_1B$, $A = AQ_1 = BQ_1$,

$B = P_2B = P_2A$ and $B = BQ_2 = AQ_2$. Then $A = P_1B = P_1AQ_2 = AQ_2 = B$. This shows that the relation $\overset{[*]}{\leqslant}$ is anti-symmetric.

Next, suppose $A \overset{[*]}{\leqslant} B$ and $B \overset{[*]}{\leqslant} C$, then by Theorem 5 (v), we have $A = AA^{[\dagger]}B = BA^{[\dagger]}A$ and $B = BB^{[\dagger]}C = CB^{[\dagger]}B$. It is clear that $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $\mathscr{R}(A^{[*]}) \subseteq \mathscr{R}(B^{[*]})$. Thus $A = AA^{[\dagger]}B = AA^{[\dagger]}BB^{[\dagger]}C = AA^{[\dagger]}C$ and $A = BA^{[\dagger]}A = CB^{[\dagger]}BA^{[\dagger]}A = CA^{[\dagger]}A$.

This shows that $A \overset{[*]}{\leqslant} C$ by Theorem 5, (v). Therefore, the relation $\overset{[*]}{\leqslant}$ is transitive. This completes the theorem. □

THEOREM 7. *Let A and B have any two complex $m \times n$ matrices. If $A \overset{[*]}{\leqslant} B$. Then*

(i) $A^{[\dagger]}BA^{[\dagger]} = A^{[\dagger]}$.

(ii) $AA^{(1)}BA^{(1)}A = A$.

Proof. By Theorem 5 (ii),

$$A = U \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} V^{-1} \quad \text{and} \quad B = U \begin{pmatrix} A_1 & 0 \\ 0 & B_4 \end{pmatrix} V^{-1}$$

for some matrix B_4 and invertible A_1. It is easy to verify that $A^{(1)} = V \begin{pmatrix} A_1^{-1} & 0 \\ 0 & L \end{pmatrix} U^{-1}$ for some L and $A^{[\dagger]} = V \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^{-1}$.

Thus,

$$\begin{aligned} A^{[\dagger]}BA^{[\dagger]} &= V \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^{-1}U \begin{pmatrix} A_1 & 0 \\ 0 & B_4 \end{pmatrix} V^{-1}V \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^{-1} \\ &= V \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^{-1} \\ &= A^{[\dagger]}. \end{aligned}$$

Also,

$$\begin{aligned} AA^{(1)}BA^{(1)}A &= U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^{-1}U \begin{pmatrix} A_1 & 0 \\ 0 & B_4 \end{pmatrix} V^{-1}V \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V^{-1} \\ &= U \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} V^{-1} \\ &= A. \quad \square \end{aligned}$$

PROPOSITION 1. *Let A and B have complex matrices of the same order. Then the following characterizations are hold:*

(i) $A \overset{[*]}{\leqslant} B \Leftrightarrow A^{[*]} \overset{[*]}{\leqslant} B^{[*]}$.

(ii) If U and V are unitary matrices of order $m\times m$ and $n\times n$ respectively then $A \overset{[*]}{\leqslant} B \Leftrightarrow UAV \overset{[*]}{\leqslant} UBV$.

Proof. It is well known that $A^{[\dagger]}$ exists if and only if $(A^{[*]})^{[\dagger]}$ exists. Also, by Theorem 5

$$\begin{aligned} A \overset{[*]}{\leqslant} B &\Leftrightarrow A^{[*]}A = A^{[*]}B \text{ and } AA^{[*]} = BA^{[*]} \\ &\Leftrightarrow (A^{[*]})^{[*]}A^{[*]} = (A^{[*]})^{[*]}B^{[*]} \text{ and } A^{[*]}(A^{[*]})^{[*]} = B^{[*]}(A^{[*]})^{[*]} \\ &\Leftrightarrow A^{[*]} \overset{[*]}{\leqslant} B^{[*]}. \end{aligned}$$

