Mathematical
Inequalities
& Applications
Volume 28, Number 1 (2025), 59–72 doi:10.7153/mia-2025-28-04

FURTHER GENERALIZATIONS OF ALZER–FONSECA–KOVAČEC TYPE INEQUALITIES AND APPLICATIONS

TRAN DINH PHUNG AND DUONG QUOC HUY*

(*Communicated by S. Varošanec*)

Abstract. In this paper, we develop a new method which allows us to establish interesting generalizations of the well-known Young-type inequalities, or inequalities between arithmetic and harmonic mean. Several attractive applications of these inequalities to matrix inequalities, determinant inequalities and unitarily invariant norm inequalities are also presented.

1. Introduction

The well-known Young inequality for two positive real numbers a and b says that, for all $\nu \in [0,1]$,

$$a\nabla_\nu b := (1-\nu)a + \nu b \geqslant a^{1-\nu}b^\nu =: a\sharp_\nu b$$

with the equality sign if and only if $a = b$ or $\nu \in \{0,1\}$.

One of the most important two weight generalizations of this inequality is the Alzer-Fonseca-Kovačec inequality discovered in 2015 of the form

$$\left(\frac{\nu}{\mu}\right)^\lambda \leqslant \frac{(a\nabla_\nu b)^\lambda - (a\sharp_\nu b)^\lambda}{(a\nabla_\mu b)^\lambda - (a\sharp_\mu b)^\lambda} \leqslant \left(\frac{1-\nu}{1-\mu}\right)^\lambda, \tag{1.1}$$

where $0 < \nu \leqslant \mu < 1$ and $\lambda \geqslant 1$, see [1] for the details. When $\lambda = 1$, the double inequality provides sharper inequalities in comparison with the original results by Kittaneh and Manasrah in [6, 7]. The idea for the proof of the left-hand side of the inequality (1.1) is to show that for each $0 < x \neq 1$, the function

$$f(\nu) = \frac{(1\nabla_\nu x)^\lambda - (1\sharp_\nu x)^\lambda}{\nu^\lambda}$$

is decreasing on $(0,1)$, which yields the claimed inequality. The right-hand side of the inequality (1.1) is proved similarly.

In 2006, Dragomir [3] established the famous double inequality in the following theorem.

Mathematics subject classification (2020): 15A45, 47A30, 15A60, 26D07.
Keywords and phrases: Young inequality, trace inequality, norm inequality, operator inequality, determinant inequality.

* Corresponding author.

THEOREM 1.1. ([3]) *Let f be a convex function defined an interval $J \subset \mathbb{R}$. If $0 < \alpha \leqslant \beta < 1$ are two weights and $x, y \in J$, we then have*

$$\frac{\alpha}{\beta} \leqslant \frac{(1-\alpha)f(x)+\alpha f(y)-f((1-\alpha)x+\alpha y)}{(1-\beta)f(x)+\beta f(y)-f((1-\beta)x+\beta y)} \leqslant \frac{1-\alpha}{1-\beta}. \tag{1.2}$$

It is remarkable that by choosing $x=0,\ y=1$, and $f(\nu)=a^{1-\nu}b^{\nu}$ with $a,b>0$, $\nu \in [0,1]$ in (1.2), we easily obtain the inequalities in (1.1) for $\lambda = 1$. Moreover, based on the techniques as in [5] and the result in the case $\lambda = 1$, we get a new proof of it without employing the tools of single variable calculus.

In 2020, by using the same idea and the techniques of single variable calculus as in [1] for the following functions

$$g(\nu) = \frac{1\nabla_\nu x - 1\sharp_\nu x}{\nu(1-\nu)} \quad \text{and} \quad h(\nu) = \frac{(1\nabla_\nu x)^2 - (1\sharp_\nu x)^2}{\nu(1-\nu)}$$

with $\nu \in (0,1)$, Y. Ren showed in [9] that, for $b > a > 0$ and $0 < \nu \leqslant \tau < 1$,

$$\frac{a\nabla_\nu b - a\sharp_\nu b}{a\nabla_\tau b - a\sharp_\tau b} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)} \quad \text{and} \quad \frac{(a\nabla_\nu b)^2 - (a\sharp_\nu b)^2}{(a\nabla_\tau b)^2 - (a\sharp_\tau b)^2} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{1.3}$$

More generally, in 2023 by utilizing the expansion

$$(1\nabla_\nu x)^m - (1\sharp_\nu x)^m = [(1\nabla_\nu x) - (1\sharp_\nu x)]\phi(\nu) \tag{1.4}$$

and showing the function ϕ is increasing on $(0,1)$ when $x \in [1,\infty)$ and the function ϕ is decreasing on $(0,1)$ when $x \in (0,1]$, C. Yang and Z. Wang [11] generalized (1.3) to

$$\frac{(a\nabla_\nu b)^m - (a\sharp_\nu b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)} \tag{1.5}$$

under the same conditions $b > a > 0$ and $0 < \nu \leqslant \tau < 1$, where m are positive integers. Recently, thanks to the same idea as of C. Yang and Z. Wang [11], Y. Ren showed in [10, Theorem 3] that, for $0 < \nu \leqslant \tau < \frac{1}{2}$, $m \in \mathbb{N}^+$ and $a > b > 0$,

$$\frac{(a\nabla_\nu b)^m - (K(h,2)^\nu a\sharp_\nu b)^m}{(a\nabla_\tau b)^m - (K(h,2)^\tau a\sharp_\tau b)^m} \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}, \tag{1.6}$$

where $h = \frac{b}{a} > 0$ and $K(h,2) = \frac{(h+1)^2}{4h}$ is the Kantorovich constant.

A natural question arises from the above mentioned results: *whether the inequalities (1.5) and (1.6) hold for real numbers $p \in (m, m+1)$ with $m \in \mathbb{N}^+$?*

Unfortunately, the method used in [10, 11] to get the inequalities (1.5) and (1.6) does not work for such real numbers p. Even an effective approach as in [5] leads only to weaker results than desired. Very recently, X. Yang, C. Yang and H. Li [12] gave an affirmative answer to the above question by showing the following three inequalities, where $a!_\nu b = ((1-\nu)a^{-1} + \nu b^{-1})^{-1}$ is meant to be the harmonic mean of reals $a, b > 0$.

