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EXTENSIONS OF CLASSICAL ANKENY–RIVLIN INEQUALITY TO THE s^{th} DERIVATIVE

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Abstract. In this paper, we present for a polynomial $p(z)$ of degree n, the s^{th} derivative $(0 \leqslant s < n)$ concept on a result due to Govil et al. [*Illinois J. Math.*, **23** (1979), 319–329]. As an application of this result, we obtain improved generalizations of the well-known theorem due to Ankeny and Rivlin which states that if $p(z)$ is a polynomial of degree n such that $p(z)$ has no zero in $|z| < 1$, then

$$\max_{|z|=R\geqslant 1} |p(z)| \leqslant \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |p(z)|.$$

Moreover, these achievements lead to enhancements of a previous result attributed to Jain [*Turk. J. Math.*, **31** (2007), 89–94] which we have compared by considering a concrete numerical example and analyzed graphically to illustrate their sharpness.

1. Introduction

Polynomial approximation is an essential concept in mathematics and applied sciences. It offers a robust method for simplifying complex functions by representing them with polynomial expressions. This technique involves constructing polynomial functions that closely mimic the behavior of more intricate functions, thereby facilitating easier analysis, computation, and problem-solving in various fields.

Various methods have been developed to tackle this challenge, each designed for specific contexts and requirements. One notable approach involves the use of Bernstein's inequality, particularly its trigonometric form, which holds significant importance in the literature for proving an inverse theorem in approximation theory (refer to Borwein and Erdélyi [6, p. 241]). The theorem estimates how accurately a polynomial of a given degree approximates a continuous function, based on its derivatives and Lipschitz constants. This effort resulted in the widely recognized Bernstein's inequality [4], which states that if t is a real trigonometric polynomial of degree n, then

$$\max_{0\leqslant\theta<2\pi} |t^{(m)}(\theta)| \leqslant n^m \max_{0\leqslant\theta<2\pi} |t(\theta)|.$$

The above inequality remains true for all complex trigonometric polynomials t of degree n, which implies, as a particular case, the following algebraic polynomial version of Bernstein's inequality on the unit disk.

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THEOREM 1. *If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1}|p'(z)| \leqslant n\max_{|z|=1}|p(z)|. \tag{1}$$

Equality holds in (1) if and only if $p(z)$ has all its zeros at the origin.

Inequality (1) can be sharpened if the zeros of $p(z)$ are restricted. In this direction, Erdös conjectured and later Lax [20] proved that if $p(z)$ has no zero in $|z|<1$, then

$$\max_{|z|=1}|p'(z)| \leqslant \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{2}$$

Inequality (2) is best possible for $p(z)=\alpha+\beta z^n$, $|\alpha|=|\beta|$.

As a generalization of (2), Malik [21] proved that if $p(z)$ has no zero in $|z|<k$, $k\geqslant 1$, then

$$\max_{|z|=1}|p'(z)| \leqslant \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{3}$$

Inequality (3) is sharp and extremal polynomial is $p(z)=(z+k)^n$.

Govil et al. [13] obtained a refinement of (3) and proved that if $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\geqslant 1$, then

$$\max_{|z|=1}|p'(z)| \leqslant \frac{n}{1+k}\left\{\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\right\}\max_{|z|=1}|p(z)|, \tag{4}$$

where

$$\lambda=\frac{k}{n}\frac{a_1}{a_0} \quad\text{and}\quad \mu=\frac{2k^2}{n(n-1)}\frac{a_2}{a_0}.$$

Inequality (3) was further generalized by Govil and Rahman [12] to the s^{th} derivative of $p(z)$ by proving that if $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\geqslant 1$, then for $1\leqslant s<n$,

$$\max_{|z|=1}|p^{(s)}(z)| \leqslant \frac{n(n-1)\dots(n-s+1)}{1+k^s}\max_{|z|=1}|p(z)|. \tag{5}$$

The result is sharp and equality retains for the polynomial $p(z)=(z+k)^n$ with $s=1$. The notion that inequality (5) holds true for $0\leqslant s<n$ was proposed by Jain in [14, Lemma 5].