Similarly, to prove (ii) it is easy to check that $A^{[\dagger]}$ exists if and only if $(UAV)^{[\dagger]}$ exist. Thus,

$$\begin{aligned} A \overset{[*]}{\leqslant} B &\Leftrightarrow A^{[*]}A = A^{[*]}B \text{ and } AA^{[*]} = BA^{[*]} \\ &\Leftrightarrow V^{[*]}A^{[*]}U^{[*]}UAV = V^{[*]}A^{[*]}U^{[*]}UBV \text{ and } UAVV^{[*]}A^{[*]}U^{[*]} = UBVV^{[*]}A^{[*]}U^{[*]} \\ &\Leftrightarrow (UAV)^{[*]}UAV = (UAV)^{[*]}UBV \text{ and } UAV(UAV)^{[*]} = UBV(UAV)^{[*]} \\ &\Leftrightarrow UAV \overset{[*]}{\leqslant} UBV. \quad \square \end{aligned}$$

THEOREM 8. *Let A, B and C be complex matrices of the same order. Then the following conditions are equivalent:*

(i) $A \overset{[*]}{\leqslant} B$.

(ii) $A^{[\dagger]}$ exists and $\mathscr{R}(C) \subseteq \mathscr{R}(B)$, $A^{[*]}C = 0$ and $CA^{[*]} = 0$, where $C = B - A$.

Proof. If $A \overset{[*]}{\leqslant} B$, then by Theorem 5, we have $A^{[*]}A = A^{[*]}B$ and $AA^{[*]} = BA^{[*]}$. This implies $A^{[*]}(B-A) = 0$ and $(B-A)A^{[*]} = 0$. Let $C = B - A$, then it follows that $\mathscr{R}(C) \subseteq \mathscr{R}(B)$ and $A^{[*]}C = 0$ and $CA^{[*]} = 0$. Conversely, if $A^{[\dagger]}$ exists and $\mathscr{R}(C) \subseteq \mathscr{R}(B)$, $A^{[*]}C = 0$ and $CA^{[*]} = 0$, where $C = B - A$, it directly follows from these conditions that $A \overset{[*]}{\leqslant} B$. $\square$

THEOREM 9. *Let A and B be any two matrices of same order. If $A \overset{-}{\leqslant} B$ and both $BA^{(1)}$ and $A^{(1)}B$ are M and N-Hermitian matrices, respectively, for some $A^{(1)} \in A\{1\}$, then $A \overset{[*]}{\leqslant} B$.*

Proof. Clearly, $AA^{(1)} = BA^{(1)} = (AA^{(1)})^{[*]}$ and $A^{(1)}A = A^{(1)}B = (A^{(1)}A)^{[*]}$. Thus, $A = AA^{(1)}A = A(A^{(1)}A)^{[*]} = AA^{[*]}(A^{(1)})^{[*]}$. It shows that $\operatorname{rank}(A) = \operatorname{rank}(AA^{[*]})$. Similarly, we can prove that $\operatorname{rank}(A) = \operatorname{rank}(A^{[*]}A)$. Thus $A^{[\dagger]}$ exists.

Also, $[Ax,By] = [AA^{(1)}Ax,By] = [Ax,(AA^{(1)})^{[*]}By] = [Ax,AA^{(1)}By] = [Ax,AA^{(1)}Ay] = [Ax,Ay]$. Thus $[Ax,(B-A)y] = 0$ for all x and y. This shows that $\mathscr{R}(A)[\perp]\mathscr{R}(B-A)$. Similarly, we can prove $\mathscr{R}(A^{[*]})[\perp]\mathscr{R}(B^{[*]} - A^{[*]})$. Thus $A \overset{[*]}{\leqslant} B$, by Theorem 4. $\square$

THEOREM 10. *Let A and B be any two $m \times n$ complex matrices. Then $A \overset{[*]}{\leqslant} B$ if and only if $A^{[\dagger]}$ exists, $A \overset{-}{\leqslant} B$ and $AB^{[*]}$ and $A^{[*]}B$ are M-Hermitian and N-Hermitian matrices, respectively.*

Proof. Necessity is straightforward. Suppose, P and Q are projections such that $A = PA = PB$ and $A = AQ = BQ$ then $AA^{[*]} = PBA^{[*]} = P(AB^{[*]})^{[*]} = PAB^{[*]} = AB^{[*]}$ and $A^{[*]}A = A^{[*]}BQ = (A^{[*]}B)^{[*]}Q = B^{[*]}AQ = B^{[*]}A$. Thus $A \overset{[*]}{\leqslant} B$. $\square$

Next, we characterize left star partial ordering in an IIPS.