1. If $0 < a < b, \lambda \geqslant 1$ and $0 < \nu \leqslant \tau < 1$, then

$$\frac{(a\nabla_\nu b)^\lambda - (a\sharp_\nu b)^\lambda}{(a\nabla_\tau b)^\lambda - (a\sharp_\tau b)^\lambda} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{1.7}$$

2. If $0 < a < b, \lambda \geqslant 1$ and $0 < \nu \leqslant \tau < 1$, then

$$\frac{(a\nabla_\nu b)^\lambda - (a!_\nu b)^\lambda}{(a\nabla_\tau b)^\lambda - (a!_\tau b)^\lambda} \leqslant \frac{a\nabla_\nu b - a!_\nu b}{a\nabla_\tau b - a!_\tau b} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{1.8}$$

3. If $0 < b < a, \lambda \geqslant 1$ and $0 < \nu \leqslant \tau < \frac{1}{2}$, then

$$\frac{(a\nabla_\nu b)^\lambda - (K(h,2)^\nu a\sharp_\nu b)^\lambda}{(a\nabla_\tau b)^\lambda - (K(h,2)^\tau a\sharp_\tau b)^\lambda} \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{1.9}$$

The method used to prove these three inequalities is not different from that of [9, 10, 11], which is difficult to apply to more general cases. In the present paper, we develop a new way to generalize the above-mentioned results for increasing convex functions.

Besides the current section the paper consists of two sections. In Section 2, we establish two series of inequalities involving increasing convex functions, the immediate consequences of which are generalizations of Alzer-Fonseca-Kovačec-type inequalities related to the Arithmetic mean, the Geometric mean and the Harmonic mean (see Theorems 2.2, 2.4, and 2.6 below). In Section 3, we propose various applications of the newly introduced results to inequalities involving matrices, unitarily invariant norms, traces and determinants.

2. Further generalizations of Alzer-Fonseca-Kovačec type inequalities

The goal of this section is to generalize the Alzer-Fonseca-Kovačec type inequalities (1.5) and (1.6) by replacing the power function by any increasing convex function.

We start this section with an important result in the following.

THEOREM 2.1. *Let a_1, a_2, b_1, b_2 with $a_1 \neq a_2$, $b_1 \neq b_2$ be real numbers selected from an interval J on which φ is a strictly increasing convex function. Let ψ be a strictly increasing convex function on the interval $\varphi(J)$. If $a_2, b_1 \in [a_1, b_2]$, the double inequality* (2.1) *below holds:*

$$\frac{a_2 - a_1}{b_2 - b_1} \geqslant \frac{\varphi(a_2) - \varphi(a_1)}{\varphi(b_2) - \varphi(b_1)} \geqslant \frac{\psi\circ\varphi(a_2) - \psi\circ\varphi(a_1)}{\psi\circ\varphi(b_2) - \psi\circ\varphi(b_1)}. \tag{2.1}$$

If $a_1, b_2 \in [b_1, a_2]$, then the above inequalities work in the reversed direction, that is

$$\frac{a_2 - a_1}{b_2 - b_1} \leqslant \frac{\varphi(a_2) - \varphi(a_1)}{\varphi(b_2) - \varphi(b_1)} \leqslant \frac{\psi\circ\varphi(a_2) - \psi\circ\varphi(a_1)}{\psi\circ\varphi(b_2) - \psi\circ\varphi(b_1)}. \tag{2.2}$$

Proof. Since $a_2, b_1 \in [a_1, b_2] \subset J$, we deduce that $a_1 \leqslant b_1,\ a_2 \leqslant b_2,\ a_1 < a_2$, and $b_1 < b_2$. This, together with the hypothesis on the strictly increasing convex property of φ on J and the inequality (3.6) in [8, p. 2], implies that $\varphi(a_1) < \varphi(a_2),\ \varphi(b_1) < \varphi(b_2)$, and

$$\frac{\varphi(a_2)-\varphi(a_1)}{a_2-a_1} \leqslant \frac{\varphi(b_2)-\varphi(b_1)}{b_2-b_1},$$

which is equivalent to

$$\frac{a_2-a_1}{b_2-b_1} \geqslant \frac{\varphi(a_2)-\varphi(a_1)}{\varphi(b_2)-\varphi(b_1)}. \tag{2.3}$$

On the other hand, we also have because of the strictly increasing property of φ on J that $\varphi(a_2), \varphi(b_1) \in [\varphi(a_1), \varphi(b_2)] \subset \varphi(J)$, which leads to $\varphi(a_1) \leqslant \varphi(b_1) < \varphi(b_2)$ and $\varphi(a_1) < \varphi(a_2) \leqslant \varphi(b_2)$. This, combined with the strictly increasing convex property of ψ on $\varphi(J)$ and the inequality (3.6) in [8, p. 2], follows that

$$\frac{\psi\circ\varphi(a_2)-\psi\circ\varphi(a_1)}{\varphi(a_2)-\varphi(a_1)} \leqslant \frac{\psi\circ\varphi(b_2)-\psi\circ\varphi(b_1)}{\varphi(b_2)-\varphi(b_1)},$$

or equivalently,

$$\frac{\psi\circ\varphi(a_2)-\psi\circ\varphi(a_1)}{\psi\circ\varphi(b_2)-\psi\circ\varphi(b_1)} \leqslant \frac{\varphi(a_2)-\varphi(a_1)}{\varphi(b_2)-\varphi(b_1)}.$$

This, together with (2.3), gives us the inequality (2.1).

The inequality (2.2) is similarly proved by utilizing the inequality (3.6) in [8, p. 2], and so we omit the details. □

Now let us mention some remarkable special cases of Theorem 2.1, which offer various generalizations of the Alzer-Fonseca-Kovačec type inequalities (1.5) and (1.6).

The first significant consequence is as follows.

