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INERTIA INDICES AND THE CONVERSE OF WEYL'S EIGENVALUE INEQUALITY FOR HERMITIAN TENSORS

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Abstract. In this paper, we use inertia indices to present a necessary and sufficient condition in order for eigenvalue inequalities to hold between two Hermitian tensors. As an application, we establish the converse of Weyl's eigenvalue inequality for Hermitian tensors. Also, we prove some classical eigenvalue inequalities for Hermitian tensors. More precisely, we extend Cauchy's interlacing theorem, Weyl's inequality, the monotonicity theorem, and the inclusion principle theorem from matrices to tensors, in a simple and unified approach.

1. Introduction

A univariate polynomial is *real-rooted* if all its coefficients and roots are real. If f is a real-rooted polynomial of degree n, we denote its roots by $r_1(f) \geqslant \ldots \geqslant r_n(f)$. Let f and g be real-rooted polynomials of degrees n and m, respectively. As defined by Sai-Nan Zheng et al. in [21], f *interlaces* g, denoted by $f \lessdot g$, if $n \leqslant m \leqslant n+1$ and $r_i(g) \geqslant r_i(f) \geqslant r_{i+1}(g)$ for all i. Also, f and g are *compatible*, denoted by $f \bowtie g$, if $\mid m-n \mid \leqslant 1$ and $r_{i-1}(g) \geqslant r_i(f) \geqslant r_{i+1}(g)$ for all i. We refer the reader to [1, 2, 8, 10, 12, 13, 14, 19, 20] for further information on interlacing and compatible polynomials.

Let A and B be Hermitian matrices. Then, $A \lessdot B$ and $A \bowtie B$ if their characteristic polynomials satisfy $det(\lambda I - A) \lessdot det(\lambda I - B)$ and $det(\lambda I - A) \bowtie det(\lambda I - B)$, respectively. In 2019, Wang proved some classical eigenvalue inequalities for Hermitian matrices, including Cauchy's interlacing theorem and Weyl's inequality, using inertia indices [21]. He proposed common generalizations of eigenvalue inequalities for (Hermitian) normalized Laplacian matrices. In matrix analysis and spectral graph theory, Hermitian matrices whose characteristic polynomials are interlacing or compatible appear frequently [21].

In this paper, we extend some classical eigenvalue inequalities from Hermitian matrices to Hermitian tensors. These include Cauchy's interlacing theorem and Weyl's inequality. Also, we introduce (p,q)-interlacing polynomials and study their properties. Using these polynomials, we prove the converse of Weyl's inequality for tensors. Recently, Wang and Zheng have shown the converse of Weyl's eigenvalue inequality for a Hermitian matrix [20].

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2. The T-product

The T-product is defined for third-order tensors. A *third-order tensor*

$$A = (a_{ijk}), \ \ 1 \leqslant i \leqslant m, \ \ 1 \leqslant j \leqslant n, \ \ 1 \leqslant k \leqslant p$$

consists of mnp entries. We denote the set of all third-order tensors over the complex field $\mathbb{C}$ (or the real field $\mathbb{R}$) by $\mathbb{C}^{m\times n\times p}$ (or $\mathbb{R}^{m\times n\times p}$).

We denote the *frontal faces* of $\mathscr{A} \in \mathbb{C}^{m\times n\times p}$ and $\mathscr{B} \in \mathbb{C}^{n\times s\times p}$, for $k = 1,\dots,p$, by $A^{(k)} = \mathscr{A}(:,:,\mathrm{k}) \in \mathbb{C}^{m\times n}$ and $B^{(k)} = \mathscr{B}(:,:,\mathrm{k}) \in \mathbb{C}^{n\times s}$, respectively. The operators bcirc, unfold and fold are defined as follows [3, 6, 7].

$$\mathrm{bcirc}(\mathscr{A}) := \begin{bmatrix} A^{(1)} & A^{(p)} & A^{(p-1)} & \dots & A^{(2)} \\ A^{(2)} & A^{(1)} & A^{(p)} & \dots & A^{(3)} \\ \vdots & \vdots & & & \vdots \\ A^{(p)} & A^{(p-1)} & \dots & A^{(2)} & A^{(1)} \end{bmatrix}, \quad \mathrm{unfold}(\mathscr{A}) := \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(p)} \end{bmatrix}.$$

DEFINITION 1. ([3, 6, 7]) The T-product of $\mathscr{A} \in \mathbb{C}^{m\times n\times p}$ and $\mathscr{B} \in \mathbb{C}^{n\times s\times p}$, denoted by $\mathscr{A} * \mathscr{B}$, is the $m \times s \times p$ tensor defined by

$$\mathscr{A} * \mathscr{B} := \mathrm{fold}(\mathrm{bcirc(A)unfold(B)}).$$

DEFINITION 2. ([3, 6, 7]) If $\mathscr{A} \in \mathbb{C}^{m\times n\times p}$, then $\mathscr{A}^T$ is an $n \times m \times p$ tensor obtained by transposing each of the frontal slices and then reversing the ordering of transposed frontal slices 2 through p. The conjugate transpose $\mathscr{A}^*$ is obtained by conjugate transposing each of the frontal slices and then reversing the ordering of transposed frontal slices 2 through p.

EXAMPLE 1. If $\mathscr{A} \in \mathbb{R}^{m\times n\times 4}$ and its frontal slices are given by $m \times n \times 4$ matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$ and $A^{(4)}$, then

$$\mathscr{A}^T = \mathrm{fold}\begin{bmatrix} {A^{(1)}}^T \\ {A^{(4)}}^T \\ {A^{(3)}}^T \\ {A^{(2)}}^T \end{bmatrix}.$$

Just as circulant matrices can be diagonalized by the discrete Fourier matrix [5], we have the following lemma for block-circulant matrices.

LEMMA 1. [7, 15] *If $\mathscr{A} \in C^{n\times n\times p}$, then*

$$(F_p \otimes I_n).bcirc(\mathscr{A}).(F_p^* \otimes I_n) = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix} \tag{1}$$

is a block-diagonal matrix, where F_p is the $p \times p$ discrete Fourier matrix, I_n is the $n \times n$ identity matrix, and $A_k \in \mathbb{C}^{n\times n}$ for $1 \leqslant k \leqslant p$. Furthermore, $A^{(1)}$, $A^{(2)}$, ..., $A^{(k)}$ are diagonal if and only if A_1, A_2, ..., A_p are diagonal.

The class of normal matrices is important in matrix analysis; it includes unitary, Hermitian and skew-Hermitian matrices. Now, we introduce the class of normal tensors based on the T_M-product.

DEFINITION 3. A tensor $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ is normal if $\mathscr{A} * \mathscr{A}^* = \mathscr{A}^* * \mathscr{A}$, and it is skew-Hermitian if $\mathscr{A}^* = -\mathscr{A}$. Also, a tensor $\mathscr{U} \in \mathbb{C}^{n\times n\times p}$ is unitary if $\mathscr{U} * \mathscr{U}^* = \mathscr{U}^* * \mathscr{U} = \mathscr{I}_{nnp}$.

Next, we define symmetric and Hermitian tensors.