The improvement and generalization of inequalities concerning complex polynomials have been extensively explored. For further insights into this area, we direct readers to a selection of recent publications, including [25], [37], [24], [23], [40], [34], [19], [33], [36], [9], [35], [41], and others. For a deeper understanding of these kinds of inequalities and their applications, we refer to the monographs of Milovanović et al. [26], Marden [22], Rahman and Schmeisser [32], and the most recent one of Gardner et al. [10, Chap. 6], where some approaches to obtaining polynomial inequalities are developed by applying the methods and results of the geometric function theory.

Inequality (1) shows how fast a polynomial of degree at most n can change, and is of interest both in mathematics, especially in approximation theory, and in the application areas such as physical systems. Various analogs of this inequality are known in which the underlying intervals, the sup-norms, and the families of polynomials are replaced by more general sets, norms, and families of functions, respectively. One such generalization is the relative growth of the polynomial $p(z)$ concerning two circles $|z| = 1$ and $|z| = R \geqslant 1$, and obtain inequalities about the dependence of sup-norms of $|p(z)|$ on $|p(Rz)|$, where $|z| = 1$.

For a polynomial $p(z)$ of degree n, in accordance with the maximum modulus principle, the ensuing result [31] holds for $R \geqslant 1$ as

$$\max_{|z|=R}|p(z)| \leqslant R^n \max_{|z|=1}|p(z)|, \tag{6}$$

with equality only for $p(z) = \alpha z^n$.

Ankeny and Rivlin [1] examined a polynomial $p(z)$ of degree n without zero in $|z| < 1$ and obtained a refinement of (6) by establishing that for $R \geqslant 1$,

$$\max_{|z|=R}|p(z)| \leqslant \frac{R^n+1}{2} \max_{|z|=1}|p(z)|. \tag{7}$$

The result is best possible with equality only for polynomials $p(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$.

As a refinement of inequality (7) by involving the leading coefficient and the constant term of the polynomial, Kumar [18] proved that if $p(z)$ is a polynomial of degree $n \geqslant 1$ with no zero in $|z| < 1$, then for $R \geqslant 1$,

$$\max_{|z|=R}|p(z)| \leqslant \frac{(R^n+1)(|a_0|+R|a_n|)}{(R+1)(|a_0|+|a_n|)} \max_{|z|=1}|p(z)|. \tag{8}$$

Equality holds in (8) for $p(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$.

Exploring the generalization of (7) to the derivative of the polynomial is intriguing. In this, Jain [14] derived the following extension of inequality (7) for the s^{th} derivative.

THEOREM 2. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geqslant 1$, then for $0 \leqslant s < n$,*

$$\max_{|z|=R}|p^{(s)}(z)| \leqslant \frac{1}{2}\left\{\frac{d^s}{dR^s}(R^n+k^n)\right\}\left(\frac{2}{1+k}\right)^n \max_{|z|=1}|p(z)|, \quad R \geqslant k \tag{9}$$

and

$$\max_{|z|=R}|p^{(s)}(z)| \leqslant \frac{1}{R^s+k^s}\left[\left\{\frac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right]\left(\frac{R+k}{1+k}\right)^n \max_{|z|=1}|p(z)|, \quad 1 \leqslant R \leqslant k. \tag{10}$$

Equality holds in (9) with $k = 1$ and $s = 0$ for $p(z) = z^n + 1$, and equality holds in (10) with $s = 1$ for $p(z) = (z+k)^n$.

REMARK 1. Some interesting implications of the inequalities for the higher-order derivatives such as (9) and (10) of Theorem 2 can be observed as discussed below:

1. When $s=0$ and $k=1$, inequality (9) reduces to inequality (7) due to Ankeny and Rivlin [1].

2. For $s=1$, and $R=1$ (which forces $k=1$) in inequality (9), we are led to the well-known inequality (2) of Erdös-Lax [20].