THEOREM 2.2. *Let $0 < \nu \leqslant \tau < 1$, φ be a strictly increasing convex function defined on some interval J containing positive real numbers a, b, and ψ be a strictly increasing convex function defined on $\varphi(J)$.*

(1) *If $b > a > 0$, then*

$$\frac{\psi\circ\varphi(a\nabla_\nu b)-\psi\circ\varphi(a\sharp_\nu b)}{\psi\circ\varphi(a\nabla_\tau b)-\psi\circ\varphi(a\sharp_\tau b)} \leqslant \frac{\varphi(a\nabla_\nu b)-\varphi(a\sharp_\nu b)}{\varphi(a\nabla_\tau b)-\varphi(a\sharp_\tau b)} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{2.4}$$

(2) *If $a > b > 0$, then*

$$\frac{\psi\circ\varphi(a\nabla_\nu b)-\psi\circ\varphi(a\sharp_\nu b)}{\psi\circ\varphi(a\nabla_\tau b)-\psi\circ\varphi(a\sharp_\tau b)} \geqslant \frac{\varphi(a\nabla_\nu b)-\varphi(a\sharp_\nu b)}{\varphi(a\nabla_\tau b)-\varphi(a\sharp_\tau b)} \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{2.5}$$

Proof. To prove the first claim, we put $a_1 = a\sharp_\nu b,\ a_2 = a\nabla_\nu b,\ b_1 = a\sharp_\tau b,\ b_2 = a\nabla_\tau b$. These numbers satisfy the conditions $a_2, b_1 \in [a_1, b_2]$, $a_1 < a_2$ and $b_1 < b_2$ in Theorem 2.1 and by Ren's first inequality (1.3) we know that the left-hand side of (2.1) is smaller or equal to $\frac{\nu(1-\nu)}{\tau(1-\tau)}$. The inequality (2.1) immediately yields the inequality (2.4).

The other inequality is proved similarly. However, we can obtain it directly from (2.4) as follows. Since $0<\nu\leqslant\tau<1$, we have $0<1-\tau\leqslant 1-\nu<1$. Thus, replacing a,b,ν and τ in the inequality (2.4) with $b,a,1-\tau$ and $1-\nu$, respectively, we get the inequality (2.5). $\square$

REMARK 2.3. If we let $\varphi(t)=t^p$ and $\psi(t)=t^{q/p}$ with $1\leqslant p\leqslant q<\infty$ and $t\in[0,\infty)$ in (2.4), then for $0<a<b$ and $0<\nu\leqslant\tau<1$,

$$\frac{(a\nabla_\nu b)^q-(a\sharp_\nu b)^q}{(a\nabla_\tau b)^q-(a\sharp_\tau b)^q}\leqslant\frac{(a\nabla_\nu b)^p-(a\sharp_\nu b)^p}{(a\nabla_\tau b)^p-(a\sharp_\tau b)^p}\leqslant\frac{a\nabla_\nu b-a\sharp_\nu b}{a\nabla_\tau b-a\sharp_\tau b}\leqslant\frac{\nu(1-\nu)}{\tau(1-\tau)}.$$

These inequalities are reverse in the case $a>b>0$ and $0<\nu\leqslant\tau<1$. These results generalize the main inequalities in [12, Theorem 2.2 and Corollary 2.3].

Next, in view of Theorem 2.1 and Ren's inequality (1.6), we obtain the second significant consequence as follows.

THEOREM 2.4. *Let φ be an increasing convex function defined on some interval J containing positive real numbers a,b, and ψ be a strictly increasing convex function defined on $\varphi(J)$. Let $K(h,2)=\frac{(h+1)^2}{4h}$ with $h=\frac{b}{a}>0$ be the Kantorovich constant.*

(1) *If $0<\nu\leqslant\tau<\frac{1}{2}$ and $a>b>0$, then*

$$\frac{\psi\circ\varphi(a\nabla_\nu b)-\psi\circ\varphi(K(h,2)^\nu a\sharp_\nu b)}{\psi\circ\varphi(a\nabla_\tau b)-\psi\circ\varphi(K(h,2)^\tau a\sharp_\tau b)}\geqslant\frac{\varphi(a\nabla_\nu b)-\varphi(K(h,2)^\nu a\sharp_\nu b)}{\varphi(a\nabla_\tau b)-\varphi(K(h,2)^\tau a\sharp_\tau b)}\geqslant\frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{2.6}$$

(2) *If $\frac{1}{2}<\nu\leqslant\tau<1$ and $b>a>0$, then*

$$\begin{aligned}\frac{\psi\circ\varphi(a\nabla_\nu b)-\psi\circ\varphi(K(h,2)^{1-\nu}a\sharp_\nu b)}{\psi\circ\varphi(a\nabla_\tau b)-\psi\circ\varphi(K(h,2)^{1-\tau}a\sharp_\tau b)}&\leqslant\frac{\varphi(a\nabla_\nu b)-\varphi(K(h,2)^{1-\nu}a\sharp_\nu b)}{\varphi(a\nabla_\tau b)-\varphi(K(h,2)^{1-\tau}a\sharp_\tau b)}\\&\leqslant\frac{\nu(1-\nu)}{\tau(1-\tau)}.\end{aligned} \tag{2.7}$$

Proof. To show the second statement of Theorem 2.4, for $\frac{1}{2}<\nu\leqslant\tau<1$, we have $0<1-\tau\leqslant 1-\nu<\frac{1}{2}$. Hence, by swapping the parameters ν,τ,a,b for $1-\tau,1-\nu,b,a$ in Ren's inequality (1.6), we get for $\frac{1}{2}<\nu\leqslant\tau<1$ and $b>a>0$,

$$\frac{a\nabla_\nu b-K(h,2)^{1-\nu}a\sharp_\nu b}{a\nabla_\tau b-K(h,2)^{1-\tau}a\sharp_\tau b}\leqslant\frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{2.8}$$

Now, we put $a_1=K(h,2)^{1-\nu}a\sharp_\nu b$, $a_2=a\nabla_\nu b$, $b_1=K(h,2)^{1-\tau}a\sharp_\tau b$, $b_2=a\nabla_\tau b$. These numbers satisfy the conditions $a_2,b_1\in[a_1,b_2]$, $a_1<a_2$ and $b_1<b_2$ in Theorem

2.1 and by (2.8) we know that the left-hand side of (2.1) is smaller or equal to $\frac{\nu(1-\nu)}{\tau(1-\tau)}$. Then thanks to the inequality (2.1), we immediately obtain the inequality (2.7).