DEFINITION 4. [9] A tensor $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ is called symmetric if $\mathscr{A}^T = \mathscr{A}$, and it is called Hermitian if $\mathscr{A}^* = \mathscr{A}$.

Thus, the class of normal tensors includes Hermitian, unitary and skew-Hermitian tensors.

THEOREM 1. [17, 18] *If $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$, then the following statements are true.*

(i) $\mathscr{A}$ *is symmetric if and only if* $bcirc(\mathscr{A}) = (bcirc(\mathscr{A}))^T$.

(ii) $\mathscr{A}$ *is Hermitian if and only if* $bcirc(\mathscr{A}) = (bcirc(\mathscr{A}))^*$.

(iii) $(\mathscr{A} * \mathscr{B})^* = \mathscr{B}^* * \mathscr{A}^*$.

(iv) $bcirc(\mathscr{A} * \mathscr{B}) = bcirc(A) bcirc(\mathscr{B})$.

(v) $\mathscr{A} * (\mathscr{B} + \mathscr{C}) = \mathscr{A} * \mathscr{B} + \mathscr{A} * \mathscr{C}$.

(vi) $(\mathscr{A} + \mathscr{B}) * \mathscr{C} = \mathscr{A} * \mathscr{C} + \mathscr{B} * \mathscr{C}$.

DEFINITION 5. (The identity tensor [3, 6, 7]) The identity tensor $\mathscr{I}_{nnp} \in \mathbb{C}^{n\times n\times p}$ is the tensor whose first frontal slice is the $n \times n$ identity matrix, while its all other frontal faces are zero. Also, $bcirc(\mathscr{I}_{nnp}) = I_{np}$, where I_{np} is the $np \times np$ identity matrix.

For a tensor $\mathscr{A} \in \mathbb{C}^{m\times n\times p}$ we can write

$$\mathscr{A} * \mathscr{I}_{nnp} = \mathscr{A} = \mathscr{I}_{mmp} * \mathscr{A}.$$

DEFINITION 6. If $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$, then a tensor $\mathscr{B}$ is called the inverse of $\mathscr{A}$ if

$$\mathscr{A} * \mathscr{B} = \mathscr{I}_{nnp}, \qquad \mathscr{B} * \mathscr{A} = \mathscr{I}_{nnp}. \tag{2}$$

In this case, we write $\mathscr{B} = \mathscr{A}^{-1}$.

By Theorem 1 (iv) and (2),

$$\text{bcirc}(\mathscr{A})\text{bcirc}(\mathscr{B}) = \text{bcirc}(\mathscr{I}_{nnp}) = I_{np}$$

and

$$\text{bcirc}(\mathscr{B})\text{bcirc}(\mathscr{A}) = \text{bcirc}(\mathscr{I}_{nnp}) = I_{np}.$$

Here, I_{np} is the $np \times np$ identity matrix. Therefore, the invertibility of $\mathscr{A}$ is equivalent to that of the matrix $\text{bcirc}(\mathscr{A})$. Also, if $\mathscr{A}$ is invertible, then

$$\text{bcirc}(\mathscr{A}^{-1}) = (\text{bcirc}(\mathscr{A}))^{-1},$$

because

$$\text{bcirc}(\mathscr{A})\text{bcirc}(\mathscr{A}^{-1}) = \text{bcirc}(\mathscr{A} * \mathscr{A}^{-1}) = I_{np} = \text{bcirc}(\mathscr{A})(\text{bcirc}(\mathscr{A}))^{-1}$$

and $\mathscr{A}$ has a unique inverse.

DEFINITION 7. (T-congruent tensors) We say that $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ and $\mathscr{B} \in \mathbb{C}^{n\times n\times p}$ are congruent if there exists a non-singular tensor $\mathscr{P}$ such that

$$\mathscr{B} = \mathscr{P}^T * \mathscr{A} * \mathscr{P}.$$

DEFINITION 8. (T-eigenvalues and T-eigenvectors) Let $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$, $\mathscr{X} \in \mathbb{C}^{n\times 1\times p}$ and $\mathscr{X} \neq 0$. If

$$\mathscr{A} * \mathscr{X} = \lambda \mathscr{X}$$

for some $\lambda \in \mathbb{C}$, then λ is called a T-eigenvalue of $\mathscr{A}$, and $\mathscr{X}$ is a T-eigenvector of $\mathscr{A}$.

It is easy to prove that for any T-eigenvalue λ of $\mathscr{A}$,

$$\text{bcirc}(\mathscr{A})\text{unfold}(\mathscr{X}) = \lambda\,\text{unfold}(\mathscr{X}). \tag{3}$$

REMARK 1. If $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$, then $\lambda(\mathscr{A} - r\mathscr{I}_{nnp}) = \lambda(\mathscr{A}) - r$ for any $r \in \mathbb{R}$.

THEOREM 2. *If $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ is a Hermitian tensor, then the T-eigenvalues of $\mathscr{A}$ are real.*

Proof. Since $\mathscr{A}$ is a Hermitian tensor, Theorem 1 shows that the eigenvalues of *bcirc*$(\mathscr{A})$ are real. Hence, by (3), the T_M-eigenvalues of $\mathscr{A}$ are also real. □

THEOREM 3. (T-EVD) *A tensor $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ is normal if and only if there exists a unitary tensor $\mathscr{U} \in \mathbb{C}^{n\times n\times p}$ such that*

$$\mathscr{A} = \mathscr{U}^* * \mathscr{D} * \mathscr{U},$$

where $\mathscr{D} \in \mathbb{C}^{n\times n\times p}$ is an F-diagonal tensor, that is, one whose all frontal faces are diagonal matrices.

Proof. Since $\mathscr{A} * \mathscr{A}^* = \mathscr{A}^* * \mathscr{A}$, $\text{bcirc}(\mathscr{A})(\text{bcirc}(\mathscr{A}))^* = (\text{bcirc}(\mathscr{A}))^*\text{bcirc}(\mathscr{A})$. By (1),

$$\text{bcirc}(\mathscr{A})(\text{bcirc}(\mathscr{A}))^* = (F_p^* \otimes I_{np}) \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_p \end{bmatrix} (F_p \otimes I_{np})(F_p^* \otimes I_{np})$$

$$\begin{bmatrix} D_1^* & & & \\ & D_2^* & & \\ & & \ddots & \\ & & & D_p^* \end{bmatrix} (F_p \otimes I_{np}) = (F_p^* \otimes I_{np}) \begin{bmatrix} D_1 D_1^* & & & \\ & D_2 D_2^* & & \\ & & \ddots & \\ & & & D_p D_p^* \end{bmatrix}$$

$$(F_p \otimes I_{np}) = (F_p^* \otimes I_{np}) \begin{bmatrix} D_1^* D_1 & & & \\ & D_2^* D_2 & & \\ & & \ddots & \\ & & & D_p^* D_p \end{bmatrix} (F_p \otimes I_{np}).$$

