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JENSEN TYPE INEQUALITIES FOR (m,M,ψ)–CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. Among various generalized classes of convexity, the class of (m,M,ψ)-convex functions, introduced by Dragomir in 2001, has attracted increasing attention recently. This class covers many other subclasses of convexity, such as the class of strongly convex functions, delta convex functions, approximately convex functions and others. In this paper, we present the Jensen and the Jensen-Steffensen type inequalities for (m,M,ψ)-convex functions. Our results extend and improve the corresponding results valid for different subclasses of convex functions. As application of the main results, we derive new lower and upper bounds estimations for some well-known mean inequalities.

1. Introduction

Recently, various generalized classes of convexity have been studied and the corresponding inequalities for these classes have been established. Among these generalizations, we point out the convexity generalization introduced by Dragomir [7]:

Let $m,M \in \mathbb{R}$, $I \subseteq \mathbb{R}$ and $\psi : I \to \mathbb{R}$ be a convex function. A function $\varphi : I \to \mathbb{R}$ is called:

- (m,ψ)-lower convex if the function $\varphi - m\psi$ is convex;
- (M,ψ)-upper convex if the function $M\psi - \varphi$ is convex;
- (m,M,ψ)-convex if it is (m,ψ)-lower convex and (M,ψ)-upper convex.

In accordance with this definition, if φ is (m,M,ψ)-convex, then $\varphi - m\psi$ and $M\psi - \varphi$ are convex and then the function $(M-m)\psi$ is convex, implying that $m \leqslant M$ whenever ψ is not trivial, *i.e.* is not the zero function.

Let us note that previous definition can be written in the following way:

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Let $m, M \in \mathbb{R}$ and $\varphi, \psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be functions such that ψ is convex. Then φ is said to be (m, ψ)-lower convex if

$$\begin{aligned} & m[\lambda \psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y)] \\ & \leqslant \lambda \varphi(x) + (1-\lambda)\varphi(y) - \varphi(\lambda x + (1-\lambda)y) \end{aligned} \tag{1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. Further, φ is said to be (M, ψ)-upper convex if

$$\begin{aligned} & \lambda \varphi(x) + (1-\lambda)\varphi(y) - \varphi(\lambda x + (1-\lambda)y) \\ & \leqslant M[\lambda \psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y)] \end{aligned} \tag{2}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If both inequalities (1) and (2) hold, then φ is said to be (m, M, ψ)-convex with $m \leqslant M$ whenever ψ is not trivial.

In the following part, we consider some particular cases of defined generalized classes of convexity.

According to [8] (see also [14]), for $\psi = id^2$, where id denotes the identity function, i.e. $id(t) = t$, $t \in I$, function φ is called m-lower convex if the function $\varphi - m \cdot id^2$ is convex and φ is called M-upper convex if the function $M \cdot id^2 - \varphi$ is convex. The same class of functions, known as convexifiable and concavifiable functions, was considered in [31] and [22].

Note that for $m = 0$ and $M = 0$ in (1) and (2), we get ordinary convexity and concavity, respectively.

Since ψ is convex function, inequality

$$\lambda \psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y) \geqslant 0$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Then in case $m > 0$, we have

$$\begin{aligned} & \varphi(\lambda x + (1-\lambda)y) \\ & \leqslant \lambda \varphi(x) + (1-\lambda)\varphi(y) - m[\lambda \psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y)] \\ & \leqslant \lambda \varphi(x) + (1-\lambda)\varphi(y), \end{aligned}$$

i.e. (m, ψ)-lower convexity implies ordinary convexity.

In particular, if $m > 0$ and $\psi = id^2$, then we come to the notion of strong convexity. The class of strongly convex functions was originally introduced in [27] and has played an important role in optimization theory. For more details on this concept as well as on new results on strong convexity, see [15], [16], [25], [28] and the references therein.

For $m < 0$ we are going in direction of consideration of approximately convex functions.

If $M < 0$, then

$$\begin{aligned} & \lambda \varphi(x) + (1-\lambda)\varphi(y) \\ & \leqslant \lambda \varphi(x) + (1-\lambda)\varphi(y) - M[\lambda \psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y)] \\ & \leqslant \varphi(\lambda x + (1-\lambda)y) \end{aligned}$$

i.e. (M,ψ)-upper convexity implies ordinary concavity.

For $M>0$, (M,ψ)-upper convex functions were investigated in [30] (see also the references therein) and are named delta convex functions. Such functions play an important role in convex analysis.

Also notice that for $M>0$ and $\psi=id^2$, corresponding (M,id^2)-upper convex functions were considered in [23] as approximately concave function.

Finally, let us mention the concept of g-convex dominated functions, introduced in [12], with g being a given convex function. Namely, function φ is called g-convex dominated if the functions $g+\varphi$ and $g-\varphi$ are convex. Note that this concept can be obtained as a particular case of (m,M,ψ)-convexity by choosing $m=-1$, $M=1$ and $\psi=g$.

In the sequel, I denotes a real interval with its interior $intI$.

We cite the following lemmas from [7] for its importance as characterizations of (m,M,ψ)-convex functions. The first one considers the supporting lines of convex functions.

LEMMA 1. *Let $\varphi,\psi\colon I\to\mathbb{R}$ be differentiable functions on $intI$ and suppose that ψ is convex on $intI$.*

a) For $m\in\mathbb{R}$, a function φ is (m,ψ)-lower convex iff

$$m\left[\psi(y)-\psi(z)-\psi'(z)(y-z)\right]\leqslant\varphi(y)-\varphi(z)-\varphi'(z)(y-z) \tag{3}$$

holds for all $y,z\in intI$.

b) For $M\in\mathbb{R}$, a function φ is (M,ψ)-upper convex iff

$$\varphi(y)-\varphi(z)-\varphi'(z)(y-z)\leqslant M\left[\psi(y)-\psi(z)-\psi'(z)(y-z)\right] \tag{4}$$

holds for all $y,z\in intI$.

c) For $m,M\in\mathbb{R}$, $m\leqslant M$, a function φ is (m,M,ψ)-convex iff both (3) and (4) hold.

The second lemma is a characterization that includes differentiability.

LEMMA 2. *Let $\varphi,\psi\colon I\to\mathbb{R}$ be twice differentiable on $intI$ and suppose ψ is convex on $intI$.*

a) For $m\in\mathbb{R}$, a function φ is (m,ψ)-lower convex on $intI$ iff

$$m\cdot\psi''(x)\leqslant\varphi''(x)\quad \text{for all } x\in intI. \tag{5}$$

b) For $M\in\mathbb{R}$, a function φ is (M,ψ)-upper convex on $intI$ iff

$$\varphi''(x)\leqslant M\cdot\psi''(x)\quad \text{for all } x\in intI. \tag{6}$$

c) *For* $m, M \in \mathbb{R}$, $m \leqslant M$, *a function* φ *is* (m, M, ψ)*-convex iff both* (5) *and* (6) *hold, i.e.*

$$m \cdot \psi''(x) \leqslant \varphi''(x) \leqslant M \cdot \psi''(x) \quad \text{for all } x \in \text{int} I.$$

More results related to the class of (m, M, ψ)-convex functions can be found in the papers [4]–[6], [10], [11], [13] and [19].

Our paper is organized as follows. In the second section, we prove useful lemmas with important characterizations of (m, ψ)-lower convex and (M, ψ)-upper convex functions. In the third section we obtain the Jensen type inequalities for (m, ψ)-lower convex, (M, ψ)-upper convex and (m, M, ψ)-convex functions. In the forth section, we prove that the same results hold under Steffensen's conditions (31), *i.e.* we prove the Jensen-Steffensen type inequalities for (m, ψ)-lower convex, (M, ψ)-upper convex and (m, M, ψ)-convex functions. In the last section we present applications of the obtained results by deriving new lower and upper bounds estimations for some well-known mean inequalities.

2. More about (m, M, ψ)-convexity

We supplement the characterizations given in Introduction with the following lemmas which provide simple consequences of the definition of the class of (m, M, ψ)-convex functions.