The inequality (2.6) follows directly from the inequality (2.7). Indeed, because of $0<\nu\leqslant\tau<\frac{1}{2}$, we get $\frac{1}{2}<1-\nu\leqslant 1-\tau<1$. Thus, replacing a,b,ν and τ in the inequality (2.7) by $b,a,1-\tau$ and $1-\nu$, respectively, we obtain the inequality (2.6). □

REMARK 2.5. In the special case $\varphi(t)=t^p$ and $\psi(t)=t^{q/p}$ with $1\leqslant p\leqslant q<\infty$ and $t\in[0,\infty)$, Theorem 2.4 yields that:

1. If $0<\nu\leqslant\tau<\frac{1}{2}$ and $a>b>0$, then

$$\frac{(a\nabla_\nu b)^q-(K(h,2)^\nu a\sharp_\nu b)^q}{(a\nabla_\tau b)^q-(K(h,2)^\tau a\sharp_\tau b)^q}\geqslant\frac{(a\nabla_\nu b)^p-(K(h,2)^\nu a\sharp_\nu b)^p}{(a\nabla_\tau b)^p-(K(h,2)^\tau a\sharp_\tau b)^p}$$
$$\geqslant\frac{a\nabla_\nu b-K(h,2)^\nu a\sharp_\nu b}{a\nabla_\tau b-K(h,2)^\tau a\sharp_\tau b}\geqslant\frac{\nu(1-\nu)}{\tau(1-\tau)}.$$

2. If $\frac{1}{2}<\nu\leqslant\tau<1$ and $b>a>0$, then

$$\frac{(a\nabla_\nu b)^q-(K(h,2)^{1-\nu} a\sharp_\nu b)^q}{(a\nabla_\tau b)^q-(K(h,2)^{1-\tau} a\sharp_\tau b)^q}\leqslant\frac{(a\nabla_\nu b)^p-(K(h,2)^{1-\nu} a\sharp_\nu b)^p}{(a\nabla_\tau b)^p-(K(h,2)^{1-\tau} a\sharp_\tau b)^p}$$
$$\leqslant\frac{a\nabla_\nu b-K(h,2)^{1-\nu} a\sharp_\nu b}{a\nabla_\tau b-K(h,2)^{1-\tau} a\sharp_\tau b}\leqslant\frac{\nu(1-\nu)}{\tau(1-\tau)}.$$

These results extend and generalize [12, Theorem 2.7].

The last main consequence of this section is as follows.

THEOREM 2.6. *Let $0<\nu\leqslant\tau<1$, φ be a strictly increasing convex function defined on some interval J containing positive real numbers a,b, and ψ be a strictly increasing convex function defined on $\varphi(J)$.*

1. If $b>a>0$, then

$$\frac{\psi\circ\varphi(a\nabla_\nu b)-\psi\circ\varphi(a!_\nu b)}{\psi\circ\varphi(a\nabla_\tau b)-\psi\circ\varphi(a!_\tau b)}\leqslant\frac{\varphi(a\nabla_\nu b)-\varphi(a!_\nu b)}{\varphi(a\nabla_\tau b)-\varphi(a!_\tau b)}\leqslant\frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{2.9}$$

2. If $a>b>0$, then

$$\frac{\psi\circ\varphi(a\nabla_\nu b)-\psi\circ\varphi(a!_\nu b)}{\psi\circ\varphi(a\nabla_\tau b)-\psi\circ\varphi(a!_\tau b)}\geqslant\frac{\varphi(a\nabla_\nu b)-\varphi(a!_\nu b)}{\varphi(a\nabla_\tau b)-\varphi(a!_\tau b)}\geqslant\frac{\nu(1-\nu)}{\tau(1-\tau)}. \tag{2.10}$$

Proof. We first prove the inequality (2.9). Set $a_1=a!_\nu b$, $a_2=a\nabla_\nu b$, $b_1=a!_\tau b$, $b_2=a\nabla_\tau b$. Observe that these numbers satisfy the conditions $a_2,b_1\in[a_1,b_2]$, $a_1<a_2$ and $b_1<b_2$ in Theorem 2.1. Then the inequality (2.1) together with the right-hand side of (1.8) implies the inequality (2.9).

One can obtain the inequality (2.10) directly from (2.9) as follows. Since $0 < \nu \leqslant \tau < 1$, we have $0 < 1-\tau \leqslant 1-\nu < 1$. Thus, replacing a, b, ν and τ in the inequality (2.9) by $b, a, 1-\tau$ and $1-\nu$, respectively, we get the inequality (2.10). $\square$

REMARK 2.7. If we choose $\varphi(t) = t^p$ and $\psi(t) = t^{q/p}$ with $1 \leqslant p \leqslant q < \infty$ and $t \in [0,\infty)$, Theorem 2.6 implies that:

1. If $b > a > 0$ and $0 < \nu \leqslant \tau < 1$, then

$$\frac{(a\nabla_\nu b)^q - (a!_\nu b)^q}{(a\nabla_\tau b)^q - (a!_\tau b)^q} \leqslant \frac{(a\nabla_\nu b)^p - (a!_\nu b)^p}{(a\nabla_\tau b)^p - (a!_\tau b)^p} \leqslant \frac{a\nabla_\nu b - a!_\nu b}{a\nabla_\tau b - a!_\tau b} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}.$$

2. If $a > b > 0$ and $0 < \nu \leqslant \tau < 1$, then

$$\frac{(a\nabla_\nu b)^q - (a!_\nu b)^q}{(a\nabla_\tau b)^q - (a!_\tau b)^q} \geqslant \frac{(a\nabla_\nu b)^p - (a!_\nu b)^p}{(a\nabla_\tau b)^p - (a!_\tau b)^p} \geqslant \frac{a\nabla_\nu b - a!_\nu b}{a\nabla_\tau b - a!_\tau b} \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}.$$

These results extend and generalize [12, Theorem 2.8].