It follows that $D_k D_k^* = D_k^* D_k$ for $1 \leqslant k \leqslant p$, that is, D_k is a normal matrix. There exists a unitary $U_k \in \mathbb{C}^{n \times n}$ such that $D_k = U_k^* \Lambda_k U_k$. Thus,

$$\text{bcirc}(\mathscr{A}) = (F_p^* \otimes I_{np}) \begin{bmatrix} U_1^* & & & \\ & U_2^* & & \\ & & \ddots & \\ & & & U_p^* \end{bmatrix} \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_p \end{bmatrix} \begin{bmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_p \end{bmatrix} (F_p \otimes I_{np})$$
$$= (F_p^* \otimes I_{np}) \begin{bmatrix} U_1^* & & & \\ & U_2^* & & \\ & & \ddots & \\ & & & U_p^* \end{bmatrix} (F_p \otimes I_{np})(F_p^* \otimes I_{np}) \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_p \end{bmatrix}$$

$$(F_p \otimes I_{np})(F_p^* \otimes I_{np}) \begin{bmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_p \end{bmatrix} (F_p \otimes I_{np}) = \text{bcirc}(\mathscr{U}^*)\text{bcirc}(\mathscr{D}_\Lambda)\text{bcirc}(\mathscr{U}),$$

that is,

$$\mathscr{A} = \mathscr{U}^* * \mathscr{D} * \mathscr{U},$$

where $\mathscr{D} := \mathscr{D}_\Lambda$. The reverse implication can be easily verified. □

DEFINITION 9. (The T-characteristic polynomial [15]) Let $\mathscr{A} \in \mathbb{C}^{n \times n \times p}$ be a complex tensor. If $bcirc(\mathscr{A})$ can be Fourier block-diagonalized as

$$\text{bcirc}(\mathscr{A}) = (F_p \otimes I_n)^* \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix} (F_p \otimes I_n),$$

then the T-characteristic polynomial $p_{\mathscr{A}}(t)$ is

$$p_{\mathscr{A}}(t) := LCM(det(\lambda I_n - A_1), \dots, det(\lambda I_n - A_p)), \tag{4}$$

where LCM means the least common multiplier, and $det(\lambda I_n - A_i)$ $(i \in \{1, \dots, p\})$ is the characteristic polynomial of the matrix A_i.

REMARK 2. If tensors $\mathscr{A}$ and $\mathscr{B}$ have the same characteristic polynomial, then there exists a unitary tensor $\mathscr{U}$ such that $\mathscr{A} = \mathscr{U}^* * \mathscr{B} * \mathscr{U}$. Since $det(\lambda I_n - A_i) = det(\lambda I_n - B_j)$ for all i, j, we may assume without loss of generality that $det(\lambda I_n - A_i) = det(\lambda I_n - B_i)$, for if $det(\lambda I_n - A_i) = det(\lambda I_n - B_j)$, then we can define a permutation matrix and the desired result follows. Thus, let $det(\lambda I_n - A_i) = det(\lambda I_n - B_i)$. Then, there exist unitary matrices U_i such that $A_i = U_i^* B_i U_i$. By (1),

$$\text{bcirc}(\mathscr{A}) = (F_p \otimes I_n)^* \begin{bmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_p \end{bmatrix} (F_p \otimes I_n)(F_p \otimes I_n)^* \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_p \end{bmatrix} (F_p \otimes I_n)$$

$$(F_p \otimes I_n)^* \begin{bmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_p \end{bmatrix}^* (F_p \otimes I_n) = \text{bcirc}(\mathscr{U})\text{bcirc}(\mathscr{B})\text{bcirc}(\mathscr{U}^*),$$

that is, $\mathscr{A} = \mathscr{U} * \mathscr{B} * \mathscr{U}^*$. The converse of this remark can be easily proved.

We arrange the T-eigenvalues of a Hermitian tensor $\mathscr{A}$ in the non-increasing order: $\lambda_1(\mathscr{A}) \geqslant \ldots \geqslant \lambda_n(\mathscr{A})$. Let $n_+(\mathscr{A})$ (respectively, $n_-(\mathscr{A})$) denote the positive (respectively, negative) inertia index, that is, the number of positive (respectively, negative) eigenvalues of $\mathscr{A}$.

In this paper, we use inertia indices to present a necessary and sufficient condition in order for eigenvalue inequalities to hold between two Hermitian tensors. Then, we apply this result to determine when the characteristic polynomials of Hermitian tensors interlace or are compatible. Let $\mathscr{A}$ and $\mathscr{B}$ be $n \times n \times p$ Hermitian tensors. We write $\mathscr{A} \lessdot \mathscr{B}$ and $\mathscr{A} \bowtie \mathscr{B}$ when their characteristic polynomials satisfy $p_{\mathscr{A}}(t) \lessdot p_{\mathscr{B}}(t)$ and $p_{\mathscr{A}}(t) \bowtie p_{\mathscr{B}}(t)$, respectively.

THEOREM 4. *Let* $\mathscr{A}, \mathscr{B} \in \mathbb{C}^{n \times n \times p}$ *be Hermitian tensors and* $m \in \mathbb{Z}$. *Then,* $\lambda_{i+m}(\mathscr{B}) \leqslant \lambda_i(\mathscr{A})$ *for all* i *if and only if* $n_+(\mathscr{B} - r\mathscr{I}_{nnp}) - n_+(\mathscr{A} - r\mathscr{I}_{nnp}) \leqslant m$ *for every* $r \in \mathbb{R}$.

Proof. Assume that $\lambda_{i+m}(\mathscr{B}) \leqslant \lambda_i(\mathscr{A})$. By Remark 1,

$$\lambda_{i+m}(\mathscr{B}-r\mathscr{I}_{nnp}) = \lambda_{i+m}(\mathscr{B}) - r \leqslant \lambda_i(\mathscr{A}) - r = \lambda_i(\mathscr{A}-r\mathscr{I}_{nnp}), \tag{5}$$

for any $r \in \mathbb{R}$. Let $n_+(\mathscr{A}-r\mathscr{I}_{nnp}) = t$. Then $\lambda_{t+1}(\mathscr{A}-r\mathscr{I}_{nnp}) \leqslant 0$, that is, $\lambda_{t+m+1}(\mathscr{B}-r\mathscr{I}_{nnp}) \leqslant 0$, by (5). It follows that $n_+(\mathscr{B}-r\mathscr{I}_{nnp}) \leqslant t+m$, or equivalently, $n_+(\mathscr{B}-r\mathscr{I}_{nnp}) - n_+(\mathscr{A}-r\mathscr{I}_{nnp}) \leqslant m$.

Conversely, assume that an index k exists such that $\lambda_{k+m}(\mathscr{B}) > \lambda_k(\mathscr{A})$. Consider some $r_k \in (\lambda_k(A), \lambda_{k+m}(B))$. Then, $\lambda_k(\mathscr{A}-r_k\mathscr{I}_{nnp}) < 0$ and $\lambda_{k+m}(\mathscr{B}-r_k\mathscr{I}_{nnp}) > 0$. Thus, $n_+(\mathscr{A}-r_k\mathscr{I}) \leqslant k-1$ and $n_+(\mathscr{B}-r_k\mathscr{I}_{nnp}) \geqslant k+m$, which imply that $n_+(\mathscr{B}-r_k\mathscr{I}_{nnp}) - n_+(\mathscr{A}-r_k\mathscr{I}_{nnp}) \geqslant m+1$. In other words, if $n_+(\mathscr{B}-r\mathscr{I}_{nnp}) - n_+(\mathscr{A}-r\mathscr{I}_{nnp}) \leqslant m$ for every $r \in \mathbb{R}$, then $\lambda_{i+m}(\mathscr{B}) \leqslant \lambda_i(\mathscr{A})$ for all i. This completes the proof. □

REMARK 3. The following statements are particularly interesting special cases of Theorem 4.