THEOREM 11. *Let A and B have any two $m \times n$ complex matrices of the same order. Then the following are equivalent:*

(i) $A \,{}^{[*]}\!\!\leqslant B$.

(ii) There exist invertible matrices U and V such that

$$A = U \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} V^{-1} \quad \textit{and} \quad B = U \begin{pmatrix} A_1 & 0 \\ 0 & B_4 \end{pmatrix} V^{-1},$$

for some matrix B_4 and invertible A_1. Moreover, $U^{-1} = SU^*M$ where S is a signature matrix with respect to U.

(iii) $A^{[*]}A = A^{[*]}B$, $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ *and* $A^{(1,2,3)}$ *exists.*

(iv) $A^{(1,2,3)}$ *exists,* $A^{(1,2,3)}A = A^{(1,2,3)}B$ *and* $\mathscr{R}(A) \subseteq \mathscr{R}(B)$.

(v) $A^{(1,2,3)}$ *exists and* $AA^{(1,2,3)}B = A = BA^{(1)}A$.

Proof. (i) $\Rightarrow$ (ii) Since $A \,{}^{[*]}\!\!\leqslant B$, there exist M-orthogonal projection P and projection Q such that $A = PA = PB$, $A = AQ = BQ$, $\mathscr{R}(P) = \mathscr{R}(A) = \mathscr{R}(AA^{[*]})$ and $\mathscr{N}(Q) = \mathscr{N}(A)$. Then by Theorem 3, P can be written as $P = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^{-1}$, where $U^{-1} = SU^*M$. Also, it is well known that any projection matrix can be written as $Q = V \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V^{-1}$. The rest of the proof is similar to Theorem 5.

(ii) $\Rightarrow$ (iii)

Suppose A and B have the decomposition mentioned in (ii). Then

$$A^{[*]}A = N^{-1}(V^*)^{-1} \begin{pmatrix} A_1^* S_1 A_1 & 0 \\ 0 & 0 \end{pmatrix} V^{-1} = A^{[*]}B.$$

Let $Y = V \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} V^{-1}$, then $BY = A$. It concludes that $\mathscr{R}(A) \subseteq \mathscr{R}(B)$. Also, $\operatorname{rank}(A^{[*]}A) = \operatorname{rank}(A_1^* S_1 A_1) = \operatorname{rank}(A_1) = \operatorname{rank}(A)$. It ensures that $A^{(1,2,3)}$ exists.

(iii) $\Leftrightarrow$ (iv) Pre-multiplying $A^{[*]}A = A^{[*]}B$ by $A^{(1,2,3)^{[*]}}$, we get $(AA^{(1,2,3)})^{[*]}\ A = (AA^{(1,2,3)})^{[*]}\ B$ implies $A = AA^{(1,2,3)}B$. Again, pre-multiply by $A^{(1,2,3)}$, we get (iv). Pre-multiplying $A^{(1,2,3)}A = A^{(1,2,3)}B$ by $A^{[*]}A$ we get (iii).

(iii) $\Rightarrow$ (v) Post- and pre-multiplying $A^{[*]}A = A^{[*]}B$ by $B^{(1)}A$ and $A^{(1,2,3)^{[*]}}$, respectively, we get $AB^{(1)}A = A$. Also, the inclusions $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $\mathscr{R}(A^{[*]}A) \subseteq \mathscr{R}(B^{[*]})$ give $A = BB^{(1)}A = AB^{(1)}B$ is equivalent to $A^{(1)}A = A^{(1)}B$ and $AA^{(1)} = BA^{(1)}$ for some $A^{(1)} \in A\{1\}$, by Remark 2. Thus, $BA^{(1)}A = BA^{(1)}B = AA^{(1)}B = AA^{(1)}A = A$. Also, $A = AA^{(1,2,3)}A = AA^{(1,2,3)}B$.

(v) $\Rightarrow$ (i) By setting, $P = AA^{(1,2,3)}$ and $Q = A^{(1)}A$ then we get the desired result. This completes the proof. $\square$

The next theorem characterizes the right star partial order. The proof is similar to Theorem 11.