3. For $s=1$ and $R=1$, inequality (10) provides inequality (3) of Malik [21].

4. Again, if we put $R=1$, inequality (10) gives inequality (5) due to Govil and Rahman [12].

5. Moreover, both inequalities (9) and (10) give a generalization of inequality (5).

Various authors have contributed numerous extensions, improvements, and generalizations concerning the maximum modulus of polynomials, being a widely studied topic, and for more information in this direction, we refer to the published papers [5], [27], [7], [17], [38], [28], [29], [15], [16], [39], [8] etc. However, there have been far fewer advancements for higher-order derivatives. Motivated by this gap, we establish new and improved bounds of inequalities (9) and (10) of Theorem 2 that incorporate specific coefficients of the polynomial in question.

The paper is organized as follows. In Section 2, we present the main results along with a remark and corollaries. In Section 3, we bring and construct some auxiliary results necessary to prove the main results. The proofs of our main results are given in Section 4. Then in Section 5, a numerical example is presented in order to graphically illustrate and compare the obtained inequalities with the ones previously known. Finally, Section 6 contains the conclusion.

2. Main results

In this paper, we introduce refinements of inequalities (9) and (10). Firstly, we prove

THEOREM 3. *If $p(z)$ is a polynomial of degree $n\geqslant 1$ having no zero in $|z|<k$, $k\geqslant 1$, then for $0\leqslant s<n$,*

$$\max_{|z|=R}|p^{(s)}(z)|\leqslant\left\{\frac{|a_0|k+|a_n|Rk^n}{(R+k)(|a_0|+|a_n|k^n)}\right\}\left\{\frac{d^s}{dR^s}(R^n+k^n)\right\}\left(\frac{2}{1+k}\right)^n \times\max_{|z|=1}|p(z)|,\quad R\geqslant k \tag{11}$$

and

$$\max_{|z|=R}|p^{(s)}(z)| \leqslant \left\{\frac{(1-|\lambda_s|)(R^2+k^2|\lambda_s|)+Rk(n-s)|\mu_s-\Lambda_s|}{(1-|\lambda_s|)(R^{s+2}+k^{s+2}+R^2k^s|\lambda_s|+R^sk^2|\lambda_s|)+Rk(R^s+k^s)(n-s)|\mu_s-\Lambda_s|}\right\} \times\left[\left\{\frac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right]\left(\frac{R+k}{1+k}\right)^n\max_{|z|=1}|p(z)|, \quad 1\leqslant R\leqslant k, \tag{12}$$

where

$$\lambda_s=\frac{k^s}{C(n,s)}\frac{a_s}{a_0}, \quad \mu_s=\frac{k^{s+1}}{C(n,s+1)}\frac{a_{s+1}}{a_0} \ \ and \ \ \Lambda_s=\frac{k^{s+1}}{C(n,s)n}\frac{a_sa_1}{a_0^2}.$$

Equality holds in (11) *with* $k=1$ *and* $s=0$ *for* $p(z)=z^n+1$*, and equality holds in* (12) *with* $R=1$ *and* $s=n-2$ *for* $p(z)=\alpha k^n+\beta z^n$, $|\alpha|=|\beta|$.

REMARK 2. Since the polynomial $p(z)=\sum_{j=0}^{n}a_jz^j$ has no zero in $|z|<k$, $k\geqslant 1$, we have

$$\left|\frac{a_0}{a_n}\right|\geqslant k^n,$$

which implies

$$\frac{(|a_0|k+|a_n|Rk^n)}{(R+k)(|a_0|+|a_n|k^n)}\leqslant\frac{1}{2} \quad \text{for } R\geqslant k.$$

In view of this fact, it follows that inequality (11) sharpens inequality (9).

Also, by Lemma 8 and for $1\leqslant R\leqslant k$, we have

$$\frac{(1-|\lambda_s|)(R^2+k^2|\lambda_s|)+Rk(n-s)|\mu_s-\Lambda_s|}{(1-|\lambda_s|)(R^{s+2}+k^{s+2}+R^2k^s|\lambda_s|+R^sk^2|\lambda_s|)+Rk(R^s+k^s)(n-s)|\mu_s-\Lambda_s|}\leqslant\frac{1}{R^s+k^s},$$

which infers that inequality (12) improves over inequality (10).