LEMMA 3. *Let* $\varphi, \psi \colon I \to \mathbb{R}$ *be functions and suppose* ψ *is convex.*

a) *If for* $m \in \mathbb{R}$, φ *is* (m, ψ)*-lower convex, then for every* $n \in \mathbb{R}$, $n < m$, φ *is* (n, ψ)*-lower convex.*

b) *If for* $M \in \mathbb{R}$, φ *is* (M, ψ)*-upper convex, then for every* $N \in \mathbb{R}$, $N > M$, φ *is* (N, ψ)*-upper convex.*

c) *If for* $m, M \in \mathbb{R}$, $m \leqslant M$, φ *is* (m, M, ψ)*-convex, then for every* $n, N \in \mathbb{R}$, $n < m \leqslant M < N$, φ *is* (n, N, ψ)*-convex.*

Proof. a) Since φ is (m, ψ)-lower convex, then

$$\begin{aligned} &m[\lambda\psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y)] + \varphi(\lambda x + (1-\lambda)y) \\ &\leqslant \lambda\varphi(x) + (1-\lambda)\varphi(y) \end{aligned}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. Further, ψ is convex and then

$$\lambda\psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y) \geqslant 0$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. Moreover, for every $n \in \mathbb{R}$, $n < m$,

$$\begin{aligned} &\lambda\varphi(x) + (1-\lambda)\varphi(y) \\ &\geqslant m[\lambda\psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y)] + \varphi(\lambda x + (1-\lambda)y) \\ &\geqslant n[\lambda\psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y)] + \varphi(\lambda x + (1-\lambda)y), \end{aligned}$$

what we need to prove.

b) Since φ is (M,ψ)-upper convex, then for every $N\in\mathbb{R},\ N>M$,

$$\begin{aligned}&\lambda\varphi(x)+(1-\lambda)\varphi(y)\\&\leqslant M\left[\lambda\psi(x)+(1-\lambda)\psi(y)-\psi(\lambda x+(1-\lambda)y)\right]+\varphi(\lambda x+(1-\lambda)y)\\&\leqslant N\left[\lambda\psi(x)+(1-\lambda)\psi(y)-\psi(\lambda x+(1-\lambda)y)\right]+\varphi(\lambda x+(1-\lambda)y)\end{aligned}$$

holds for all $x,y\in I$ and $\lambda\in[0,1]$.

c) This part is combination of a) and b). □

LEMMA 4. *Let $\varphi,\psi\colon I\to\mathbb{R}$ be functions such that ψ is a convex one. Let $m,M\in\mathbb{R}$.*

a) If φ is (m,ψ)-lower convex, then it is continuous on $\mathrm{int}I$ and has finite left and right derivatives at each point of $\mathrm{int}I$. Further, for every $x,y\in \mathrm{int}I,\ x\leqslant y$, holds

$$\begin{aligned}\varphi'_-(x)-m\cdot\psi'_-(x)&\leqslant\varphi'_+(x)-m\cdot\psi'_+(x)\\&\leqslant\varphi'_-(y)-m\cdot\psi'_-(y)\leqslant\varphi'_+(y)-m\cdot\psi'_+(y).\end{aligned}\tag{7}$$

Additionally, if φ and ψ are differentiable, then for every $x,y\in \mathrm{int}I,\ x\leqslant y$, holds

$$m\cdot(\psi'(y)-\psi'(x))\leqslant\varphi'(y)-\varphi'(x).\tag{8}$$

b) If φ is (M,ψ)-upper convex, then it is continuous on $\mathrm{int}I$ and has finite left and right derivatives at each point of $\mathrm{int}I$. Further, for every $x,y\in \mathrm{int}I,\ x\leqslant y$, holds

$$\begin{aligned}M\cdot\psi'_-(x)-\varphi'_-(x)&\leqslant M\cdot\psi'_+(x)-\varphi'_+(x)\\&\leqslant M\cdot\psi'_-(y)-\varphi'_-(y)\leqslant M\cdot\psi'_+(y)-\varphi'_+(y).\end{aligned}\tag{9}$$

Additionaly, if φ and ψ are differentiable, then for every $x,y\in \mathrm{int}I,\ x\leqslant y$, holds

$$\varphi'(y)-\varphi'(x)\leqslant M\cdot\left(\psi'(y)-\psi'(x)\right).\tag{10}$$

Proof. a) As an easy consequence of the Stolz theorem (see [24, p. 25.]) applied to the convex function ψ and $g=\varphi-m\psi$, we have that ψ and g are continuous on $\mathrm{int}I$ and have finite left and right derivatives at each point of $\mathrm{int}I$. Then the same holds for the function $\varphi=g+m\psi$. Moreover, by the Stolz theorem, functions $\psi'_-,\ \psi'_+,\ g'_-$ and g'_+ are *nondecreasing* and for every $x,y\in \mathrm{int}I,\ x\leqslant y$, we have

$$\psi'_-(x)\leqslant\psi'_+(x)\leqslant\psi'_-(y)\leqslant\psi'_+(y)$$

and

$$g'_-(x)\leqslant g'_+(x)\leqslant g'_-(y)\leqslant g'_+(y),$$

where the last relations are equivalent to (7).

If φ and ψ are differentiable, then $\varphi'(x)=\varphi'_-(x)=\varphi'_+(x)$ and $\psi'(x)=\psi'_-(x)=\psi'_+(x)$, *i.e.* $g'(x)=g'_-(x)=g'_+(x)$ holds for all $x\in intI$. Further, for every $x,y\in intI$, $x\leqslant y$, we have

$$g'(x)\leqslant g'(y),$$

which is equivalent to (8).

b) We proceed analogously and omitting details, but this time observing the convex function $h=M\psi-\varphi$. □

Note that inequalities (3) and (4) follow from the fact that the differentiable functions $g=\varphi-m\psi$ and $h=M\psi-\varphi$ are convex iff

$$g(y)-g(z)\geqslant g'(z)(y-z),$$
$$h(y)-h(z)\geqslant h'(z)(y-z)$$

hold for all $y,z\in intI$. Instead of the assumption of differentiability we can take any $c(z)\in[g'_-(z),g'_+(z)]$, $d(z)\in[h'_-(z),h'_+(z)]$, where $y,z\in intI$, and

$$g(y)-g(z)\geqslant c(z)(y-z), \tag{11}$$
$$h(y)-h(z)\geqslant d(z)(y-z)$$

hold for all $y,z\in intI$.

Without loss of generality we can set $c(z)=g'_+(z)=\varphi'_+(z)-m\psi'_+(z)$ and $d(z)=h'_+(z)=M\psi'_+(z)-\varphi'_+(z)$ and then (11) are equivalent to

$$m\left[\psi(y)-\psi(z)-\psi'_+(z)(y-z)\right]\leqslant\varphi(y)-\varphi(z)-\varphi'_+(z)(y-z)$$
$$\varphi(y)-\varphi(z)-\varphi'_+(z)(y-z)\leqslant M\left[\psi(y)-\psi(z)-\psi'_+(z)(y-z)\right]$$

for all $y,z\in intI$.

In the rest of the paper, unless otherwise stated, (a,b) denotes a real interval such that $-\infty\leqslant a<b\leqslant\infty$.

For $\varphi,\psi\colon(a,b)\to\mathbb{R}$, without loss of generality, we may set $\varphi'(x)=\varphi'_+(x)$ and $\psi'(x)=\psi'_+(x)$, for any $x\in(a,b)$.

LEMMA 5. *Let $\varphi,\psi\colon(a,b)\to\mathbb{R}$ be functions such that ψ is convex and let $m\in\mathbb{R}$. If φ is (m,ψ)-lower convex, then for a fixed $z\in(a,b)$, the functions $F_1,\bar{F}_1\colon(a,b)\to\mathbb{R}$, defined by*

$$F_1(y)=\varphi(y)-\varphi(z)-\varphi'(z)(y-z)-m\left[\psi(y)-\psi(z)-\psi'(z)(y-z)\right]$$

and

$$\bar{F}_1(y)=\varphi(z)-\varphi(y)-\varphi'(y)(z-y)-m\left[\psi(z)-\psi(y)-\psi'(y)(z-y)\right]$$

are nonnegative on (a,b), nonincreasing on $(a,z]$ and nondecreasing on $[z,b)$.

Proof. Nonnegativity of the functions F_1 and $\bar{F}_1$ follows from (3).

First we prove the statement for the function F_1.

Let $a < y_1 < y_2 \leqslant z$.