THEOREM 12. *Let A and B have any two $m \times n$ complex matrices of the same order. Then the following are equivalent:*

(i) $A \leqslant^{[*]} B$.

(ii) There exists invertible matrices U and V such that

$$A = U \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} V^{-1} \quad \text{and} \quad B = U \begin{pmatrix} A_1 & 0 \\ 0 & B_4 \end{pmatrix} V^{-1},$$

for some matrix B_4 and invertible A_1. Moreover, $V^{-1} = RV^*N$ where R is signature matrix with respect to V.

(iii) $AA^{[*]} = BA^{[*]}$, $\mathscr{R}(A^{[*]}) \subseteq \mathscr{R}(B^{[*]})$, $A^{(1,2,4)}$ *exists.*

(iv) $AA^{(1,2,4)} = BA^{(1,2,4)}$, $\mathscr{R}(A^{[*]}) \subseteq \mathscr{R}(B^{[*]})$, $A^{(1,2,4)}$ *exists.*

(v) $A^{(1,2,4)}AB = A = BAA^{(1)}$.

5. Space, diamond and plus matrix partial orders

In this section, we introduce the space, diamond and plus order in an IIPS and study some of their properties.

DEFINITION 4. Let A and B be any two matrices of same order. We call A is below B with respect to

- Space order, denoted $A \overset{[S]}{\leqslant} B$ if $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $\mathscr{R}(A^{[*]}) \subseteq \mathscr{R}(B^{[*]})$.
- Diamond order, denoted $A \overset{[\diamond]}{\leqslant} B$ if $A^{[\dagger]}$ exists, $A \overset{[S]}{\leqslant} B$ and $AA^{[*]}A = AB^{[*]}A$.
- Plus order, denoted $A \overset{[+]}{\leqslant} B$ if $A^{[\dagger]}$ exists, $A \overset{[S]}{\leqslant} B$ and $A = PBQ$, where P and Q are some projections.

REMARK 3. The relation $\overset{[S]}{\leqslant}$ is not a partial order on $\mathbb{C}^{m\times n}$. Since, $\overset{[S]}{\leqslant}$ is reflexive and transitive but not an anti-symmetric.

THEOREM 13. *The relation* $\overset{[\diamond]}{\leqslant}$ *is a partial order on* $\mathbb{C}^{m\times n}$.

Proof. Clearly $\overset{[\diamond]}{\leqslant}$ is reflexive. Now, we assume $A \overset{[\diamond]}{\leqslant} B$ and $B \overset{[\diamond]}{\leqslant} A$. Then by Definition 4, we have $AA^{[*]}A = AB^{[*]}A$, $BB^{[*]}B = BA^{[*]}B$, $\mathscr{R}(A) = \mathscr{R}(B)$ and $\mathscr{R}(A^{[*]}) = \mathscr{R}(B^{[*]})$. Thus there exist invertible matrices X and Y, such that $B = AY = XA$. Since $AA^{[*]}A = AB^{[*]}A = AA^{[*]}X^{[*]}A$. Pre- and post-multiplying by $A^{[\dagger]}$, we get $A^{[*]} = A^{[*]}X^{[*]}AA^{[\dagger]} = (XA)^{[*]}AA^{[\dagger]} = B^{[*]}AA^{[\dagger]} = B^{[*]}$. It concludes that $B = A$. Finally, let $A \overset{[\diamond]}{\leqslant} B$ and $B \overset{[\diamond]}{\leqslant} C$, we have $AA^{[*]}A = AB^{[*]}A$ and $BB^{[*]}B = BC^{[*]}B$. Thus there exist matrices X and Y, such that $A = BY = XB$. It follows that $AA^{[*]}A = AB^{[*]}A = XBB^{[*]}BY = XBC^{[*]}BY = AC^{[*]}A$ and hence $A \overset{[\diamond]}{\leqslant} C$. This completes the proof. $\square$

PROPOSITION 2. *Let A and B be matrices of the same order. Then the following characterizations are hold:*

(i) $A \overset{[\diamond]}{\leqslant} B \Leftrightarrow A^{[*]} \overset{[\diamond]}{\leqslant} B^{[*]}$.