3. Some applications

We begin this section with some preliminaries on the theory of matrices. Let M_n be a set of all $n \times n$ complex matrices. A Hermitian matrix $A \in M_n$ is called positive semidefinite (positive definite), written $A \geqslant 0$ ($A > 0$), if $z^*Az \geqslant 0$ for all $z \in \mathbb{C}^n$ (if $z^*Az > 0$ for all $z \in \mathbb{C}^n \setminus \{0\}$, respectively), and let us denote respectively by

$$M_n^+ = \{A \in M_n : \ A \text{ is positive semidefinite}\}$$

and

$$M_n^{++} = \{A \in M_n : \ A \text{ is positive definite}\}.$$

For two Hermitian matrices $A, B \in M_n$, we use the notation $A \geqslant B$ to indicate that $A - B \geqslant 0$. The spectral theorem for Hermitian matrices says that (see [4, Theorem 2.5.6]) if $\lambda_1, \ldots, \lambda_n$ are the not necessarily distinct eigenvalues of a Hermitian matrix $A \in M_n$, there then exists a unitarily matrix $U \in M_n$ such that $A = U\Lambda U^*$, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Moreover, if f is a continuous function defined on an interval J containing $\lambda_1, \ldots, \lambda_n$, one defines $f(A) = Uf(\Lambda)U^*$, where $f(\Lambda) = \operatorname{diag}(f(\lambda_1), \ldots, f(\lambda_n))$. The absolute value of $A \in M_n$ is defined as $|A| = (A^*A)^{1/2}$, and the singular values $s_1(A) \geqslant \cdots \geqslant s_n(A)$ of A are the eigenvalues of $|A|$.

Next, a norm $|||\cdot|||$ on M_n is called unitarily invariant if $|||UAV||| = |||A|||$ for all unitary matrices $U, V \in M_n$. Several typical examples of unitarily invariant norms are as follows: For $A \in M_n$,

1. the trace norm: $\operatorname{tr}(|A|) = \sum_{j=1}^n s_j(A)$;
2. the spectral norm: $\|A\| = s_1(A)$;
3. the Schatten p-norm: $\|A\|_p = \left(\sum_{j=1}^n s_j^p(A)\right)^{1/p}$ for any $p \in [1,\infty)$;

4. the Hilbert-Schmidt (or the Frobenious) norm: $\|A\|_2 = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{\frac{1}{2}}$.

Finally, we denote the ν-weighted arithmetic mean, geometric and harmonic mean of two positive definite matrices $A,B \in M^{++}$ and $\nu \in [0,1]$ by

$$A\nabla_\nu B = (1-\nu)A + \nu B, \quad A\sharp_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}},$$

and

$$A!_\nu B = ((1-\nu)A^{-1} + \nu B^{-1})^{-1} = A^{1/2}[(1-\nu)I + \nu(A^{-1/2}BA^{-1/2})^{-1}]^{-1}A^{1/2},$$

here, $I \in M_n$ is the unit matrix. The same notations as above are also employed when $\nu \notin [0,1]$. In particular, we write $A\nabla B$, $A\sharp B$ and $A!B$ respectively for the case $\nu = \frac{1}{2}$.

3.1. Matrix inequalities

The matrix version of the Young inequality says that, for any matrices $A,B \in M_n^{++}$,

$$A\nabla_\nu B \geqslant A\sharp_\nu B.$$

The two weight refinement and reverse of this inequality was established by Alzer, Fonseca and Kovačec in [1] in the form

$$\frac{\nu}{\tau}(A\nabla_\tau B - A\sharp_\tau B) \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant \frac{1-\nu}{1-\tau}(A\nabla_\tau B - A\sharp_\tau B),$$

where $A,B \in M_n^{++}$ and $0 < \nu \leqslant \tau < 1$. Its exponential version was proved by Choi in [2] in the form

$$A\sharp_m(A\nabla_\nu B) \geqslant A\sharp_{\nu m}B + (2r_0)^m[A\sharp_m(A\nabla B) - A\sharp_{m/2}B],$$
$$A\sharp_m(A\nabla_\nu B) \leqslant A\sharp_{\nu m}B + (2R_0)^m[A\sharp_m(A\nabla B) - A\sharp_{m/2}B],$$

where $A,B \in M_n^{++}$, $m \in \mathbb{N}^+$, $r_0 = \min\{\nu, 1-\nu\}$, and $R_0 = \max\{\nu, 1-\nu\}$. The proving idea of these inequalities is to rely on the following.

LEMMA 3.1. (see [5, Lemma 3.3]) *Let $X \in M_n$ be any Hermitian matrix with its spectrum in some interval $J \subset \mathbb{R}$. If f and g are continuous real-valued functions defined on J such that $f(t) \geqslant g(t)$ for all $t \in J$, then $f(X) \geqslant g(X)$.*

Now, we will provide some generalizations of the above inequalities.

THEOREM 3.2. *Let $A,B \in M_n^{++}$, $\lambda \geqslant 1$ and $0 < \nu \leqslant \tau < 1$. Then, the following assertions hold.*

(1) *If $B \geqslant A$, then*

$$A\sharp_\lambda(A\nabla_\nu B) - A\sharp_{\nu\lambda}B \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[A\sharp_\lambda(A\nabla_\tau B) - A\sharp_{\tau\lambda}B],$$

and

$$A\sharp_\lambda(A\nabla_\nu B) - A\sharp_\lambda(A!_\nu B) \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[A\sharp_\lambda(A\nabla_\tau B) - A\sharp_\lambda(A!_\tau B)].$$

(2) *If* $A \geqslant B$, *then*

$$A\sharp_\lambda (A\nabla_\nu B) - A\sharp_{\nu\lambda} B \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[A\sharp_\lambda (A\nabla_\tau B) - A\sharp_{\tau\lambda} B],$$

and

$$A\sharp_\lambda (A\nabla_\nu B) - A\sharp_\lambda (A!_\nu B) \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[A\sharp_\lambda (A\nabla_\tau B) - A\sharp_\lambda (A!_\tau B)].$$

Proof. We give a detailed proof of the first inequality and the other one are proved similarly. In view of (1.7), we have that, for $x \geqslant 1$,

$$[(1-\nu)+\nu x]^\lambda - x^{\nu\lambda} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}\{[(1-\tau)+\tau x]^\lambda - x^{\tau\lambda}\}.$$