By (8) we have $\varphi'(y_2) + m \cdot (\psi'(z) - \psi'(y_2)) \leqslant \varphi'(z)$. Multiplying it with $(y_1 - y_2) < 0$, we get

$$\varphi'(y_2)(y_1 - y_2) + m \cdot \left[\psi'(z) - \psi'(y_2)\right](y_1 - y_2) \geqslant \varphi'(z)(y_1 - y_2). \tag{12}$$

Let us calculate $F_1(y_1) - F_1(y_2)$. It holds

$$\begin{aligned} & F_1(y_1) - F_1(y_2) \\ & = \varphi(y_1) - \varphi(z) - \varphi'(z)(y_1 - z) - m\left[\psi(y_1) - \psi(z) - \psi'(z)(y_1 - z)\right] \\ & \quad - \varphi(y_2) + \varphi(z) + \varphi'(z)(y_2 - z) + m\left[\psi(y_2) - \psi(z) - \psi'(z)(y_2 - z)\right] \\ & = \varphi(y_1) - \varphi(y_2) - \varphi'(z)(y_1 - y_2) - m\left[\psi(y_1) - \psi(y_2) - \psi'(z)(y_1 - y_2)\right]. \end{aligned} \tag{13}$$

Further, by using (12), we have

$$\begin{aligned} & \varphi(y_1) - \varphi(y_2) - \varphi'(z)(y_1 - y_2) - m\left[\psi(y_1) - \psi(y_2) - \psi'(z)(y_1 - y_2)\right] \\ & \geqslant \varphi(y_1) - \varphi(y_2) - \varphi'(y_2)(y_1 - y_2) \\ & \quad - m \cdot \left[\psi'(z) - \psi'(y_2)\right](y_1 - y_2) - m\left[\psi(y_1) - \psi(y_2) - \psi'(z)(y_1 - y_2)\right] \\ & = \varphi(y_1) - \varphi(y_2) - \varphi'(y_2)(y_1 - y_2) - m \cdot \left[\psi(y_1) - \psi(y_2) - \psi'(y_2)(y_1 - y_2)\right] \\ & \geqslant 0, \end{aligned}$$

where nonnegativity follows from (3). This proves that F_1 is *nonincreasing on* $(a, z]$.

Now, let $z \leqslant y_1 < y_2 < b$.

Then by (8) we have $\varphi'(z) + m \cdot (\psi'(y_1) - \psi'(z)) \leqslant \varphi'(y_1)$. Multiplying it with $(y_1 - y_2) < 0$, we get

$$\varphi'(y_1)(y_1 - y_2) - m \cdot \left[\psi'(y_1) - \psi'(z)\right](y_1 - y_2) \leqslant \varphi'(z)(y_1 - y_2). \tag{14}$$

Now, by using (14), we have

$$\begin{aligned} & F_1(y_1) - F_1(y_2) \\ & = \varphi(y_1) - \varphi(y_2) - \varphi'(z)(y_1 - y_2) - m\left[\psi(y_1) - \psi(y_2) - \psi'(z)(y_1 - y_2)\right] \\ & \leqslant \varphi(y_1) - \varphi(y_2) - \varphi'(y_1)(y_1 - y_2) + m \cdot \left[\psi'(y_1) - \psi'(z)\right](y_1 - y_2) \\ & \quad - m\left[\psi(y_1) - \psi(y_2) - \psi'(z)(y_1 - y_2)\right] \\ & = \varphi(y_1) - \varphi(y_2) - \varphi'(y_1)(y_1 - y_2) - m\left[\psi(y_1) - \psi(y_2) - \psi'(y_1)(y_1 - y_2)\right] \\ & \leqslant 0, \end{aligned} \tag{15}$$

where nonpositivity follows from (3).

This proves that F_1 is nondecreasing on $[z, b)$.

The statement for the function $\bar{F}_1$ we can prove in an analogous way, so we omite it. □

LEMMA 6. *Let $\varphi, \psi \colon (a,b) \to \mathbb{R}$ be functions such that ψ is convex and let $M \in \mathbb{R}$. If φ is (M,ψ)-upper convex, then for a fixed $z \in (a,b)$, functions $F_2, \bar{F}_2 \colon (a,b) \to \mathbb{R}$, defined by*

$$F_2(y) = \varphi(y) - \varphi(z) - \varphi'(z)(y-z) - M\left[\psi(y) - \psi(z) - \psi'(z)(y-z)\right]$$

and

$$\bar{F}_2(y) = \varphi(z) - \varphi(y) - \varphi'(y)(z-y) - M\left[\psi(z) - \psi(y) - \psi'(y)(z-y)\right]$$

are nonpositive on (a,b), nondecreasing on $(a,z]$ and nonincreasing on $[z,b)$.

Proof. Nonpositivity of the functions F_2 and $\bar{F}_2$ follows from (4).

Further, we proceed analogously as in proof of Lemma 5, only instead of (8) and (3), we use (10) and (4), respectively. We omit the details. □

3. The Jensen type inequalities

In this section we deal with the Jensen type inequalities for (m,ψ)-lower convex, (M,ψ)-upper convex and (m,M,ψ)-convex functions.

THEOREM 1. *Let $\boldsymbol{x} = (x_1,\ldots,x_n) \in (a,b)^n$ and $\boldsymbol{a} = (a_1,\ldots,a_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^n a_i = 1$ with $\bar{x} = \sum_{i=1}^n a_i x_i$. Let $m, M \in \mathbb{R}$ and suppose $\varphi, \psi \colon (a,b) \to \mathbb{R}$ are functions such that ψ is convex.*

a) If φ is an (m,ψ)-lower convex function, then for any $d, e \in (a,b)$,

$$\begin{aligned}
&\varphi(d) + \varphi'(d)(\bar{x}-d) + m\left(\sum_{i=1}^n a_i \psi(x_i) - \psi(d) - \psi'(d)(\bar{x}-d)\right) \\
&\leqslant \sum_{i=1}^n a_i \varphi(x_i) \qquad (16)\\
&\leqslant \varphi(e) - \sum_{i=1}^n a_i \varphi'(x_i)(e-x_i) \\
&\quad - m\left(\psi(e) - \sum_{i=1}^n a_i \psi(x_i) - \sum_{i=1}^n a_i \psi'(x_i)(e-x_i)\right).
\end{aligned}$$

b) If φ is an (M,ψ)-upper convex function, then for any $e, d \in (a,b)$,

$$\begin{aligned}
&M\left(\sum_{i=1}^n a_i \psi(x_i) + \sum_{i=1}^n a_i \psi'(x_i)(e-x_i) - \psi(e)\right) + \varphi(e) - \sum_{i=1}^n a_i \varphi'(x_i)(e-x_i) \\
&\leqslant \sum_{i=1}^n a_i \varphi(x_i) \qquad (17)\\
&\leqslant M\left(\sum_{i=1}^n a_i \psi(x_i) - \psi(d) - \psi'(d)(\bar{x}-d)\right) + \varphi(d) + \varphi'(d)(\bar{x}-d).
\end{aligned}$$

c) If φ is (m,M,ψ)-convex, $m \leqslant M$, then both (16) and (17) hold.

Proof. a) Applying (3) to $z=d$ and $y=x_i$, $i\in\{1,\dots,n\}$, we have

$$m\left[\psi(x_i)-\psi(d)-\psi'(d)(x_i-d)\right]\leqslant \varphi(x_i)-\varphi(d)-\varphi'(d)(x_i-d). \tag{18}$$

Multiplying (18) with a_i and summing over i, $i=1,\dots,n$, we get

$$m\left(\sum_{i=1}^n a_i\psi(x_i)-\psi(d)-\psi'(d)(\bar{x}-d)\right)\leqslant \sum_{i=1}^n a_i\varphi(x_i)-\varphi(d)-\varphi'(d)(\bar{x}-d). \tag{19}$$

On the other side, applying (3) to $y=e$ and $z=x_i$, $i\in\{1,\dots,n\}$, we have

$$m\left[\psi(e)-\psi(x_i)-\psi'(x_i)(e-x_i)\right]\leqslant \varphi(e)-\varphi(x_i)-\varphi'(x_i)(e-x_i)$$

and multiplying it with a_i and summing over i, $i=1,\dots,n$, we get

$$m\left(\psi(e)-\sum_{i=1}^n a_i\psi(x_i)-\sum_{i=1}^n a_i\psi'(x_i)(e-x_i)\right) \tag{20}$$
$$\leqslant \varphi(e)-\sum_{i=1}^n a_i\varphi(x_i)-\sum_{i=1}^n a_i\varphi'(x_i)(e-x_i).$$

Now, combining inequalities (19) and (20), we get (16).

b) We proceed analogously, only instead of (3) we use (4).

c) This case is proved by a) and b) combined. □

As an easy consequence of the previous theorem we get the following corollary.