(ii) If U and V are M-unitary and N-unitary matrices, respectively. Then $A \overset{[\diamond]}{\leqslant} B \Leftrightarrow UAV \overset{[\diamond]}{\leqslant} UBV$.

Proof. Similar to Proposition 1. $\square$

THEOREM 14. *Let A and B have two complex matrices of the same order. If $A^{[\dagger]}$ exists, then the following are equivalent:*

(i) $AA^{[\dagger]}BA^{[\dagger]}A = A$.

(ii) $AA^{[*]}A = AB^{[*]}A$.

(iii) $A^{[\dagger]}BA^{[\dagger]} = A^{[\dagger]}$.

(iv) $A^{[\dagger]}BA^{[\dagger]} \in A\{1\}$.

Proof. (i) $\Rightarrow$ (ii) Suppose $AA^{[\dagger]}BA^{[\dagger]}A = A$. Pre- and post-multiplying by $A^{[*]}$, we get $AA^{[*]}A = AB^{[*]}A$.

(ii) $\Rightarrow$ (iii) Let $A^{[*]}BA^{[*]} = A^{[*]}AA^{[*]}$. Pre-multiplying by $(A^{[*]}A)^{[\dagger]}$ and post-multiplying by $(AA^{[*]})^{[\dagger]}$, we get $(A^{[*]}A)^{[\dagger]}A^{[*]}BA^{[*]}(AA^{[*]})^{[\dagger]} = (A^{[*]}A)^{[\dagger]}A^{[*]}AA^{[*]}(AA^{[*]})^{[\dagger]}$. Thus $A^{[\dagger]}BA^{[\dagger]} = A^{[\dagger]}AA^{[\dagger]} = A^{[\dagger]}$.

(iii) $\Rightarrow$ (iv) Since $A^{[\dagger]}BA^{[\dagger]} = A^{[\dagger]} \in A\{1,2,3,4\}$. Thus $A^{[\dagger]}BA^{[\dagger]} \in A\{1\}$.

(iv) $\Rightarrow$ (i) Since $A^{[\dagger]}BA^{[\dagger]} \in A\{1\}$, thus by Definition of $\{1\}$ inverse we get, $AA^{[\dagger]}BA^{[\dagger]}A = A$. $\square$

THEOREM 15. *If A and B are complex matrices of the same order. If $A^{[\dagger]}$ and $B^{[\dagger]}$ exists. Then the following conditions are equivalent:*

(i) $A \overset{[\diamond]}{\leqslant} B$.

(ii) $BB^{[\dagger]}AA^{[\dagger]} = AA^{[\dagger]}$, $B^{[\dagger]}BA^{[\dagger]}A = A^{[\dagger]}A$ *and* $A^{[\dagger]}BA^{[\dagger]} = A^{[\dagger]}$.

Proof. (i) $\Rightarrow$ (ii) Suppose $A \overset{[\diamond]}{\leqslant} B$, then $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $\mathscr{R}(A^{[*]}) \subseteq \mathscr{R}(B^{[*]})$ and $AA^{[*]}A = AB^{[*]}A$. By Theorem 14, $A^{[\dagger]} = A^{[\dagger]}BA^{[\dagger]}$. Since $\mathscr{R}(A) \subseteq \mathscr{R}(B)$, we have $BB^{[\dagger]}A = A$ and hence $BB^{[\dagger]}AA^{[\dagger]} = AA^{[\dagger]}$. Similar way, we can prove the other one.

(ii) $\Rightarrow$ (i) It is clear that $\mathscr{R}(A) = \mathscr{R}(AA^{[\dagger]}) = \mathscr{R}(BB^{[\dagger]}AA^{[\dagger]}) \subseteq \mathscr{R}(B)$ and similarly, $\mathscr{R}(A^{[*]}) \subseteq \mathscr{R}(B^{[*]})$. Finally, pre- and post-multiplying the equation $A^{[\dagger]}BA^{[\dagger]} = A^{[\dagger]}$ by $A^{[*]}A$ and $AA^{[*]}$, it gives $A^{[*]}BA^{[*]} = A^{[*]}AA^{[*]}$. Thus $A \overset{[\diamond]}{\leqslant} B$. $\square$