Thus, applying Lemma 3.1 for $X \in M_n^{++}$ with $X \geqslant I$, we deduce

$$[(1-\nu)I+\nu X]^\lambda - X^{\nu\lambda} \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}\{[(1-\tau)I+\tau X]^\lambda - X^{\tau\lambda}\}, \tag{3.1}$$

where $I \in M_n$ is the unit matrix. Since $B \geqslant A > 0$, we have $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geqslant I$. By taking $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (3.1) and multiplying both-sides of the obtained inequality by $A^{\frac{1}{2}}$, we get the claimed inequality. This completes the proof of the theorem. □

THEOREM 3.3. *Let* $A,B \in M_n$, $\lambda \geqslant 1$, $0 < m \leqslant \alpha \leqslant \beta \leqslant M < \infty$ *and* $K(\cdot,2)$ *be defined as in Theorem 2.4. Then, the following statements hold.*

(1) *If* $0 < \nu \leqslant \tau < \frac{1}{2}$ *and* $MI \geqslant A \geqslant \beta I > \alpha I \geqslant B \geqslant mI > 0$, *then*

$$A\sharp_\lambda (A\nabla_\nu B) - K(\alpha/\beta,2)^{\nu\lambda} A\sharp_{\nu\lambda} B \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[A\sharp_\lambda (A\nabla_\tau B) - K(m/M,2)^{\nu\lambda} A\sharp_{\tau\lambda} B].$$

(2) *If* $\frac{1}{2} < \nu \leqslant \tau < 1$ *and* $MI \geqslant B \geqslant \beta I > \alpha I \geqslant A \geqslant mI > 0$, *then*

$$\frac{\tau(1-\tau)}{\nu(1-\nu)}[A\sharp_\lambda (A\nabla_\nu B) - K(M/m,2)^{(1-\nu)\lambda} A\sharp_{\nu\lambda} B] \leqslant A\sharp_\lambda (A\nabla_\tau B) - K(\beta/\alpha,2)^{(1-\nu)\lambda} A\sharp_{\tau\lambda} B.$$

Proof. It follows from (1.9) that, for all $x \in [\frac{m}{M},\frac{\alpha}{\beta}] \subset (0,1]$, we have

$$((1-\nu)+\nu x)^\lambda - (K(x,2)^\nu x^\nu)^\lambda \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[((1-\tau)+\tau x)^\lambda - (K(x,2)^\tau x^\tau)^\lambda].$$

Combining with the fact that the function $K(\cdot,2)$ is decreasing on $(0,1]$, we infer that, for all $x\in[\frac{m}{M},\frac{\alpha}{\beta}]$,

$$\begin{aligned}((1-\nu)+\nu x)^{\lambda}&-K(\alpha/\beta,2)^{\nu\lambda}x^{\nu\lambda}\\ &\geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[((1-\tau)+\tau x)^{\lambda}-(K(x,2)^{\tau}x^{\tau})^{\lambda}]\\ &\geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[((1-\tau)+\tau x)^{\lambda}-K(m/M,2)^{\tau\lambda}x^{\tau\lambda}].\end{aligned}$$

By Lemma 3.1, we find that, for $X\in M_n^{++}$ with $X\leqslant I$,

$$((1-\nu)I+\nu X)^{\lambda}-K(\alpha/\beta,2)^{\nu\lambda}X^{\nu\lambda}\geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[((1-\tau)I+\tau X)^{\lambda}-K(m/M,2)^{\tau\lambda}X^{\tau\lambda}]. \tag{3.2}$$

Since A and B satisfy the condition (1), it is easy to deduce that the spectrum of $A^{-\frac12}BA^{-\frac12}$ is in $[\frac{m}{M},\frac{\alpha}{\beta}]$. Indeed, it follows from the condition (1) and [4, Theorem 7.7.2(a)] that

$$MA^{-1}\geqslant I\geqslant \beta A^{-1}>\alpha A^{-1}\geqslant A^{-\frac12}BA^{-\frac12}\geqslant mA^{-1},$$

where we have just used the fact that $(A^{-\frac12})^*=A^{-\frac12}$. Moreover, these inequalities imply that

$$mA^{-1}=\frac{m}{M}(MA^{-1})\geqslant\frac{m}{M}I\quad\text{and}\quad \alpha A^{-1}=\frac{\alpha}{\beta}(\beta A^{-1})\leqslant\frac{\alpha}{\beta}I.$$

Thus, we obtain $\frac{\alpha}{\beta}I\geqslant A^{-\frac12}BA^{-\frac12}\geqslant\frac{m}{M}I$. Now, a combination of the inequality (3.2) and arguments as in the proof of Theorem 3.2 yields the desired inequality.

The other inequality is proved similarly and so, we omit the details. □

3.2. Determinant inequalities

In 2018, Choi [2] showed that, for all $A,B\in M_n^{++}$, $\nu\in[0,1]$ and $m\in\mathbb{N}^+$,

$$[\det(A\nabla_\nu B)]^m\geqslant[\det(A\sharp_\nu B)]^m+(2r_0)^m\Big[\Big(\frac{\det A^{1/n}+\det B^{1/n}}{2}\Big)^{mn}-[\det(AB)]^{m/2}\Big],$$

where $r_0=\min\{\nu,1-\nu\}$. In this subsection, we establish a general refinement and reverse of this inequality as follows.