COROLLARY 1. *Let the assumptions of Theorem 1 hold.*

a) If φ is (m,ψ)-lower convex, then

$$m\left(\sum_{i=1}^n a_i\psi(x_i)-\psi(\bar{x})\right) \tag{21}$$
$$\leqslant \sum_{i=1}^n a_i\varphi(x_i)-\varphi(\bar{x})$$
$$\leqslant \sum_{i=1}^n a_i\varphi'(x_i)(x_i-\bar{x})-m\left(\psi(\bar{x})-\sum_{i=1}^n a_i\psi(x_i)-\sum_{i=1}^n a_i\psi'(x_i)(\bar{x}-x_i)\right).$$

b) If φ is (M,ψ)-upper convex, then

$$M\left(\sum_{i=1}^n a_i\psi(x_i)+\sum_{i=1}^n a_i\psi'(x_i)(\bar{x}-x_i)-\psi(\bar{x})\right)-\sum_{i=1}^n a_i\varphi'(x_i)(\bar{x}-x_i)$$
$$\leqslant \sum_{i=1}^n a_i\varphi(x_i)-\varphi(\bar{x}) \tag{22}$$
$$\leqslant M\left(\sum_{i=1}^n a_i\psi(x_i)-\psi(\bar{x})\right).$$

c) If φ is (m,M,ψ)-convex, $m \leqslant M$, then

$$m\left(\sum_{i=1}^{n} a_i \psi(x_i) - \psi(\bar{x})\right) \leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \leqslant M\left(\sum_{i=1}^{n} a_i \psi(x_i) - \psi(\bar{x})\right) \tag{23}$$

and

$$\begin{aligned} &M\left(\sum_{i=1}^{n} a_i \psi(x_i) + \sum_{i=1}^{n} a_i \psi'(x_i)(\bar{x}-x_i) - \psi(\bar{x})\right) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) + \sum_{i=1}^{n} a_i \varphi'(x_i)(\bar{x}-x_i) \\ &\leqslant m\left(\sum_{i=1}^{n} a_i \psi(x_i) + \sum_{i=1}^{n} a_i \psi'(x_i)(\bar{x}-x_i) - \psi(\bar{x})\right). \end{aligned} \tag{24}$$

Proof. a) If we set $d = e = \bar{x}$, then (16) becomes

$$\begin{aligned} &\varphi(\bar{x}) + m\left(\sum_{i=1}^{n} a_i \psi(x_i) - \psi(\bar{x})\right) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) \\ &\leqslant \varphi(\bar{x}) - \sum_{i=1}^{n} a_i \varphi'(x_i)(\bar{x}-x_i) - m\left(\psi(\bar{x}) - \sum_{i=1}^{n} a_i \psi(x_i) - \sum_{i=1}^{n} a_i \psi'(x_i)(\bar{x}-x_i)\right) \end{aligned}$$

which is equivalent to (21).

b) If we set $d = e = \bar{x}$, then (17) becomes

$$\begin{aligned} &M\left(\sum_{i=1}^{n} a_i \psi(x_i) + \sum_{i=1}^{n} a_i \psi'(x_i)(\bar{x}-x_i) - \psi(\bar{x})\right) + \varphi(\bar{x}) - \sum_{i=1}^{n} a_i \varphi'(x_i)(\bar{x}-x_i) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) \\ &\leqslant M\left(\sum_{i=1}^{n} a_i \psi(x_i) - \psi(\bar{x})\right) + \varphi(\bar{x}) \end{aligned}$$

which is equivalent to (22).

c) Combining inequalities (21) and (22), we get (23) and (24). □

REMARK 1. Note that if ψ is a zero function, *i.e.* $\psi(t) = 0$, for all $t \in (a,b)$, then φ is convex in the usual sense and (16) becomes

$$\varphi(d) + \varphi'(d)(\bar{x}-d) \leqslant \sum_{i=1}^{n} a_i \varphi(x_i) \leqslant \varphi(e) - \sum_{i=1}^{n} a_i \varphi'(x_i)(e-x_i), \tag{25}$$

i.e. we get the Jensen type inequalities for convex functions. Specially, (21) becomes

$$0 \leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \leqslant \sum_{i=1}^{n} a_i \varphi'(x_i)(x_i - \bar{x}), \tag{26}$$

where the first inequality in (26), i.e.

$$\varphi(\bar{x}) \leqslant \sum_{i=1}^{n} a_i \varphi(x_i) \tag{27}$$

is Jensen's inequality for convex functions and the second inequality in (26) is a counterpart of Jensen's inequality for convex functions, proved in [9].

If $m = c > 0$ and $\psi = id^2$, then (16) becomes

$$\begin{aligned} &\varphi(d) + \varphi'(d)(\bar{x} - d) + c \sum_{i=1}^{n} a_i (x_i - d)^2 \\ &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) \\ &\leqslant \varphi(e) - \sum_{i=1}^{n} a_i \varphi'(x_i)(e - x_i) - c \sum_{i=1}^{n} a_i (x_i - e)^2, \end{aligned} \tag{28}$$

i.e. we get the Jensen type inequalities for strongly convex functions which accompanies an integral version of (28) for strongly convex functions observed in [29].

Furthermore, the first inequality in (21) is Jensen's inequality for (m, ψ)-lower convex functions, while the second inequality in (22) is Jensen's inequality for (M, ψ)-upper convex functions. Jensen's inequality for (m, M, ψ)-convex function φ is inequality (23). On the other hand, the Jessen type inequalities for (m, M, ψ)-convex functions, i.e. generalizations of (23) for positive linear functionals were proved in [6].

If $m = c > 0$ and $\psi = id^2$, then (21) becomes

$$\begin{aligned} c \sum_{i=1}^{n} a_i (x_i - \bar{x})^2 &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi'(x_i)(x_i - \bar{x}) - c \sum_{i=1}^{n} a_i (x_i - \bar{x})^2. \end{aligned} \tag{29}$$

The first inequality in (29), i.e.

$$\varphi(\bar{x}) + c \sum_{i=1}^{n} a_i (x_i - \bar{x})^2 \leqslant \sum_{i=1}^{n} a_i \varphi(x_i)$$

is Jensen's inequality for strongly convex functions with modulus c (see [25]).

For $M = c > 0$ and $\psi = id^2$, inequality (22) becomes

$$\begin{aligned} c \sum_{i=1}^{n} a_i (\bar{x} - x_i)^2 - \sum_{i=1}^{n} a_i \varphi'(x_i)(\bar{x} - x_i) &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \\ &\leqslant c \sum_{i=1}^{n} a_i (x_i - \bar{x})^2. \end{aligned} \tag{30}$$

The second inequality in (30), i.e.

$$\sum_{i=1}^{n} a_i\varphi(x_i) \leqslant \varphi(\bar{x}) + c\sum_{i=1}^{n} a_i\left(x_i-\bar{x}\right)^2$$

is Jensen's inequality for approximately concave functions, obtained in [23].

4. The Jensen-Steffensen type inequalities

The assumption "$\boldsymbol{a}=(a_1,\ldots,a_n)$ is nonnegative n-tuple" can be relaxed at the expense of more restrictions on the n-tuple $\boldsymbol{x}$. Namely, if $\boldsymbol{a}=(a_1,\ldots,a_n)$ is a real n-tuple that satisfies

$$0 \leqslant A_j = \sum_{i=1}^{j} a_i \leqslant A_n, \quad j=1,\ldots,n, \quad A_n=1, \tag{31}$$

then for any monotonic (increasing or decreasing) n-tuple $\boldsymbol{x}=(x_1,\ldots,x_n)\in(a,b)^n$ we have

$$\bar{x}=\textstyle\sum_{i=1}^{n} a_i x_i \in (a,b),$$

and for any convex function $\varphi\colon (a,b)\to\mathbb{R}$, inequality (27) still holds. Inequality (27) considered under conditions (31) is known as the Jensen-Steffensen inequality for convex functions (see [26]). The Jensen-Steffensen inequality is a proper generalization of Jensen's inequality since nonnegative weights $\boldsymbol{a}$ satisfy Steffensen's conditions (31) in every order, which means that for nonnegative weights the monotonicity condition on $\boldsymbol{x}$ becomes irrelevant.

In this section we prove the Jensen-Steffensen type inequalities for (m,ψ)-lower convex, (M,ψ)-upper convex functions and (m,M,ψ)-convex functions, *i.e.* we prove that inequalities from the previous section hold under Steffensen's conditions (31).

THEOREM 2. *Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ be any monotonic n-tuple (increasing or decreasing) in $(a,b)^n$ and $\mathbf{a}=(a_1,\ldots,a_n)$ be a real n-tuple such that (31) holds. Let $m,M\in\mathbb{R}$ and $\psi,\varphi\colon (a,b)\to\mathbb{R}$ be functions such that ψ is convex.*

a) If φ is (m,ψ)-lower convex, then (16) holds for all $d,e\in(a,b)$. In particular, (21) holds.

b) If φ is (M,ψ)-upper convex, then (17) holds for all $d,e\in(a,b)$. In particular, (22) holds.

c) If φ is (m,M,ψ)-convex, $m\leqslant M$, then both (16) and (17) hold. In particular, (23) and (24) hold.