Putting $s=0$ in Theorem 3, we get the following generalization of inequality (8) due to Kumar [18].

COROLLARY 1. *If* $p(z)$ *is a polynomial of degree* $n\geqslant 1$ *having no zero in* $|z|<k$, $k\geqslant 1$*, then*

$$\max_{|z|=R}|p(z)|\leqslant\frac{(R^n+k^n)(|a_0|k+|a_n|Rk^n)}{(R+k)(|a_0|+|a_n|k^n)}\left(\frac{2}{1+k}\right)^n\max_{|z|=1}|p(z)|, \quad R\geqslant k$$

and

$$\max_{|z|=R}|p(z)|\leqslant\left(\frac{R+k}{1+k}\right)^n\max_{|z|=1}|p(z)|, \quad 1\leqslant R\leqslant k.$$

Putting $k=1$ in Theorem 3, we get the following result which is the s^{th} derivative generalization of inequality (8) due to Kumar [18].

COROLLARY 2. *If $p(z)$ is a polynomial of degree $n\geqslant 1$ having no zero in $|z|<1$, then for $R\geqslant 1$ and $0\leqslant s<n$,*

$$\max_{|z|=R}|p^{(s)}(z)|\leqslant \frac{(|a_0|+|a_n|R)}{(R+1)(|a_0|+|a_n|)}\left\{\frac{d^s}{dR^s}(R^n+1)\right\}\max_{|z|=1}|p(z)|.$$

Moreover, we enhance the results of Theorem 3 by incorporating the minimum modulus of the polynomial on $|z|=k$. More precisely, we prove

THEOREM 4. *If $p(z)$ is a polynomial of degree $n\geqslant 1$ having no zero in $|z|<k$, $k\geqslant 1$, then for some fixed complex number γ with $|\gamma|<1$, and $0\leqslant s<n$,*

$$\max_{|z|=R}|p^{(s)}(z)|\leqslant \left\{\frac{|a_0-\gamma m|k+|a_n|Rk^n}{(R+k)(|a_0-\gamma m|+|a_n|k^n)}\right\}\left\{\frac{d^s}{dR^s}(R^n+k^n)\right\}\times\left(\frac{2}{1+k}\right)^n\left\{\max_{|z|=1}|p(z)|-|\gamma|m\right\},\quad R\geqslant k \tag{13}$$

and

$$\max_{|z|=R}|p^{(s)}(z)| \leqslant \left\{\frac{(1-|\lambda_s'|)(R^2+k^2|\lambda_s'|)+Rk(n-s)|\mu_s'-\Lambda_s'|}{(1-|\lambda_s'|)(R^{s+2}+k^{s+2}+R^2k^s|\lambda_s'|+R^sk^2|\lambda_s'|)+Rk(R^s+k^s)(n-s)|\mu_s'-\Lambda_s'|}\right\} \times\left[\left\{\frac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right]\left(\frac{R+k}{1+k}\right)^n\left\{\max_{|z|=1}|p(z)|-|\gamma|m\right\},\quad 1\leqslant R\leqslant k, \tag{14}$$

where

$$\lambda_s'=\frac{k^s}{C(n,s)}\frac{a_s}{(a_0-\gamma m)},\quad \mu_s'=\frac{k^{s+1}}{C(n,s+1)}\frac{a_{s+1}}{(a_0-\gamma m)},\quad \Lambda_s'=\frac{k^{s+1}}{C(n,s)n}\frac{a_sa_1}{(a_0-\gamma m)^2}$$

and $m=\min_{|z|=k}|p(z)|$.

Equality holds in (13) with $k=1$ and $s=0$ for $p(z)=z^n+1$, and equality holds in (14) with $R=1$ and $s=n-2$ for $p(z)=\alpha k^n+\beta z^n$, $|\alpha|=|\beta|$.