THEOREM 3.4. *Let $A,B\in M_n^{++}$, $0<\nu\leqslant\tau<1$ and $\varphi:[0,\infty)\to\mathbb{R}$ be an increasing convex function.*

(1) *If $A\geqslant B\geqslant 0$, then*

$$\begin{aligned}\varphi\circ\det(A\nabla_\nu B)&-\varphi\circ\det(A\sharp_\nu B)\\ &\geqslant\frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi([(\det A^{\frac1n})\nabla_\tau(\det B^{\frac1n})]^n)-\varphi\circ\det(A\sharp_\tau B)],\end{aligned}$$

and

$$\varphi \circ \det(A\nabla_\nu B) - \varphi(\det(A)!_\nu \det(B)) \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi([(\det A^{\frac{1}{n}})\nabla_\tau(\det B^{\frac{1}{n}})]^n) - \varphi(\det(A)!_\tau \det(B))].$$

(2) *If* $B \geqslant A \geqslant 0$, *then*

$$\varphi \circ \det(A\nabla_\tau B) - \varphi \circ \det(A\sharp_\tau B) \geqslant \frac{\tau(1-\tau)}{\nu(1-\nu)}[\varphi([(\det A^{\frac{1}{n}})\nabla_\nu(\det B^{\frac{1}{n}})]^n) - \varphi \circ \det(A\sharp_\nu B)],$$

and

$$\varphi \circ \det(A\nabla_\tau B) - \varphi(\det(A)!_\tau \det(B)) \geqslant \frac{\tau(1-\tau)}{\nu(1-\nu)}[\varphi([(\det A^{\frac{1}{n}})\nabla_\nu(\det B^{\frac{1}{n}})]^n) - \varphi(\det(A)!_\nu \det(B))].$$

To prove this theorem, we need the following inequality of Minkowski for determinants.

LEMMA 3.5. (Minkowski's inequality, [4, Theorem 7.8.21]) *Let* $A, B \in M_n$ *be positive definite. Then*

$$[\det(A+B)]^{1/n} \geqslant (\det A)^{1/n} + (\det B)^{1/n}.$$

Proof of Theorem 3.4. We give a proof of the case $A \geqslant B \geqslant 0$. By using Lemma 3.5, the fact that the function φ is increasing and Theorem 2.2(2) with $a = \det(A)^{\frac{1}{n}}$ and $b = \det(B)^{\frac{1}{n}}$, we have

$$\begin{aligned}
\varphi \circ \det(A\nabla_\nu B) &= \psi(\det(A\nabla_\nu B)^{\frac{1}{n}}) \\
&\geqslant \psi(\det((1-\nu)A)^{\frac{1}{n}} + \det(\nu B)^{\frac{1}{n}}) \\
&= \psi([\det(A)^{\frac{1}{n}}]\nabla_\nu[\det(B)^{\frac{1}{n}}]) \\
&\geqslant \psi([\det(A)\sharp_\nu \det(B)]^{1/n}) \\
&\quad + \frac{\nu(1-\nu)}{\tau(1-\tau)}[\psi([\det(A)^{\frac{1}{n}}]\nabla_\tau[\det(B)^{\frac{1}{n}}]) - \psi([\det(A)\sharp_\tau \det(B)]^{1/n})] \\
&= \varphi \circ \det(A\sharp_\nu B) \\
&\quad + \frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi([\det(A)^{\frac{1}{n}}\nabla_\tau \det(B)^{\frac{1}{n}}]^n) - \varphi \circ \det(A\sharp_\tau B)],
\end{aligned}$$

where $\psi(t) = \varphi(t^n)$ is an increasing convex function defined on $[0,\infty)$.

Utilizing the same method, we can obtain the other inequality, hence we omit the details. □

3.3. Trace and unitarily invariant norm inequalities

In 2018, Choi [2] gave trace and unitarily invariant norm inequalities for $A,B \in M_n^{++}$ and $X \in M_n$ of the forms

$$[\mathrm{tr}(A\nabla_\nu B)]^m - (\mathrm{tr}\,|A\sharp_\nu B|)^m \geqslant (2r_0)^m\{[\mathrm{tr}(A\nabla B)]^m - [\mathrm{tr}(A)\sharp\,\mathrm{tr}(B)]^m\}$$

and

$$[|||AX|||\,\nabla_\nu\,|||XB|||]^m - |||A^{1-\nu}XB^\nu|||^m \geqslant (2r_0)^m\Big\{\big(|||AX|||\,\nabla\,|||XB|||\big)^m - (|||AX|||\,\sharp\,|||XB|||)^m\Big\},$$

where $m \in \mathbb{N}^+$, $\nu \in [0,1]$ and $r_0 = \min\{\nu, 1-\nu\}$. Two weight generalizations of the above inequalities are given in [12, Theorems 3.2 and 3.3]. In order to show these inequalities, the authors used fundamental results in the following.

LEMMA 3.6. ([5, p. 381]) *Let $A,B,X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant \nu \leqslant 1$, then*

$$|||A^{1-\nu}XB^\nu||| \leqslant |||AX|||^{1-\nu}\,|||XB|||^\nu .$$

In particular, we have

$$\mathrm{tr}\,|A^{1-\nu}B^\nu| \leqslant \mathrm{tr}(A)^{1-\nu}\,\mathrm{tr}(B)^\nu.$$

Now, we provide several generalizations for these results.

THEOREM 3.7. *Let $A,B \in M_n^+$, $0 < \nu \leqslant \tau < 1$ and $\varphi : [0,\infty) \to \mathbb{R}$ be an increasing convex function. Then, the following statements hold.*

(1) *If $B \geqslant A \geqslant 0$, then*

$$\varphi \circ \mathrm{tr}(A\nabla_\nu B) - \varphi(\mathrm{tr}(A)\sharp_\nu\,\mathrm{tr}(B)) \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi \circ \mathrm{tr}(A\nabla_\tau B) - \varphi \circ \mathrm{tr}(|A\sharp_\tau B|)],$$

and

$$\varphi \circ \mathrm{tr}(A\nabla_\nu B) - \varphi(\mathrm{tr}(A)!_\nu\,\mathrm{tr}(B)) \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi \circ \mathrm{tr}(A\nabla_\tau B) - \varphi(\mathrm{tr}(A)!_\tau\,\mathrm{tr}(B))].$$

(2) *If $A \geqslant B \geqslant 0$, then*

$$\varphi \circ \mathrm{tr}(A\nabla_\nu B) - \varphi(\mathrm{tr}(A)\sharp_\nu\,\mathrm{tr}(B)) \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi \circ \mathrm{tr}(A\nabla_\tau B) - \varphi \circ \mathrm{tr}(|A\sharp_\tau B|)],$$

and

$$\varphi \circ \operatorname{tr}(A\nabla_\nu B) - \varphi(\operatorname{tr}(A)!_\nu \operatorname{tr}(B)) \geqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi \circ \operatorname{tr}(A\nabla_\tau B) - \varphi(\operatorname{tr}(A)!_\tau \operatorname{tr}(B))].$$