(i) $\mathscr{A} \lessdot \mathscr{B}$ if and only if $0 \leqslant n_+(\mathscr{B}-r\mathscr{I}_{nnp}) - n_+(\mathscr{A}-r\mathscr{I}_{nnp}) \leqslant 1$ for every $r \in \mathbb{R}$.

(ii) $\mathscr{A} \bowtie \mathscr{B}$ if and only if $|\, n_+(\mathscr{B}-r\mathscr{I}_{nnp}) - n_+(\mathscr{A}-r\mathscr{I}_{nnp}) \,| \leqslant 1$ for every $r \in \mathbb{R}$.

LEMMA 2. *The operator bcirc is linear, that is,*

$$bcirc(\alpha\mathscr{A}+\beta\mathscr{B}) = \alpha bcirc(\mathscr{A}) + \beta bcirc(\mathscr{B}),$$

where $\mathscr{A}$ and $\mathscr{B}$ are of the same size, and $\alpha, \beta \in \mathbb{C}$.

The truth of Lemma 2 follows from [11, Theorem 3]. Note that $det_T(\mathscr{A}) := det(\text{bcirc}(\mathscr{A}))$.

THEOREM 5. (The inclusion principle [4, Theorem 4.3.28]) *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and partitioned as*

$$A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} \quad B \in \mathbb{C}^{m\times m},\ C \in \mathbb{C}^{n-m,n-m},\ D \in \mathbb{C}^{m\times n-m}.$$

Then, $\lambda_{n-m+i}(A) \leqslant \lambda_i(B) \leqslant \lambda_i(A)$ for all i.

Next, we extend the inclusion principle. We consider the structures of two Hermitian tensors similar to Theorem 5. Suppose that $\mathscr{B} \in \mathbb{C}^{m\times m\times p}$, $\mathscr{C} \in \mathbb{C}^{n-m\times n-m\times p}$, and $\mathscr{D} \in \mathbb{C}^{m\times n-m\times p}$ are Hermitian, and that $B^{(k)} \in C^{m\times m}$, $C^{(k)} \in \mathbb{C}^{n-m,n-m}$ and $D^{(k)} \in \mathbb{C}^{m\times n-m}$ are their frontal faces, for $1 \leqslant k \leqslant p$, respectively. We obtain the following result.

THEOREM 6. *Let $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ be Hermitian, and $A^{(k)} \in \mathbb{C}^{n\times n}$ be its frontal face partitioned as*

$$A^{(k)} = \begin{bmatrix} B^{(k)} & C^{(k)} \\ C^{(k)*} & D^{(k)} \end{bmatrix}.$$

Then, $\lambda_{p(n-m)+i}(\mathscr{A}) \leqslant \lambda_i(\mathscr{B}) \leqslant \lambda_i(\mathscr{A})$ for all i.

Proof. Let r be a real number such that $det_T(\mathscr{B} - r\mathscr{I}) \neq 0$. By (1),

$$bcirc(\mathscr{B}) = (F_p \otimes I_m)^* \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_p \end{bmatrix} (F_p \otimes I_m).$$

Thus, $det(B_i - rI_n) \neq 0$ for $1 \leqslant i \leqslant p$. Also,

$$bcirc(\mathscr{A}) = (F_p \otimes I_n)^* \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix} (F_p \otimes I_n) \tag{6}$$

$$= (F_p \otimes I_n)^* \begin{bmatrix} \begin{bmatrix} B_1 & C_1 \\ C_1^* & D_1 \end{bmatrix} & & & \\ & \begin{bmatrix} B_2 & C_2 \\ C_2^* & D_2 \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} B_p & C_p \\ C_p^* & D_p \end{bmatrix} \end{bmatrix} (F_p \otimes I_n), \tag{7}$$

where D_i is a linear combination of the frontal faces of $\mathscr{D}$. Also,

$$bcirc(\mathscr{C}) = (F_p \otimes I_n)^* \begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_P \end{bmatrix} (F_p \otimes I_n).$$

Thus, by (6) and (7),

$$bcirc(\mathscr{A} - r\mathscr{I}_{nnp}) =$$

$$(F_p \otimes I_n)^* \begin{bmatrix} \begin{bmatrix} B_1 - rI_m & C_1 \\ C_1^* & D_1 - \lambda I_{n-m} \end{bmatrix} & & & \\ & \begin{bmatrix} B_2 - rI_m & C_2 \\ C_2^* & D_2 - rI_{n-m} \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} B_p - rI_m & C_p \\ C_p^* & D_p - rI_{n-m} \end{bmatrix} \end{bmatrix}$$

$$\times (F_p \otimes I_n)$$

because

$$\begin{bmatrix} B_i - rI_m & C_i \\ C_i^* & D_i - rI_{n-m} \end{bmatrix} \cong \begin{bmatrix} B_i - rI_m & 0 \\ 0 & D_i - rI_{n-m} - C_i^*(B_i - rI_m)^{-1}C_i \end{bmatrix},$$

for any $1 \leqslant i \leqslant p$. So,

$$n_+(\mathscr{B} - r\mathscr{I}_{nnp}) \leqslant n_+(\mathscr{A} - r\mathscr{I}_{nnp}) \leqslant n_+(\mathscr{B} - r\mathscr{I}_{nnp}) + p(n-m). \quad (8)$$

Clearly, there are only finitely many real numbers r such that $det_T(\mathscr{B} - r\mathscr{I}_{nnp}) = 0$. Hence, (8) holds for all $r \in \mathbb{R}$. Thus, $\lambda_{p(n-m)+i}(\mathscr{A}) \leqslant \lambda_i(\mathscr{B}) \leqslant \lambda_i(\mathscr{A})$ by Theorem 4. □

Cauchy's interlacing theorem is a special case of the inclusion principle for matrices. We obtain the following version of Cauchy's interlacing theorem for tensors, which is a special case of Theorem 6.

COROLLARY 1. *Let $\mathscr{B} \in \mathbb{C}^{(n-1)\times(n-1)\times p}$ be Hermitian and $B^{(k)} \in \mathbb{C}^{(n-1)\times(n-1)}$ be its frontal faces, for $k = 1, \ldots, p$. Similarly, let $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ and let $A^{(k)} \in \mathbb{C}^{n\times n}$ be its frontal faces partitioned as*

$$A^{(k)} = \begin{bmatrix} B^{(k)} & x^{(k)} \\ (y^{(k)})^* & a^{(k)} \end{bmatrix},$$

where $x^{(k)}, y^{(k)} \in \mathbb{C}^n$ and $a^{(k)} \in \mathbb{R}$, for $k = 1, 2, \ldots, p$. Then, $\mathscr{B} \lessdot \mathscr{A}$.