Proof. Without any loss of generality, we may assume that $x_1\leqslant x_2\leqslant\ldots\leqslant x_n$.

Let $\bar{x}=\sum_{i=1}^{n} a_i x_i$. Under the assumptions of theorem, we have $x_1\leqslant\bar{x}\leqslant x_n$ (for the proof see [3].)

a) Let $F_1(y) = \varphi(y) - \varphi(d) - \varphi'(d)(y-d) - m[\psi(y) - \psi(d) - \psi'(d)(y-d)]$.
From Lemma 5 we have $F_1(x_i) \geqslant 0$ for all $i \in \{1, \ldots, n\}$.
Comparing d with $x_1, \ldots, x_n$ we need to consider three cases.

Case 1. $x_n < d < b$: In this case $x_i \in (a,d)$ for all $i = 1, \ldots, n$. Hence, according to Lemma 5 we have

$$F_1(x_1) \geqslant F_1(x_2) \geqslant \ldots \geqslant F_1(x_n) \geqslant 0.$$

Denoting $A_0 = 0$ it follows

$$a_i = A_i - A_{i-1}, \qquad i = 1, \ldots, n,$$

and therefore

$$\begin{aligned}\sum_{i=1}^{n} a_i F_1(x_i) &= \sum_{i=1}^{n} (A_i - A_{i-1}) F_1(x_i) \\ &= A_1 F_1(x_1) + (A_2 - A_1) F_1(x_2) + \ldots + (A_n - A_{n-1}) F_1(x_n) \\ &= \sum_{i=1}^{n-1} A_i (F_1(x_i) - F_1(x_{i+1})) + A_n F_1(x_n) \\ &\geqslant 0.\end{aligned}$$

Case 2. $a \leqslant d < x_1$: In this case $x_i \in (d,b)$ for all $i = 1, \ldots, n$. Hence, according to Lemma 5 we have

$$0 \leqslant F_1(x_1) \leqslant F_1(x_2) \leqslant \ldots \leqslant F_1(x_n).$$

Denoting $\overline{A}_{n+1} = 0$ it follows

$$\begin{aligned}\overline{A}_k &= \sum_{i=k}^{n} a_i = A_n - A_{k-1}, \qquad k = 1, \ldots, n, \\ a_i &= \overline{A}_i - \overline{A}_{i+1}, \qquad i = 1, \ldots, n,\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{i=1}^{n} a_i F_1(x_i) &= \sum_{i=1}^{n} \left(\overline{A}_i - \overline{A}_{i+1}\right) F_1(x_i) \\ &= \overline{A}_1 F_1(x_1) + \sum_{i=2}^{n} \overline{A}_i (F_1(x_i) - F_1(x_{i-1})) \\ &\geqslant 0.\end{aligned}$$

Case 3. $x_1 \leqslant d \leqslant x_n$: In this case there exists $k \in \{1, \ldots, n-1\}$ such that $x_k \leqslant d \leqslant x_{k+1}$.

By Lemma 5 we get

$$F_1(x_1) \geqslant F_1(x_2) \geqslant \ldots \geqslant F_1(x_k) \geqslant 0 \quad \text{and} \quad 0 \leqslant F_1(x_{k+1}) \leqslant F_1(x_{k+2}) \leqslant \ldots \leqslant F_1(x_n).$$

Then

$$\begin{aligned}\sum_{i=1}^{n} a_i F_1(x_i) &= \sum_{i=1}^{k} a_i F_1(x_i) + \sum_{i=k+1}^{n} a_i F_1(x_i)\\ &= \sum_{i=1}^{k-1} A_i \left(F_1(x_i) - F_1(x_{i+1})\right) + A_k F_1(x_k)\\ &\quad + \overline{A}_{k+1} F_1(x_{k+1}) + \sum_{i=k+2}^{n} \overline{A}_i \left(F_1(x_i) - F_1(x_{i-1})\right)\\ &\geqslant 0.\end{aligned}$$

In all three cases we have

$$\begin{aligned}&\sum_{i=1}^{n} a_i F_1(x_i)\\ &= \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(d) - \varphi'(d)(\overline{x} - d) - m\left(\sum_{i=1}^{n} a_i \psi(x_i) - \psi(d) - \psi'(d)(\overline{x} - d)\right)\\ &\geqslant 0,\end{aligned}$$

and therefore, the first inequality in (16) holds.

Now, let $\bar{F}_1(x_i) = \varphi(e) - \varphi(x_i) - \varphi'(x_i)(e - x_i) - m[\psi(e) - \psi(x_i) - \psi'(x_i)(e - x_i)]$, $i = 1, \ldots, n$.

Analogously, applying Lemma 5, we can prove that

$$\begin{aligned}\sum_{i=1}^{n} a_i \bar{F}_1(x_i) &= \varphi(e) - \sum_{i=1}^{n} a_i \varphi(x_i) - \sum_{i=1}^{n} a_i \varphi'(x_i)(e - x_i)\\ &\quad - m\left[\psi(e) - \sum_{i=1}^{n} a_i \psi(x_i) - \sum_{i=1}^{n} a_i \psi'(x_i)(e - x_i)\right]\\ &\geqslant 0,\end{aligned}$$

i.e. the second inequality in (16) holds.

Inserting $d = e = \overline{x}$ in (16), we get (21).

b) Analogously as in the proof of case a), we show that

$$\begin{aligned}\sum_{i=1}^{n} a_i F_2(x_i) &= \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(d) - \varphi'(d)(\overline{x} - d)\\ &\quad - M\left(\sum_{i=1}^{n} a_i \psi(x_i) - \psi(d) - \psi'(d)(\overline{x} - d)\right)\\ &\leqslant 0\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n} a_i \bar{F}_2(x_i) &= \varphi(e) - \sum_{i=1}^{n} a_i \varphi(x_i) - \sum_{i=1}^{n} a_i \varphi'(x_i)(e - x_i) \\ &\quad - M\left(\psi(e) - \sum_{i=1}^{n} a_i \psi(x_i) - \sum_{i=1}^{n} a_i \psi'(x_i)(e - x_i)\right) \\ &\leqslant 0,\end{aligned}$$

i.e. (17) holds.

Inserting $d = e = \bar{x}$ in (17), we get (22).

c) This case is obtained by a) and b) combined. □

REMARK 2. Note that in the trivial case of ψ being a zero function, function φ is convex in the usual sense and (16) reduces to the Jensen-Steffensen type inequalities for convex functions, i.e. inequality (25) holds under Steffensen's conditions (31) and its particular case is (26). Similar inequalities for convex functions were proved in [20]. For related results see also [1], [2], [17], [18] and [21].

If $m = c > 0$ and $\psi = id^2$, then (16) considered under (31) assumes form as in (28), i.e. we get the Jensen-Steffensen type inequalities for strongly convex functions, with particular case (29). These accompany analogous integral versions for strongly convex functions, proved in [29].

5. Applications

First we present the Jensen type inequalities for twice differentiable (m,ψ)-lower convex, (M,ψ)-upper convex and (m,M,ψ)-convex functions with some specified forms of the function ψ.