THEOREM 16. *The relation $\overset{[+]}{\leqslant}$ is a partial order on $\mathbb{C}^{m\times n}$.*

Proof. Choose $P = AA^{(1)}$ and $Q = A^{(1)}A$, then $PAQ = AA^{(1)}AA^{(1)}A = A$. Clearly $\overset{[+]}{\leqslant}$ is reflexive. Now, we assume $A \overset{[+]}{\leqslant} B$ and $B \overset{[+]}{\leqslant} A$. By Definition 4, we have $A = PBQ$ and $B = PAQ$, such that $A = PBQ = PPAQQ = PAQ = B$. Finally, let $A \overset{[+]}{\leqslant} B$ and $B \overset{[+]}{\leqslant} C$. We have $A = PBQ$, $B = PCQ$, then $A = PBQ = PPCQQ = PCQ$, which gives $A \overset{[+]}{\leqslant} C$. Thus $\overset{[+]}{\leqslant}$ is a partial order. $\square$

The next theorem establishes a relationship between the star, diamond and plus partial orders.

THEOREM 17. *Let A and B be any two $m\times n$ complex matrices. Then*

$$A \overset{[*]}{\leqslant} B \Rightarrow A \overset{[\diamond]}{\leqslant} B \Rightarrow A \overset{[+]}{\leqslant} B.$$

Conversely, if $AB^{[*]}$ and $A^{[*]}B$ are M-Hermitian and N-Hermitian matrices, respectively, then

$$A \overset{[*]}{\leqslant} B \Leftrightarrow A \overset{[\diamond]}{\leqslant} B \Leftrightarrow A \overset{[+]}{\leqslant} B.$$

Proof. Let $A \overset{[*]}{\leqslant} B$, then by Theorem 5, $AA^{[*]} = BA^{[*]}$ and $A^{[*]}A = B^{[*]}A$. Clearly, $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $\mathscr{R}(A^{[*]}) \subseteq \mathscr{R}(B^{[*]})$. Thus $A \overset{[S]}{\leqslant} B$. Also, by Theorem 5, we have $A^{[\dagger]}A = A^{[\dagger]}B$. Pre- and post-multiplying by $A^{[*]}A$ and $A^{[*]}$, respectively, we get $AA^{[*]}A = AB^{[*]}A$. It concludes that $A \overset{[\diamond]}{\leqslant} B$.

Suppose $A \overset{[\diamond]}{\leqslant} B$, then $A^{[*]}AA^{[*]} = A^{[*]}BA^{[*]}$. Pre- and post-multiplying by $(A^{[\dagger]})^{[*]}$, we get $AA^{[\dagger]}BA^{[\dagger]}A = A$. Clearly, $P = AA^{[\dagger]}$ and $Q = A^{[\dagger]}A$ are projections and $A = PBQ$. Thus $A \overset{[+]}{\leqslant} B$.

Conversely, let $A \overset{[\diamond]}{\leqslant} B$ and $AB^{[*]}$ and $A^{[*]}B$ are Hermitian. Then $AA^{[*]} = AA^{[*]}AA^{[\dagger]} = AB^{[*]}AA^{[\dagger]} = AA^{[\dagger]}(AB^{[*]})^{[*]} = AB^{[*]}$. Similarly, we can prove $A^{[*]}A = B^{[*]}A$. It concludes $A \overset{[*]}{\leqslant} B$.

Suppose $A \overset{[+]}{\leqslant} B$ and set $P = AA^{[\dagger]}$ and $Q = A^{[\dagger]}A$, then we have $AA^{[*]} = PBQA^{[*]} = AA^{[\dagger]}BA^{[\dagger]}AA^{[*]} = AA^{[\dagger]}BA^{[*]} = AA^{[\dagger]}(AB^{[*]})^{[*]} = AB^{[*]}$. Similarly, we can prove $A^{[*]}A = B^{[*]}A$. It concludes $A \overset{[*]}{\leqslant} B$. $\square$

Acknowledgement. The first author thanks Prof. K. C. Sivakumar, IIT Madras, India, for his valuable suggestions during the preparation of the results presented in Section 3. The authors wish to thank anonymous referee(s) for their comments and suggestions which helped to improve the presentation.