PROPOSITION 7. [21] *Let $A, B \in \mathbb{C}^{q\times l}$ be Hermitian matrices. Then, $n_+(A + B) \leqslant n_+(A) + n_+(B)$.*

PROPOSITION 8. *Let $\mathscr{A}, \mathscr{B} \in \mathbb{C}^{n\times n\times p}$ be Hermitian tensors. Then, $n_+(\mathscr{A} + \mathscr{B}) \leqslant n_+(\mathscr{A}) + n_+(\mathscr{B})$.*

Proof. Since $bcirc(\mathscr{A})$ is a linear operator, $\text{bcirc}(\mathscr{A} + \mathscr{B}) = \text{bcirc}(\mathscr{A}) + \text{bcirc}(\mathscr{B})$. By Proposition 7,

$$n_+(\text{bcirc}(\mathscr{A} + \mathscr{B})) \leqslant n_+(\text{bcirc}(\mathscr{A})) + n_+(\text{bcirc}(\mathscr{B})),$$

that is, $n_+(\mathscr{A} + \mathscr{B}) \leqslant n_+(\mathscr{A}) + n_+(\mathscr{B})$. □

PROPOSITION 9. *Let $\mathscr{A}, \mathscr{B}$ be Hermitian tensors and $m \in \mathbb{Z}$.*

(i) If $n_+(\mathscr{B}) \leqslant m$, then $\lambda_{i+m}(\mathscr{A} + \mathscr{B}) \leqslant \lambda_i(\mathscr{A})$ for all i.

(ii) If $n_-(\mathscr{B}) \leqslant m$, then $\lambda_{i+m}(\mathscr{A}) \leqslant \lambda_i(\mathscr{A} + \mathscr{B})$ for all i.

Proof. For every $r \in \mathbb{R}$, $n_+(\mathscr{A} + \mathscr{B} - r\mathscr{I}_{nnp}) - n_+(\mathscr{A} - r\mathscr{I}_{nnp}) \leqslant n_+(\mathscr{B})$ by Proposition 8. Thus, (i) follows from Theorem 4, and (ii) follows from (i) if we note that $n_+(-\mathscr{B}) = n_-(\mathscr{B})$. □

We define an inner product $\langle .,. \rangle_T$ on $\mathbb{C}^{n\times 1\times p}$ by

$$\langle \mathscr{X}, \mathscr{Y} \rangle_T = \sum_{j=1}^{n} \text{unfold}(\mathscr{X})(j)\overline{\text{unfold}(\mathscr{Y})}(j) = \langle \text{unfold}(\mathscr{X}), \text{unfold}(\mathscr{Y}) \rangle, \quad (9)$$

where $\langle .,. \rangle$ is the Euclidean inner product and $\mathscr{X}, \mathscr{Y} \in \mathbb{C}^{n\times 1\times p}$. Let $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ be given. We say that $\mathscr{A}$ is T-positive semi-definite if

$$\langle \mathscr{A} * \mathscr{X}, \mathscr{X} \rangle_T \geqslant 0, \qquad \text{for } \mathscr{X} \neq 0,$$

where

$$\langle \mathscr{A} * \mathscr{X}, \mathscr{X} \rangle_T = \langle \text{unfold}(\mathscr{A} * \mathscr{X}), \text{unfold}(\mathscr{Y}) \rangle = \langle \text{bcirc}(\mathscr{A})\text{unfold}(\mathscr{X}), \text{unfold}(\mathscr{Y}) \rangle.$$

The monotonicity theorem. Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian, and suppose that B is positive semi-definite. Then,

$$\lambda_i(A) \leqslant \lambda_i(A+B).$$

In what follows, we extend the monotonicity theorem.

COROLLARY 2. *Suppose that $\mathscr{A}, \mathscr{B} \in \mathbb{C}^{n\times n\times p}$ are Hermitian, and that $\mathscr{B}$ is T-positive semi-definite. Then, $\lambda_i(\mathscr{A}) \leqslant \lambda_i(\mathscr{A}+\mathscr{B})$.*

COROLLARY 3. *Let $\mathscr{A}, \mathscr{B} \in \mathbb{C}^{n\times n\times p}$ be Hermitian. If $n_+(\mathscr{B}) \leqslant p$ and $n_-(\mathscr{B}) \leqslant q$, then*

$$\lambda_{i+q}(\mathscr{A}) \leqslant \lambda_i(\mathscr{A}+\mathscr{B}) \leqslant \lambda_{i-p}(\mathscr{A}). \tag{10}$$

Next, we present an extension of Weyl's inequality to tensors.

COROLLARY 4. *If $\mathscr{A}, \mathscr{B} \in \mathbb{C}^{n\times n\times p}$ are Hermitian, then*

$$\lambda_{i+j-1}(\mathscr{A}+\mathscr{B}) \leqslant \lambda_i(\mathscr{A}) + \lambda_j(\mathscr{B}). \tag{11}$$

Proof. Clearly, $n_+(\mathscr{B} - \lambda_j(\mathscr{B})\mathscr{I}_{nnp}) \leqslant j-1$. Thus, $\lambda_{i+j-1}(\mathscr{A}+\mathscr{B}-\lambda_j(\mathscr{B})\mathscr{I}_{nnp}) \leqslant \lambda_i(\mathscr{A})$ by Proposition 9, that is, $\lambda_{i+j-1}(\mathscr{A}+\mathscr{B}) \leqslant \lambda_i(\mathscr{A}) + \lambda_j(\mathscr{B})$. $\square$

COROLLARY 5. *Let $\mathscr{A}$ be a Hermitian tensor and $\mathscr{X} \in \mathbb{C}^{n\times 1\times p}$. Then $\mathscr{A} \lessdot (\mathscr{A} + \mathscr{X} * \mathscr{X}^*)$, that is,*

$$\lambda_i(\mathscr{A}) \leqslant \lambda_i(\mathscr{A} + \mathscr{X} * \mathscr{X}^*) \leqslant \lambda_{i-1}(\mathscr{A}).$$

Another useful, special case of Corollary 3 is the following result on compatible polynomials.

COROLLARY 6. *Let $\mathscr{A}, \mathscr{B} \in \mathbb{C}^{n\times n\times p}$ be Hermitian. If $n_+(\mathscr{B}) = n_-(\mathscr{B}) = 1$, then $\mathscr{A} \bowtie (\mathscr{A}+\mathscr{B})$, that is,*

$$\lambda_{i+1}(\mathscr{A}) \leqslant \lambda_i(\mathscr{A}+\mathscr{B}) \leqslant \lambda_{i-1}(\mathscr{A}).$$

Suppose that $\mathscr{P}, \mathscr{A} \in \mathbb{C}^{n\times n\times p}$ are Hermitian tensors and define

$$f(\lambda; \mathscr{P}, \mathscr{A}) = det_T(\lambda \mathscr{P} - \mathscr{A}).$$

Then, $f(\lambda; \mathscr{I}, \mathscr{A})$ is the characteristic polynomial of $\text{bcirc}(\mathscr{A})$. For a real-rooted polynomial f, let $n_+(f), n_-(f)$ and $n_0(f)$ denote the number of positive, negative and zero roots of f, respectively. Call $(n_+(f), n_-(f), n_0(f))$ the inertia index of f. Clearly, if f is the characteristic polynomial of a Hermitian tensor $\mathscr{A}$, then the inertia index of f coincides with that of $\mathscr{A}$. We obtain the following result.