Proof. Let us consider the case $B \geqslant A \geqslant 0$. By using Theorem 2.2(1) with $a = \operatorname{tr}(A)$ and $b = \operatorname{tr}(B)$, we infer that

$$\begin{aligned}
&\varphi \circ \operatorname{tr}(A\nabla_\nu B) - \varphi(\operatorname{tr}(A)\sharp_\nu \operatorname{tr}(B)) \\
&\quad = \varphi(\operatorname{tr}(A)\nabla_\nu \operatorname{tr}(B)) - \varphi(\operatorname{tr}(A)\sharp_\nu \operatorname{tr}(B)) \\
&\quad \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi(\operatorname{tr}(A)\nabla_\tau \operatorname{tr}(B)) - \varphi(\operatorname{tr}(A)\sharp_\tau \operatorname{tr}(B))] \\
&\quad = \frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi \circ \operatorname{tr}(A\nabla_\tau B) - \varphi(\operatorname{tr}(A)\sharp_\tau \operatorname{tr}(B))] \\
&\quad \leqslant \frac{\nu(1-\nu)}{\tau(1-\tau)}[\varphi \circ \operatorname{tr}(A\nabla_\tau B) - \varphi \circ \operatorname{tr}(|A\sharp_\tau B|)].
\end{aligned}$$

The other inequalities are proved similarly, so we omit the details. □

THEOREM 3.8. *Let $A,B,X \in M_n$ with $A,B \in M_n^+$, $0 < \nu \leqslant \tau < 1$ and $\varphi : [0,\infty) \to \mathbb{R}$ be an increasing convex function. Then, for every unitarily invariant norm $|||\cdot|||$, the following statements hold.*

(1) *If $|||XB||| \geqslant |||AX|||$, then*

$$\varphi(|||AX|||\,\nabla_\tau\,|||XB|||) - \varphi(|||A^{1-\tau}XB^\tau|||) \geqslant \frac{\tau(1-\tau)}{\nu(1-\nu)}[\varphi(|||AX|||\,\nabla_\nu\,|||XB|||) - \varphi(|||AX|||\,\sharp_\nu\,|||XB|||)].$$

(2) *If $|||AX||| \geqslant |||XB|||$, then*

$$\varphi(|||AX|||\,\nabla_\nu\,|||XB|||) - \varphi(|||A^{1-\nu}XB^\nu|||) \geqslant \frac{\nu(1-\nu)}{\nu(1-\tau)}[\varphi(|||AX|||\,\nabla_\tau\,|||XB|||) - \varphi(|||AX|||\,\sharp_\tau\,|||XB|||)].$$

Proof. We present a proof of the first inequality, namely $|||XB||| \geqslant |||AX||| \geqslant 0$. Relying on Lemma 3.6, Theorem 2.2(1) and the fact that the function φ is increasing, we have

$$\begin{aligned}
&\varphi(\,|||AX|||\,\nabla_\tau\,|||XB|||) - \varphi(|||A^{1-\tau}XB^\tau|||) \\
&\quad \geqslant \varphi(|||AX|||\,\nabla_\tau\,|||XB|||) - \varphi(|||AX|||\,\sharp_\tau\,|||XB|||) \\
&\quad \geqslant \frac{\tau(1-\tau)}{\nu(1-\nu)}[\varphi(|||AX|||\,\nabla_\nu\,|||XB|||) - \varphi(|||AX|||\,\sharp_\nu\,|||XB|||].
\end{aligned}$$

The other inequality is proved similarly and we omit the details. □

Acknowledgement. The authors wish to express their gratitude to the anonymous referee for insightful suggestions and comments which helped to improve the quality of this paper. Duong Quoc Huy is funded by the Research Project of Vietnam Ministry of Education and Training under grant number B2022-DQN-02.

REFERENCES

[1] H. ALZER, C. M. FONSECA AND A. KOVAČEC, *Young-type inequalities and their matrix analogues*, Linear Multilinear Alebra **63**, 3 (2015), 622–635.

[2] D. CHOI, *A generalization of Young-type inequalities*, Math. Inequal. Appl. **21**, 1 (2018), 99–106.

[3] S. S. DRAGOMIR, *Bounds for the normalised Jensen functional*, Bull. Austral. Math. Soc. **74**, 3 (2006), 471–478.

[4] R. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge UP, 2013.

[5] D. Q. HUY, D. T. T. VAN AND D. T. XINH, *Some generalizations of real power form for Young-type inequalities and their applications*, Linear Algebra App. **656**, 1 (2023), 368–384.

[6] F. KITTANEH AND Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361**, 1 (2010), 262–269.

[7] F. KITTANEH AND Y. MANASRAH, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra **59**, 9 (2011), 1031–1037.

[8] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and new inequalities in analysis*, Math. Appl. (East European Ser.), **61** Kluwer Academic Publishers Group, Dordrecht, 1993.

[9] Y. REN, *Some results of Young-type inequalities*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114**, 143 (2020), 10.

[10] Y. REN, *A generalized refinement of Young's inequality*, J. Math. Inequal. **17**, 4 (2023), 1463–1470.

[11] C. YANG AND Z. WANG, *Some new improvements of Young's inequalities*, J. Math. Inequal. **17**, 1 (2023), 315–328.

[12] X. YANG, C. YANG AND H. LI, *Further improvements for Young's inequalities on the Arithmetic, Geometric, and Harmonic mean*, J. Math. Inequal. **18**, 1 (2024), 315–328.

(Received May 27, 2024)

Tran Dinh Phung
Department of Mathematics and Statistics
University of Finance – Marketing
778 Nguyen Kiem, Ho Chi Minh City, Vietnam
e-mail: td.phung@ufm.edu.vn

Duong Quoc Huy
Department of Mathematics
Tay Nguyen University
567 Le Duan, Buon Ma Thuot, Dak Lak, Vietnam
e-mail: duongquochuy@ttn.edu.vn