PROPOSITION 1. *Let $I \subseteq (0,\infty)$ be an open interval, $\mathbf{x} = (x_1,\ldots,x_n) \in I^n$ and $\mathbf{a} = (a_1,\ldots,a_n)$ be nonnegative n-tuple such that $\sum_{i=1}^{n} a_i = 1$ and $\bar{x} = \sum_{i=1}^{n} a_i x_i$. Let $\varphi\colon I \to \mathbb{R}$ be a twice differentiable function and $g_p\colon I \to \mathbb{R}$ be defined by $g_p(t) = \varphi''(t)t^{2-p}$, where $p \in (-\infty,0) \cup (1,\infty)$.*

a) If $\inf_{t\in I} g_p(t) = \gamma > -\infty$, then

$$\begin{aligned}&\frac{\gamma}{p(p-1)}\left(\sum_{i=1}^{n} a_i x_i^p - \bar{x}^p\right) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \qquad (32)\\ &\leqslant \sum_{i=1}^{n} a_i \varphi'(x_i)(x_i - \bar{x}) - \frac{\gamma}{p(p-1)}\left(\bar{x}^p - p\bar{x}\sum_{i=1}^{n} a_i x_i^{p-1} + (p-1)\sum_{i=1}^{n} a_i x_i^p\right).\end{aligned}$$

b) If $\sup_{t\in I} g_p(t) = \delta < \infty$, *then*

$$\frac{\delta}{p(p-1)}\left(p\bar{x}\sum_{i=1}^{n} a_i x_i^{p-1} + (1-p)\sum_{i=1}^{n} a_i x_i^{p} - \bar{x}^p\right) - \sum_{i=1}^{n} a_i\varphi'(x_i)(\bar{x}-x_i) \leqslant \sum_{i=1}^{n} a_i\varphi(x_i) - \varphi(\bar{x}) \leqslant \frac{\delta}{p(p-1)}\left(\sum_{i=1}^{n} a_i x_i^p - \bar{x}^p\right). \tag{33}$$

c) If $-\infty < \gamma \leqslant g_p(t) \leqslant \delta < \infty$, *for all* $t \in I$, *then*

$$\frac{\gamma}{p(p-1)}\left(\sum_{i=1}^{n} a_i x_i^p - \bar{x}^p\right) \leqslant \sum_{i=1}^{n} a_i\varphi(x_i) - \varphi(\bar{x}) \leqslant \frac{\delta}{p(p-1)}\left(\sum_{i=1}^{n} a_i x_i^p - \bar{x}^p\right) \tag{34}$$

and

$$\frac{\delta}{p(p-1)}\left(p\bar{x}\sum_{i=1}^{n} a_i x_i^{p-1} + (1-p)\sum_{i=1}^{n} a_i x_i^{p} - \bar{x}^p\right) \leqslant \sum_{i=1}^{n} a_i\varphi(x_i) - \varphi(\bar{x}) + \sum_{i=1}^{n} a_i\varphi'(x_i)(\bar{x}-x_i) \leqslant \frac{\gamma}{p(p-1)}\left(p\bar{x}\sum_{i=1}^{n} a_i x_i^{p-1} + (1-p)\sum_{i=1}^{n} a_i x_i^{p} - \bar{x}^p\right). \tag{35}$$

Proof. a) Let's consider the function $h_p(t) = \varphi(t) - \frac{\gamma}{p(p-1)}t^p$. Then

$$h_p''(t) = \varphi''(t) - \gamma t^{p-2} = t^{p-2}\left(t^{2-p}\varphi''(t) - \gamma\right) = t^{p-2}(g_p(t) - \gamma) \geqslant 0,$$

i.e. h_p is convex and then the function φ is $\left(\frac{\gamma}{p(p-1)}, (\cdot)^p\right)$-lower convex.

Now, applying (21) to the $\left(\frac{\gamma}{p(p-1)}, (\cdot)^p\right)$-lower convex function φ, we get (32).

b) Let us consider the function $i_p(t) = \frac{\delta}{p(p-1)}t^p - \varphi(t)$. Then

$$i_p''(t) = \delta t^{p-2} - \varphi''(t) = t^{p-2}(\delta - t^{2-p}\varphi''(t)) = t^{p-2}(\delta - g_p(t)) \geqslant 0,$$

i.e. i_p is convex and then the function φ is $\left(\frac{\delta}{p(p-1)}, (\cdot)^p\right)$-upper convex.

Now, applying (22) to the $\left(\frac{\delta}{p(p-1)}, (\cdot)^p\right)$-upper convex function φ, we get (33).

c) This case is a combination of a) and b). □

REMARK 3. If $-\infty < \gamma \leqslant \varphi''(t) \leqslant \delta < \infty$, for all $t \in I$, then for $p = 2$, as a direct consequence of (34) and (35) from Proposition 1, we get

$$\frac{\gamma}{2}\left(\sum_{i=1}^{n} a_i x_i^2 - \bar{x}^2\right) \leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \leqslant \frac{\delta}{2}\left(\sum_{i=1}^{n} a_i x_i^2 - \bar{x}^2\right) \tag{36}$$

and

$$\begin{aligned}\frac{\delta}{2}\left(2\bar{x}\sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} a_i x_i^2 - \bar{x}^2\right) &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) + \sum_{i=1}^{n} a_i \varphi'(x_i)(\bar{x} - x_i) \\ &\leqslant \frac{\gamma}{2}\left(2\bar{x}\sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} a_i x_i^2 - \bar{x}^2\right).\end{aligned} \tag{37}$$

Related results for $\left(\frac{\gamma}{2}, \frac{\delta}{2}, (\cdot)^2\right)$-convex functions, but in the context of so called α-lower and β-upper convex functions, can be found in [8].

PROPOSITION 2. *Let $I \subseteq (0,\infty)$ be an open interval, $\mathbf{x} = (x_1,\ldots,x_n) \in I^n$ and $\mathbf{a} = (a_1,\ldots,a_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^{n} a_i = 1$ and $\bar{x} = \sum_{i=1}^{n} a_i x_i$. Let $\varphi\colon I \to \mathbb{R}$ be a twice differentiable function and $g\colon I \to \mathbb{R}$ be defined by $g(t) = t^2\varphi''(t)$.*

a) If $\inf_{t\in I} g(t) = \gamma > -\infty$, *then*

$$\begin{aligned}\ln\left(\frac{\bar{x}}{\prod_{i=1}^{n} x_i^{a_i}}\right)^{\gamma} &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi'(x_i)(x_i - \bar{x}) + \gamma\left(\ln\frac{\bar{x}}{\prod_{i=1}^{n} x_i^{a_i}} - \bar{x}\sum_{i=1}^{n}\frac{a_i}{x_i} + 1\right).\end{aligned} \tag{38}$$

b) If $\sup_{t\in I} g(t) = \delta < \infty$, *then*

$$\begin{aligned}&\delta\left(\ln\frac{\bar{x}}{\prod_{i=1}^{n} x_i^{a_i}} - \bar{x}\sum_{i=1}^{n}\frac{a_i}{x_i} + 1\right) - \sum_{i=1}^{n} a_i \varphi'(x_i)(\bar{x} - x_i) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \\ &\leqslant \ln\left(\frac{\bar{x}}{\prod_{i=1}^{n} x_i^{a_i}}\right)^{\delta}.\end{aligned} \tag{39}$$

c) If $-\infty < \gamma \leqslant g(t) \leqslant \delta < \infty$, *for all* $t \in I$, *then*

$$\ln\left(\frac{\bar{x}}{\prod_{i=1}^{n} x_i^{a_i}}\right)^{\gamma} \leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \leqslant \ln\left(\frac{\bar{x}}{\prod_{i=1}^{n} x_i^{a_i}}\right)^{\delta} \tag{40}$$

and

$$\delta\left(\ln\frac{\bar{x}}{\prod_{i=1}^{n}x_i^{a_i}}-\bar{x}\sum_{i=1}^{n}\frac{a_i}{x_i}+1\right)\leqslant\sum_{i=1}^{n}a_i\varphi(x_i)-\varphi(\bar{x})+\sum_{i=1}^{n}a_i\varphi'(x_i)(\bar{x}-x_i)$$
$$\leqslant\gamma\left(\ln\frac{\bar{x}}{\prod_{i=1}^{n}x_i^{a_i}}-\bar{x}\sum_{i=1}^{n}\frac{a_i}{x_i}+1\right). \qquad (41)$$

Proof. a) Let's consider the function $h(t)=\varphi(t)-\gamma(-\ln t)$. Then

$$h''(t)=\varphi''(t)-\gamma\frac{1}{t^2}=\frac{1}{t^2}\left(t^2\varphi''(t)-\gamma\right)=\frac{1}{t^2}(g(t)-\gamma)\geqslant 0,$$

i.e. h is convex and then the function φ is $(\gamma,-\ln(\cdot))$-lower convex.

Now, applying (21) to the $(\gamma,-\ln(\cdot))$-lower convex function φ, we get (38).

b) Let's consider the function $i(t)=\delta(-\ln t)-\varphi(t)$. Then

$$i''(t)=\delta\frac{1}{t^2}-\varphi''(t)=\frac{1}{t^2}(\delta-t^2\varphi''(t))=\frac{1}{t^2}(\delta-g(t))\geqslant 0,$$

i.e. i is convex and then the function φ is $(\delta,-\ln(\cdot))$-upper convex.