REFERENCES

[1] J. K. BAKSALARY, *A Relationship Between the Star and Minus Orderings*, Linear Algebra and its Applications, **82**, (1986), 163–167.

[2] J. K. BAKSALARY AND J. HAUKE, *A further algebraic version of Cochran's theorem and matrix partial orderings*, Linear Algebra and its Applications, **127**, (1990), 157–169.

[3] J. K. BAKSALARY AND S. K. MITRA, *Left-Star and Right-Star Partial Orderings*, Linear Algebra and its Applications, **149**, (1991), 73–89.

[4] A. BEN ISRAEL AND T. N. E. GREVILLE, *Generalized inverses: Theory and applications*, 2nd edition CMS books in mathematics, New York (NY): Springer-Verlag; **15**, 2003.

[5] J. BOGNAR, *Indefinite Inner Product Spaces*, Springer, 1974.

[6] G. DOLINAR AND J. MARVOT, *Star partial order on $\mathscr{B}(\mathscr{H})$*, Linear Algebra and its Applications, **434**, (2011), 319–326.

[7] G. DOLINAR, A. GUTERMAN AND J. MARVOT, *Monotone Transformations on $\mathscr{B}(\mathscr{H})$ with Respect to the Left-star and the Right-star Order*, Mathematical Inequalities & Applications, **17**, (2014), 573–589.

[8] M. P. DRAZIN, *Natural structures on semigroups with involution*, Bulletin American Mathematical Society, **84**, (1978), 139–141.

[9] I. GOHBERG, P. LANCASTER AND L. RODMAN, *Indefinite linear algebra and applications*, Birkhauser Verlag, Basel, 2005.

[10] R. E. HARTWIG, *How to partially order regular elements?*, Japanese Journal of Mathematics, **25**, (1980), 1–13.

[11] S. HASSI, *A Singular value decomposition of matrices in a space with an indefinite scalar product*, Annales Academiae Scientiarum Fennaicae, Series A, Dissertationes, 79, 1992.

[12] J. HAUKE AND A. MARKIEWICZ, *On orderings induced by the Lowner partial ordering*, Applicationes Mathematicae, **22**, (1994), 145–154.

[13] K. KAMARAJ, *Certain Aspects of Matrices in Indefinite Inner Product Space*, Ph.D. Thesis, 2006.

[14] K. KAMARAJ, P. SAM JOHNSON AND K. ATHIRA SATHEESH, *Reverse Order Law for Generalized Inverses with Indefinite Hermitian Weights*, Filomat, **37**, (2023), 699–709.

[15] K. KAMARAJ AND K. C. SIVAKUMAR, *Moore-Penrose inverse in an indefinite inner product space*, Journal of Applied Mathematics and Computing, **19**, (2005), 297–310.

[16] K. KAMARAJ AND K. C. SIVAKUMAR, *Spectral theorem for normal matrices in an Indefinite Inner Product Space*, Journal of Analysis, **12**, (2004), 143–152.

[17] M. LAURA ARIAS, ALEJANDRA MAESTRIPIERI, *On partial orders of operators*, Annals of Functional Analysis, **14**, (2023), 1–17.
[18] S. K. MITRA, *Matrix partial orders through generalized inverses: unified theory*, Linear Algebra and its Applications, **148**, (1991), 237–263.
[19] S. K. MITRA, P. BHIMASANKARAM AND S. B. MALIK, *Matrix partial orders, shorted operators and applications*, World Scientific Publishing Company, 2010.
[20] P. SEMRL, *Automorphisms of $\mathscr{B}(\mathscr{H})$ with respect to minus partial order*, Jounal of Mathematical Analysis and Applications, **369**, (2010), 205–213.
[21] I. M. STANISEV, *Star partial order in indefinite inner product spaces*, Quaestiones Mathematicae, **45**, (2020), 213–221.

(Received September 2, 2023)

K. Kamaraj
Department of Mathematics
University College of Engineering, BIT Campus
Tiruchirappalli, 620 024, Tamilnadu, India
e-mail: krajkj@yahoo.com

A. Karpagam
Department of Mathematics
SRM Valliammai Engineering College
Kattankulathur, Chennai-603 203, Tamilnadu, India
e-mail: mohankarpu@gmail.com