COROLLARY 7. *Suppose that* $\mathscr{P},\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ *are Hermitian, and that* $\mathscr{P}$ *is* T*-positive definite. Then, the following statements are true.*

(i) $f(\lambda;\mathscr{P},\mathscr{A})$ *is a real-rooted polynomial in* λ*.*

(ii) $f(\lambda;\mathscr{P},\mathscr{A})$ *has the same inertia index as* $\mathscr{A}$*.*

(iii) $f(\lambda;\overline{\mathscr{P}},\overline{\mathscr{A}}) \lessdot f(\lambda;\mathscr{P},\mathscr{A})$*, where* $\overline{\mathscr{P}}$ *(respectively,* $\overline{\mathscr{A}}$*) is the tensor obtained from* $\mathscr{P}$ *(respectively,* $\mathscr{A}$*) by deleting the last row and column of frontal faces* $\mathscr{P}$ *(respectively,* $\overline{\mathscr{A}}$*).*

Proof. Let $\mathscr{B} = \mathscr{P}^{-1/2} * \mathscr{A} * \mathscr{P}^{-1/2}$, where $\mathscr{P}^{1/2} * \mathscr{P}^{1/2} = \mathscr{P}$. Then, $\mathscr{B}$ is a Hermitian tensor which is T-congruent to $\mathscr{A}$. Moreover, $f(\lambda;\mathscr{P},\mathscr{A}) = det_T(\lambda\mathscr{P} - \mathscr{A}) = det_T(\mathscr{P})det_T(\lambda\mathscr{I}_{nnp} - \mathscr{B})$. So, $f(\lambda;\mathscr{P},\mathscr{A})$ has the same roots as the characteristic polynomial $det_T(\lambda\mathscr{I}_{nnp} - \mathscr{B})$ of $\mathscr{B}$. Thus, (i) and (ii) follow.

Similarly, let $\mathscr{C} = \overline{\mathscr{P}}^{-1/2} * \mathscr{A} * \overline{\mathscr{P}}^{-1/2}$. Then, $f(\lambda;\overline{\mathscr{P}},\overline{\mathscr{A}})$ has the same roots as $det_T(\lambda\mathscr{I}_{nnp} - \mathscr{C})$. So, to prove (iii) it suffices to show that $\mathscr{C} \lessdot \mathscr{B}$. We prove this by Remark 3 (i). Let $r \in \mathbb{R}$. Note that $\mathscr{B} - r\mathscr{I}_{nnp} = \mathscr{P}^{-1/2} * (\mathscr{A} - r\mathscr{P}) * \mathscr{P}^{-1/2}$. Hence, $n_+(\mathscr{B} - r\mathscr{I}_{nnp}) = n_+(\mathscr{A} - r\mathscr{P})$. Similarly, $n_+(\mathscr{C} - r\mathscr{I}_{nnp}) = n_+(\overline{\mathscr{A}} - r\overline{\mathscr{P}})$. Since $\overline{\mathscr{A}} - r\overline{\mathscr{P}}$ is the $(n-1)\times(n-1)\times p$ tensor obtained from $\mathscr{A} - r\mathscr{P}$, Theorem 1 and Remark 3 (i) allow us to deduce that $0 \leqslant n_+(\mathscr{A} - r\mathscr{P}) - n_+(\overline{\mathscr{A}} - r\overline{\mathscr{P}}) \leqslant 1$. Thus, $0 \leqslant n_+(\mathscr{B} - r\mathscr{I}_{nnp}) - n_+(\mathscr{C} - r\mathscr{I}_{nnp}) \leqslant 1$ and so $\mathscr{C} \lessdot \mathscr{B}$, again by Remark 3 (i). □

REMARK 4. Let $\mathscr{P}_k$ (respectively, $\mathscr{A}_k$) be a $k\times k\times p$ tensor obtained from $\mathscr{P}$ (respectively, $\mathscr{A}$) and $f_k(\lambda) = f(\lambda;\mathscr{P}_k,\mathscr{A}_k)$. Then, each $f_k(\lambda)$ interlaces $f(\lambda;\mathscr{P},\mathscr{A})$. It follows that $\sum_k c_k f_k(\lambda)$ is real-rooted for all $c_k \geqslant 0$.

3. The converse of Weyl's eigenvalue inequality

In this section, we consider a more general problem, namely, to find the converse of Weyl's inequality. First, we present some definitions and fix our notation. Let $\mathscr{R}$ (respectively, $\mathscr{R}_n$) denote the set of real polynomials (respectively, polynomials of degree n) with only real roots and with positive leading coefficients. In particular, let $\mathscr{R}^{(1)}$ denote the set of monic polynomials in $\mathscr{R}$. For g $\in \mathscr{R}$, we use $r_i(g)$ to denote its roots, and we arrange them in the non-increasing order: $r_1(g) \geqslant \cdots \geqslant r_n(g)$. For convenience, set $r_i(g) = +\infty$ for $i < 1$ and $r_i(g) = -\infty$ for $i > deg\ g$.

We refer to (10) as Weyl's inequality, because it is equivalent to (11). Assume that (10) holds for any $\mathscr{A}$ and $\mathscr{B}$. Noting that $n_+(\mathscr{B} - \lambda_j(\mathscr{B}) * \mathscr{I}_{nnp}) \leqslant j-1$ we obtain

$$\lambda_{i+j-1}(\mathscr{A}+\mathscr{B}) = \lambda_{i+j-1}(\mathscr{A}+\mathscr{B}-\lambda_j(\mathscr{B}) * \mathscr{I}_{nnp}) + \lambda_j(\mathscr{B}) \leqslant \lambda_i(\mathscr{A}) + \lambda_j(\mathscr{B}).$$

Let $\mathscr{A}$ and $\mathscr{B}$ be Hermitian tensors. Write $\mathscr{A} \lessdot_q^p \mathscr{B}$ whenever their characteristic polynomials satisfy $p_{\mathscr{A}}(t) \lessdot_q^p p_{\mathscr{B}}(t)$. Using this notation, Weyl's inequality (10) can be restated as follows.

Weyl's inequality. Let $\mathscr{A}$ and $\mathscr{B}$ be Hermitian tensors of the same size. Assume that $n_+(\mathscr{B}-\mathscr{A}) \leqslant p$ and $n_-(\mathscr{B}-\mathscr{A}) \leqslant q$. Then, $\mathscr{A} \lessdot_q^p \mathscr{B}$.

For $f \in \mathscr{R}^{(1)}$, denote by $\mathscr{H}(f)$ the set of Hermitian tensors whose characteristic polynomials are equal to f. The aim of this note is to establish the converse of Weyl's inequality.

THEOREM 10. *Let $f,g \in \mathscr{R}^{(1)}$ have the same degree, and $f \lessdot_q^p g$. Then, there exist $\mathscr{A} \in \mathscr{H}(f)$ and $\mathscr{B} \in \mathscr{H}(g)$ such that $n_+(\mathscr{B}-\mathscr{A}) \leqslant p$ and $n_-(\mathscr{B}-\mathscr{A}) \leqslant q$.*

In the next section, we investigate the property of being (p,q)-interlacing. Some known results on interlacing and compatible polynomials will be extended to (p,q)-interlacing polynomials.