Now, applying (22) to the $(\delta,-\ln(\cdot))$-upper convex function φ, we get (39).

c) This case is a combination of a) and b). □

PROPOSITION 3. *Let $I\subseteq(0,\infty)$ be an open interval, $\mathbf{x}=(x_1,\dots,x_n)\in I^n$ and $\mathbf{a}=(a_1,\dots,a_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^{n}a_i=1$ and $\bar{x}=\sum_{i=1}^{n}a_ix_i$. Let $\varphi\colon I\to\mathbb{R}$ be a twice differentiable function and $g\colon I\to\mathbb{R}$ be defined by $g(t)=t\varphi''(t)$.*

a) If $\inf_{t\in I}g(t)=\gamma>-\infty$, then

$$\ln\left(\frac{\prod_{i=1}^{n}x_i^{a_ix_i}}{\bar{x}^{\bar{x}}}\right)^{\gamma} \qquad (42)$$
$$\leqslant\sum_{i=1}^{n}a_i\varphi(x_i)-\varphi(\bar{x})$$
$$\leqslant\sum_{i=1}^{n}a_i\varphi'(x_i)(x_i-\bar{x})+\gamma\bar{x}\ln\frac{\prod_{i=1}^{n}x_i^{a_i}}{\bar{x}}.$$

b) If $\sup_{t\in I}g(t)=\delta<\infty$, then

$$\delta\bar{x}\ln\frac{\prod_{i=1}^{n}x_i^{a_i}}{\bar{x}}-\sum_{i=1}^{n}a_i\varphi'(x_i)(\bar{x}-x_i)$$
$$\leqslant\sum_{i=1}^{n}a_i\varphi(x_i)-\varphi(\bar{x})$$
$$\leqslant\ln\left(\frac{\prod_{i=1}^{n}x_i^{a_ix_i}}{\bar{x}^{\bar{x}}}\right)^{\delta}. \qquad (43)$$

c) If $-\infty < \gamma \leqslant g(t) \leqslant \delta < \infty$, for all $t \in I$, then

$$\ln\left(\frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\bar{x}^{\bar{x}}}\right)^{\gamma} \leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \leqslant \ln\left(\frac{\prod_{i=1}^{n} x_i^{a_i x_i}}{\bar{x}^{\bar{x}}}\right)^{\delta} \tag{44}$$

and

$$\begin{aligned} &\delta \bar{x} \ln \frac{\prod_{i=1}^{n} x_i^{a_i}}{\bar{x}} \\ &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) + \sum_{i=1}^{n} a_i \varphi'(x_i)(\bar{x} - x_i) \\ &\leqslant \gamma \bar{x} \ln \frac{\prod_{i=1}^{n} x_i^{a_i}}{\bar{x}}. \end{aligned} \tag{45}$$

Proof. a) Let's consider the function $h(t) = \varphi(t) - \gamma t \ln t$. Then

$$h''(t) = \varphi''(t) - \gamma \frac{1}{t} = \frac{1}{t}\left(t\varphi''(t) - \gamma\right) = \frac{1}{t}(g(t) - \gamma) \geqslant 0,$$

i.e. h is convex and then the function φ is $(\gamma, (\cdot)\ln(\cdot))$-lower convex.

Now, applying (21) to the $(\gamma, (\cdot)\ln(\cdot))$-lower convex function φ, we get

$$\begin{aligned} &\gamma\left(\sum_{i=1}^{n} a_i x_i \ln x_i - \bar{x} \ln \bar{x}\right) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi'(x_i)(x_i - \bar{x}) - \gamma\left(\bar{x} \ln \bar{x} - \sum_{i=1}^{n} a_i x_i \ln x_i - \sum_{i=1}^{n} \frac{a_i}{x_i}(\bar{x} - x_i)\right), \end{aligned}$$

what is equivalent to (42).

b) Let's consider the function $\delta t \ln t - \varphi(t)$. Then

$$i''(t) = \delta \frac{1}{t} - \varphi''(t) = \frac{1}{t}\left(\delta - t\varphi''(t)\right) = \frac{1}{t}(\delta - g(t)) \geqslant 0,$$

i.e. i is convex and then the function φ is $(\delta, (\cdot)\ln(\cdot))$-upper convex.

So, applying (22) to such function we obtain

$$\begin{aligned} &\delta\left(\sum_{i=1}^{n} a_i x_i \ln x_i + \sum_{i=1}^{n} \frac{a_i}{x_i}(\bar{x} - x_i) - \bar{x} \ln \bar{x}\right) - \sum_{i=1}^{n} a_i \varphi'(x_i)(\bar{x} - x_i) \\ &\leqslant \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \\ &\leqslant \delta\left(\sum_{i=1}^{n} a_i x_i \ln x_i - \bar{x} \ln \bar{x}\right) \end{aligned}$$

what is equivalent to (43).

c) This case is a combination of a) and b). □

REMARK 4. Making use of Theorem 2 we could obtain analogous results as in the previous propositions, concerning Steffensen's conditions (31).

Now we use the previous propositions to derive new lower and upper bounds for the well known mean inequalities. For this purpose, let us recall that for $0 < l < L$, $\mathbf{x} = (x_1, \ldots, x_n) \in (l, L)^n$ and a nonnegative n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ such that $\sum_{i=1}^n a_i = 1$, weighted power mean of order $s \in \mathbb{R}$ is defined by

$$M_s(\boldsymbol{x}, \boldsymbol{a}) := \begin{cases} \left(\sum_{i=1}^n a_i x_i^s\right)^{\frac{1}{s}}, & s \neq 0 \\ \prod_{i=1}^n x_i^{a_i}, & s = 0 \\ \min\{x_1, \ldots, x_n\}, & s \to -\infty \\ \max\{x_1, \ldots, x_n\} & s \to \infty. \end{cases} \tag{46}$$

In particular, classical weighted means are then given as follows:

- arithmetic mean $A(\boldsymbol{x}, \boldsymbol{a}) = M_1(\boldsymbol{x}, \boldsymbol{a}) = \sum_{i=1}^n a_i x_i$,
- geometric mean $G(\boldsymbol{x}, \boldsymbol{a}) = M_0(\boldsymbol{x}, \boldsymbol{a}) = \prod_{i=1}^n x_i^{a_i}$,
- harmonic mean $H(\boldsymbol{x}, \boldsymbol{a}) = M_{-1}(\boldsymbol{x}, \boldsymbol{a}) = \frac{1}{\sum_{i=1}^n \frac{w_i}{x_i}}$.

EXAMPLE 1. Let $0 < l < L$ and let functions $\varphi, g_p \colon (l, L) \to \mathbb{R}$ be such that $g_p(t) = \varphi''(t) t^{2-p}$, where $p \in (-\infty, 0) \cup (1, \infty)$. Applying Proposition 1 to particular cases of the function φ, we get the following results.

a) If $\varphi(t) = -\ln t$, then $\inf_{t \in (l,L)} g_p(t) = L^{-p}$ and $\sup_{t \in (l,L)} g_p(t) = l^{-p}$.

Applying (34) and (35), we get

$$\begin{aligned} &\frac{L^{-p}}{p(p-1)} \left(M_p^p(\boldsymbol{x}, \boldsymbol{a}) - A^p(\boldsymbol{x}, \boldsymbol{a})\right) \\ &\leqslant \ln \frac{A(\boldsymbol{x}, \boldsymbol{a})}{G(\boldsymbol{x}, \boldsymbol{a})} \\ &\leqslant \frac{l^{-p}}{p(p-1)} \left(M_p^p(\boldsymbol{x}, \boldsymbol{a}) - A^p(\boldsymbol{x}, \boldsymbol{a})\right) \end{aligned}$$

and

$$\begin{aligned} &\frac{l^{-p}}{p(p-1)} \Big(pA(\boldsymbol{x}, \boldsymbol{a}) M_{p-1}^{p-1}(\boldsymbol{x}, \boldsymbol{a}) + (1-p) M_p^p(\boldsymbol{x}, \boldsymbol{a}) - A^p(\boldsymbol{x}, \boldsymbol{a})\Big) \\ &\leqslant \ln \frac{A(\boldsymbol{x}, \boldsymbol{a})}{G(\boldsymbol{x}, \boldsymbol{a})} - \frac{A(\boldsymbol{x}, \boldsymbol{a})}{H(\boldsymbol{x}, \boldsymbol{a})} + 1 \\ &\leqslant \frac{L^{-p}}{p(p-1)} \Big(pA(\boldsymbol{x}, \boldsymbol{a}) M_{p-1}^{p-1}(\boldsymbol{x}, \boldsymbol{a}) + (1-p) M_p^p(\boldsymbol{x}, \boldsymbol{a}) - A^p(\boldsymbol{x}, \boldsymbol{a})\Big). \end{aligned}$$