DEFINITION 10. [20] Let $f,g \in \mathscr{R}$ and $p,q \in \mathbb{N}$. The polynomial f is said to (p,q)-interlace the polynomial g, denoted by $f \lessdot_q^p g$, if

$$r_{i+p}(g) \leqslant r_i(f) \leqslant r_{i-q}(g),$$

for all $i \in \mathbb{Z}$.

The following properties immediately follow from the definition.

PROPOSITION 11. [20]

(a) $f \lessdot_q^p f$ for any p and q.

(b) $f \lessdot_q^p g$ is equivalent to $g \lessdot_p^q f$.

(c) $f \lessdot_q^p h$ and $h \lessdot_t^s g$ imply $f \lessdot_{q+t}^{p+s} g$.

(d) $f \lessdot_q^p g$ implies $f \lessdot_t^s g$ for any $s \geqslant p$ and $t \geqslant q$.

(e) $f \lessdot_q^p g$ implies -p $\leqslant$ deg f - deg g $\leqslant$ q.

When $f \lessdot_1^1 g$, we say that f and g are compatible, and we simply write $f \bowtie g$. When $f \lessdot_0^1 g$, we say that f interlaces g, and we denote this by writing $f \lessdot g$. Let $n(f,r)$ be the number of real roots of $f(x)$ in the interval $[r,+\infty)$. It is well-known that f interlaces g if and only if $n(f,r) \leqslant n(g,r) \leqslant n(f,r)+1$ for any r $\in \mathbb{R}$. A result proved by Chudnovsky and Seymour ([1, Theorem 3.4]) states that f and g are compatible if and only if $|\, n(g,r)-n(f,r) \,| \leqslant 1$ for any r $\in \mathbb{R}$. We show that the converse of Proposition 11 (c) is also true.

LEMMA 3. [20] *Suppose that $f \lessdot_q^p g$. For $0 \leqslant s \leqslant p$ and $0 \leqslant t \leqslant q$, let $k = \max\{deg\ f - t, deg\ g - p + s\}$ and $m = \min\{deg\ f + s, deg\ g + q - t\}$. Then, for each integer $d \in [k,m]$ there exists a real-rooted polynomial h of degree d such that $f \lessdot_t^s h$ and $h \lessdot_{q-t}^{p-s} g$.*

Let $\mathscr{A} \in C^{n\times n\times p}$ be a complex tensor. The T-*rank* of A is defined by

$$rank_T(A) = rank(bcirc(\mathscr{A})).$$

For convenience, we agree to define a Hermitian tensor $\mathscr{A}$ as $\mathscr{A} = \sum_{i=1}^{p} \alpha_i * \alpha_i^*$, where $\alpha_i \in \mathbb{C}^{n\times 1\times p}$. This is equivalent to $\mathrm{bcirc}(\mathscr{A}) = \sum_{i=1}^{p} \mathrm{bcirc}(\alpha_i)\mathrm{bcirc}(\alpha_i^*)$, which we simply denote by $\mathscr{A} = \sum_p$. Also, we use the convention that $\sum_0 = 0$. Obviously, $\sum_p$ is T-positive semi-definite and of T-rank at most p. It is also clear that $\sum_{p+q} = \sum_p + \sum_q$, and that $\mathscr{U} * \sum_p * \mathscr{U}^* = \sum_p$ for any unitary tensor.

LEMMA 4. *Let $\mathscr{A}$ be a Hermitian tensor. Then, $n_+(\mathscr{A}) \leqslant p$ and $n_-(\mathscr{A}) \leqslant q$ if and only if $\mathscr{A} = \sum_p - \sum_q$.*

Proof. Let $\mathscr{A} = \sum_p - \sum_q$. Then, we can write

$$n_+(\sum_p - \sum_q) \leqslant n_+(\sum_p) \leqslant rank_T(\sum_p) \leqslant p,$$

and

$$n_-(\sum_p - \sum_q) = n_+(\sum_q - \sum_p) \leqslant q.$$

Conversely, let $n_+(\mathscr{A}) \leqslant p$ and $n_-(\mathscr{A}) \leqslant q$. By Theorem 3, $\mathscr{A} = \mathscr{U} * \mathscr{D} * \mathscr{U}^*$ or $\mathrm{bcirc}(\mathscr{A}) = \mathrm{bcirc}(\mathscr{U})\mathrm{bcirc}(\mathscr{D})\mathrm{bcirc}(\mathscr{U}^*)$. Thus, by (1),

$$\mathrm{bcirc}(\mathscr{A}) = (F_p \otimes I_n)^* \begin{bmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_p \end{bmatrix} (F_p \otimes I_n)(F_p \otimes I_n)^* \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_p \end{bmatrix} (F_p \otimes I_n)$$

$$(F_p \otimes I_n)^* \begin{bmatrix} U_1^* & & & \\ & U_2^* & & \\ & & \ddots & \\ & & & U_p^* \end{bmatrix} (F_p \otimes I_n)$$

$$= (F_p \otimes I_n)^* \underbrace{\begin{bmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_p \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_p \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} U_1^* & & & \\ & U_2^* & & \\ & & \ddots & \\ & & & U_p^* \end{bmatrix} (F_p \otimes I_n)}_{V^*}$$

$$= VDV^*.$$

Since $\mathscr{D}$ is F-diagonal, D_i is diagonal. Thus, D is a diagonal matrix. By (1), V is the unitary matrix defined by

$$V = \mathrm{bcirc}(\mathscr{U})(F_p \otimes I_n)^*.$$

By Sylvester's law of inertia for Hermitian matrices,

$$\mathrm{bcirc}(\mathscr{A}) = \sum_i \lambda(D) V_i V_i^* = \sum_i \lambda_i(D)(\mathrm{bcirc}(\mathscr{U}))_i(\mathrm{bcirc}(\mathscr{U}))_i^* = \sum_p - \sum_q,$$

where $(\mathrm{bcirc}(\mathscr{U}))_i$ is the ith column of $\mathrm{bcirc}(\mathscr{U})$. $\square$

Note that $(\mathscr{A} * \mathscr{B})^{(k)} = A^{(k)} B^{(k)}$ for $\mathscr{A}, \mathscr{B} \in \mathbb{C}^{n\times n\times p}$.

LEMMA 5. *Let f, $g \in \mathscr{R}_n^{(1)}$ and $f \lessdot g$. Then, there exist a Hermitian tensor $\mathscr{A}$ and a complex vector α such that $\mathscr{A} \in \mathscr{H}(f)$ and $\mathscr{A} + \alpha * \alpha^* \in \mathscr{H}(g)$.*

Proof. First, consider the special case where f and g are coprime. Then, by the fact $f \lessdot g$, f only has simple roots. Let $r_i = r_i(f)$, $f_i(x) = \frac{f(x)}{x-r_i}$ and $g(x) = f(x) + \sum_{i=1}^n c_i f_i(x)$. Then, $g(r_i) = c_i f_i(r_i)$. Note that $\mathrm{sign}[f_i(r_i)] = (-1)^{(i-1)}$ and $\mathrm{sign}[g(r_i)] = (-1)^i$. Hence, $c_i < 0$. Define $\mathscr{A} = (a_{ijk}) \in \mathbb{C}^{n\times n\times p}$ such that $A^{(1)} = diag(r_1, r_2, \ldots, r_n)$ and $A^{(k)} = 0$ for $2 \leqslant k \leqslant p$, and $\alpha \in \mathbb{C}^{n\times 1\times p}$, where $\alpha^{(1)} = (a_1, a_2, \ldots, a_n)^T, a_i = \sqrt{-c_i}$, and $\alpha^{(k)} = 0$ for $2 \leqslant k \leqslant p$. Then, $det(xI_n - A^{(1)}) = f(x)$. By (1),

$$det(xI_{np} - \mathrm{bcirc}(\mathscr{A})) = det(xI_{np} - D_A) = f^n(x),$$

where $D_A = diag(A_1, A_2, \ldots, A_p)$ in (1). It follows that $det(xI_n - A_i) = f(x)$, for all i. On the other hand, by (4), $p_{\mathscr{A}}(x) = f(x)$. Thus, $\mathscr{A} \in \mathscr{H}(f)$. We can write

$$det(xI_n - A^{(1)} - \alpha^{(1)}(\alpha^*)^{(1)}) = det(xI_n - A^{(1)}) - \sum_{i=1}^n a_i^2 \prod_{j\neq i}(x - r_j) = g(x).$$

Similarly, $\mathscr{A} + \alpha * \alpha^* \in \mathscr{H}(f)$.

Next, consider the general case. Let $f = (f,g)f_1$ and $g = (f,g)g_1$. Then, $f_1 \lessdot g_1$ and $(f_1, g_1) = 1$. Thus, there exist Hermitian tensors $\mathscr{A}_1 \in \mathscr{H}(f_1)$ and $\mathscr{A}_1 + \alpha_1 * \alpha_1^* \in \mathscr{H}(g_1)$. Assume that $(f,g) = \prod_{j=1}^m (x - s_j)$ and define $\mathscr{A} \in \mathbb{C}^{n\times n\times p}$ such that $A^{(1)} = diag(A_1^{(1)}, s_1, \ldots, s_m)$ and $A^{(k)} = 0$ for $2 \leqslant k \leqslant p$. Also, define $\alpha \in \mathbb{C}^{n\times 1\times p}$ such that $\alpha^{(1)} = (\alpha_1^{(1)}, 0, \ldots, 0)$ and $\alpha^{(k)} = 0$ for $2 \leqslant k \leqslant p$. Then, $\mathscr{A} \in \mathscr{H}(f)$ and $\mathscr{A} + \alpha * \alpha^* \in \mathscr{H}(g)$. This completes the proof. $\square$

LEMMA 6. *Let f, $g \in \mathscr{R}_n^{(1)}$ and $f \lessdot_0^p g$. Then, there exist Hermitian tensors $\mathscr{A} \in \mathscr{H}(f)$ and $\mathscr{B} \in \mathscr{H}(g)$ such that $\mathscr{B} - \mathscr{A} = \sum_p$.*

Proof. We proceed by induction on p. For $p = 1$, the statement follows from Lemma 5. Now, suppose that $p > 1$ and $f \lessdot_0^p g$. Then, by Lemma 3, there exists $h \in \mathscr{R}_n^{(1)}$ such that $f \lessdot_0^{p-1} h$ and $h \lessdot_0^1 g$. By the fact $f \lessdot_0^{p-1} h$ and the induction hypothesis, there exist $\mathscr{A} \in \mathscr{H}(f)$ and $\mathscr{C} \in \mathscr{H}(h)$ such that $\mathscr{C} - \mathscr{A} = \sum_{p-1}$. By the fact $h \lessdot_0^1 g$ and Lemma 5, there exist $\mathscr{B}_1 \in \mathscr{H}(g)$ and $\mathscr{C}_1 \in \mathscr{H}(h)$ such that $\mathscr{B}_1 - \mathscr{C}_1 = \sum_1$. Since the Hermitian tensors $\mathscr{C}$ and $\mathscr{C}_1$ have the same characteristic polynomial, by Remark 2 there exists a unitary tensor $\mathscr{U}$ such that $\mathscr{U} * \mathscr{C}_1 * \mathscr{U}^*$. Define $\mathscr{B} = \mathscr{U} * \mathscr{B}_1 * \mathscr{U}^*$. Then, $\mathscr{B} \in \mathscr{H}(g)$ and

$$\mathscr{B} - \mathscr{C} = \mathscr{U} * (\mathscr{B}_1 - \mathscr{C}_1) * \mathscr{U}^* = \mathscr{U} * \sum_1 * \mathscr{U}^* = \sum_1.$$

It follows that

$$\mathscr{B} - \mathscr{A} = (\mathscr{B} - \mathscr{C}) + (\mathscr{C} - \mathscr{A}) = \sum_1 + \sum_{p-1} = \sum_p.$$

Thus, the proof is complete by induction. $\square$

Proof of Theorem 10. By Lemma 4, it suffices to show that if $f \lessdot_q^p g$, then there exist Hermitian tensors $\mathscr{A} \in \mathscr{H}(f)$ and $\mathscr{B} \in \mathscr{H}(g)$ such that $\mathscr{B} - \mathscr{A} = \sum_p - \sum_q$.

If $p = 0$ or $q = 0$, then the statement follows from Lemma 6. Next, assume that $p, q > 0$. Then, by Lemma 3, there exists $h \in R_n^{(1)}$ such that $f \lessdot_0^p h$ and $g \lessdot_0^q h$. By the fact $f \lessdot_0^p h$ and Lemma 6, there exist $\mathscr{A} \in \mathscr{H}(f)$ and $\mathscr{C} \in \mathscr{H}(h)$ such that $\mathscr{C} - \mathscr{A} = \sum_p$. By the fact $g \lessdot_0^q h$ and Lemma 6, there exist $\mathscr{B}_1 \in \mathscr{H}(g)$ and $\mathscr{C}_1 \in \mathscr{H}(h)$ such that $\mathscr{C}_1 - \mathscr{B}_1 = \sum_q$. Since the Hermitian tensors $\mathscr{C}$ and $\mathscr{C}_1$ have the same characteristic polynomial, there exists a unitary tensor $\mathscr{U}$ such that $\mathscr{C} = \mathscr{U} * \mathscr{C}_1 * \mathscr{U}^*$. Define $\mathscr{B} = \mathscr{U} * \mathscr{B}_1 * \mathscr{U}^*$. Then, $\mathscr{B} \in \mathscr{H}(g)$ and

$$\mathscr{B} - \mathscr{C} = \mathscr{U} * (\mathscr{B}_1 - \mathscr{C}_1) * \mathscr{U}^* = \mathscr{U} * \sum_q * \mathscr{U}^* = \sum_q.$$

It follows that

$$\mathscr{B} - \mathscr{A} = (\mathscr{B} - \mathscr{C}) + (\mathscr{C} - \mathscr{A}) = \sum_p - \sum_q.$$

This completes the proof. $\square$

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