If $p=2$, then

$$\frac{1}{2L^2}\left(M_2^2(\boldsymbol{x},\boldsymbol{a})-A^2(\boldsymbol{x},\boldsymbol{a})\right)\leqslant \ln\frac{A(\boldsymbol{x},\boldsymbol{a})}{G(\boldsymbol{x},\boldsymbol{a})} \leqslant \frac{1}{2l^2}\left(M_2^2(\boldsymbol{x},\boldsymbol{a})-A^2(\boldsymbol{x},\boldsymbol{a})\right)$$

and

$$\frac{1}{2l^2}\left(A^2(\boldsymbol{x},\boldsymbol{a})-M_2^2(\boldsymbol{x},\boldsymbol{a})\right)\leqslant \ln\frac{A(\boldsymbol{x},\boldsymbol{a})}{G(\boldsymbol{x},\boldsymbol{a})}-\frac{A(\boldsymbol{x},\boldsymbol{a})}{H(\boldsymbol{x},\boldsymbol{a})}+1 \leqslant \frac{1}{2L^2}\left(A^2(\boldsymbol{x},\boldsymbol{a})-M_2^2(\boldsymbol{x},\boldsymbol{a})\right).$$

b) Let $\varphi(t)=t\ln t$. Then for $p\in(-\infty,0)$, we have $\inf_{t\in(l,L)} g_p(t)=l^{1-p}$ and $\sup_{t\in(l,L)} g_p(t)=L^{1-p}$.

Applying (34) and (35), we get

$$\frac{l^{1-p}}{p(p-1)}\left(M_p^p(\boldsymbol{x},\boldsymbol{a})-A^p(\boldsymbol{x},\boldsymbol{a})\right) \leqslant \ln\frac{\prod_{i=1}^n x_i^{a_i x_i}}{(A(\boldsymbol{x},\boldsymbol{a}))^{A(\boldsymbol{x},\boldsymbol{a})}} \leqslant \frac{L^{1-p}}{p(p-1)}\left(M_p^p(\boldsymbol{x},\boldsymbol{a})-A^p(\boldsymbol{x},\boldsymbol{a})\right) \tag{47}$$

and

$$\frac{L^{1-p}}{p(p-1)}\left(pA(\boldsymbol{x},\boldsymbol{a})M_{p-1}^{p-1}(\boldsymbol{x},\boldsymbol{a})+(1-p)M_p^p(\boldsymbol{x},\boldsymbol{a})-A^p(\boldsymbol{x},\boldsymbol{a})\right) \leqslant \ln\frac{\prod_{i=1}^n x_i^{a_i x_i}}{(A(\boldsymbol{x},\boldsymbol{a}))^{A(\boldsymbol{x},\boldsymbol{a})}}+\frac{A(\boldsymbol{x},\boldsymbol{a})}{H(\boldsymbol{x},\boldsymbol{a})}-1 \leqslant \frac{l^{1-p}}{p(p-1)}\left(pA(\boldsymbol{x},\boldsymbol{a})M_{p-1}^{p-1}(\boldsymbol{x},\boldsymbol{a})+(1-p)M_p^p(\boldsymbol{x},\boldsymbol{a})-A^p(\boldsymbol{x},\boldsymbol{a})\right). \tag{48}$$

If $p\in(1,\infty)$, then $\inf_{t\in(l,L)} g_p(t)=L^{1-p}$ and $\sup_{t\in(l,L)} g_p(t)=l^{1-p}$ and then inequalities (47) and (48) are reversed.

EXAMPLE 2. Let $0<l<L$ and the functions $\varphi,g\colon (l,L)\to\mathbb{R}$ be such that $\varphi(t)=t\ln t$ and $g(t)=t^2\varphi''(t)=t$. Then $\inf_{t\in(l,L)} g(t)=l$ and $\sup_{t\in(l,L)} g(t)=L$.

Applying (40) and (41), we get

$$\ln\left(\frac{A(\boldsymbol{x},\boldsymbol{a})}{G(\boldsymbol{x},\boldsymbol{a})}\right)^l \leqslant \ln\frac{\prod_{i=1}^n x_i^{a_i x_i}}{(A(\boldsymbol{x},\boldsymbol{a}))^{A(\boldsymbol{x},\boldsymbol{a})}} \leqslant \ln\left(\frac{A(\boldsymbol{x},\boldsymbol{a})}{G(\boldsymbol{x},\boldsymbol{a})}\right)^L$$

and

$$L\left(\ln\frac{A(\boldsymbol{x},\boldsymbol{a})}{G(\boldsymbol{x},\boldsymbol{a})}-\frac{A(\boldsymbol{x},\boldsymbol{a})}{H(\boldsymbol{x},\boldsymbol{a})}+1\right)\leqslant \ln\frac{\prod_{i=1}^{n}x_i^{a_ix_i}}{(A(\boldsymbol{x},\boldsymbol{a}))^{A(\boldsymbol{x},\boldsymbol{a})}}+\frac{A(\boldsymbol{x},\boldsymbol{a})}{H(\boldsymbol{x},\boldsymbol{a})}-1)$$
$$\leqslant l\left(\ln\frac{A(\boldsymbol{x},\boldsymbol{a})}{G(\boldsymbol{x},\boldsymbol{a})}-\frac{A(\boldsymbol{x},\boldsymbol{a})}{H(\boldsymbol{x},\boldsymbol{a})}+1\right).$$

EXAMPLE 3. Let $0<l<L$ and let functions $\varphi, g_p\colon (l,L)\to\mathbb{R}$ be such that $g_p(t)=t\varphi''(t)$, where $p\in(-\infty,0)\cup(1,\infty)$. Applying Proposition 3 to particular case of the function φ, we get the following results.

a) If $\varphi(t)=-\ln t$, then $\inf_{t\in(l,L)} g_p(t)=\frac{1}{L}$ and $\sup_{t\in(l,L)} g_p(t)=\frac{1}{l}$.
Now applying (44) and (45), we have

$$\ln\left(\frac{\prod_{i=1}^{n}x_i^{a_ix_i}}{A(\boldsymbol{x},\boldsymbol{a})^{A(\boldsymbol{x},\boldsymbol{a})}}\right)^{\frac{1}{L}}\leqslant \ln\frac{A(\boldsymbol{x},\boldsymbol{a})}{G(\boldsymbol{x},\boldsymbol{a})}\leqslant \ln\left(\frac{\prod_{i=1}^{n}x_i^{a_ix_i}}{A(\boldsymbol{x},\boldsymbol{a})^{A(\boldsymbol{x},\boldsymbol{a})}}\right)^{\frac{1}{l}}$$

and

$$\frac{A(\boldsymbol{x},\boldsymbol{a})}{l}\ln\frac{G(\boldsymbol{x},\boldsymbol{a})}{A(\boldsymbol{x},\boldsymbol{a})}$$
$$\leqslant \ln\frac{A(\boldsymbol{x},\boldsymbol{a})}{G(\boldsymbol{x},\boldsymbol{a})}-\frac{A(\boldsymbol{x},\boldsymbol{a})}{H(\boldsymbol{x},\boldsymbol{a})}+1$$
$$\leqslant \frac{A(\boldsymbol{x},\boldsymbol{a})}{L}\ln\frac{G(\boldsymbol{x},\boldsymbol{a})}{A(\boldsymbol{x},\boldsymbol{a})}.$$

b) If $\varphi(t)=e^t$, then $\inf_{t\in(l,L)} g_p(t)=le^l$ and $\sup_{t\in(l,L)} g_p(t)=Le^L$.
Applying (44) and (45), we get

$$\ln\left(\frac{\prod_{i=1}^{n}x_i^{a_ix_i}}{\bar{x}^{\bar{x}}}\right)^{le^l}\leqslant\sum_{i=1}^{n}a_ie^{x_i}-e^{\bar{x}}\leqslant \ln\left(\frac{\prod_{i=1}^{n}x_i^{a_ix_i}}{\bar{x}^{\bar{x}}}\right)^{Le^L}$$

and

$$Le^LA(\boldsymbol{x},\boldsymbol{a})\ln\frac{G(\boldsymbol{x},\boldsymbol{a})}{A(\boldsymbol{x},\boldsymbol{a})}$$
$$\leqslant\sum_{i=1}^{n}a_ie^{x_i}-e^{A(\boldsymbol{x},\boldsymbol{a})}+\sum_{i=1}^{n}a_ie^{x_i}(A(\boldsymbol{x},\boldsymbol{a})-x_i)$$
$$\leqslant le^lA(\boldsymbol{x},\boldsymbol{a})\ln\frac{G(\boldsymbol{x},\boldsymbol{a})}{A(\boldsymbol{x},\boldsymbol{a})}.$$

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