Putting $s=0$ in Theorem 4, we get the following result that gives improvement of Corollary 1.

COROLLARY 3. *If $p(z)$ is a polynomial of degree $n\geqslant 1$ having no zero in $|z|<k$, $k\geqslant 1$, then for some fixed complex number γ with $|\gamma|<1$,*

$$\max_{|z|=R}|p(z)|\leqslant \frac{(R^n+k^n)(|a_0-\gamma m|k+|a_n|Rk^n)}{(R+k)(|a_0-\gamma m|+|a_n|k^n)}\left(\frac{2}{1+k}\right)^n\times\left\{\max_{|z|=1}|p(z)|-|\gamma|m\right\},\quad R\geqslant k$$

and

$$\max_{|z|=R}|p(z)| \leqslant 2\left\{\frac{(1-|\lambda_0'|)(R^2+k^2|\lambda_0'|)+Rkn|\mu_0'-\Lambda_0'|}{(1-|\lambda_0'|)(R^2+k^2+R^2|\lambda_0'|+k^2|\lambda_0'|)+2Rkn|\mu_0'-\Lambda_0'|}\right\} \times\left(\frac{R+k}{1+k}\right)^n\left\{\max_{|z|=1}|p(z)|-|\gamma|m\right\}, \quad 1\leqslant R\leqslant k,$$

where

$$\lambda_0'=\frac{a_0}{(a_0-\gamma m)}, \quad \mu_0'=\frac{k}{n}\frac{a_1}{(a_0-\gamma m)}, \quad \Lambda_0'=\frac{k}{n}\frac{a_0a_1}{(a_0-\gamma m)^2}$$

and $m=\min_{|z|=k}|p(z)|$.

Putting $k=1$ in Theorem 4, we get an improved s^{th} derivative generalization of inequality (8) due to Kumar [18].

COROLLARY 4. *If $p(z)$ is a polynomial of degree $n\geqslant 1$ having no zero in $|z|<1$, then for $R\geqslant 1$, for some fixed complex number γ with $|\gamma|<1$, and $0\leqslant s<n$,*

$$\max_{|z|=R}|p^{(s)}(z)| \leqslant \frac{(|a_0-\gamma m_1|+|a_n|R)}{(R+1)(|a_0-\gamma m_1|+|a_n|)}\left\{\frac{d^s}{dR^s}(R^n+1)\right\}\left\{\max_{|z|=1}|p(z)|-|\gamma|m_1\right\},$$

where $m_1=\min_{|z|=1}|p(z)|$.

Taking $s=0$ in Corollary 4, we get the following result which is an improvement in inequality (8) due to Kumar [18].

COROLLARY 5. *If $p(z)$ is a polynomial of degree $n\geqslant 1$ having no zero in $|z|<1$, then for $R\geqslant 1$, and for some fixed complex number γ with $|\gamma|<1$,*

$$\max_{|z|=R}|p(z)| \leqslant \frac{(R^n+1)(|a_0-\gamma m_1|+|a_n|R)}{(R+1)(|a_0-\gamma m_1|+|a_n|)}\left\{\max_{|z|=1}|p(z)|-|\gamma|m_1\right\},$$

where $m_1=\min_{|z|=1}|p(z)|$.

REMARK 3. It is interesting to observe that implementing similar parameter values as discussed in Remark 1 on the inequalities of Theorem 3, we have improved bounds over the inequalities of Theorem 2 due to Jain [14], except for the case when $s=1$ and $R=1$ in inequality (9), whereas more improved bounds are given by Theorem 4 in every case.

3. Lemmas

For the proofs of the theorems, we require the following lemmas. The first lemma is a generalization of the well-known Schwarz Lemma and is due to Osserman [30].

LEMMA 1. *Let* $f(z)$ *be analytic in* $|z|<1$ *such that* $|f(z)|<1$ *for* $|z|<1$, *and* $f(0)=0$. *Then*

$$|f(z)|\leqslant |z|\frac{|z|+|f'(0)|}{1+|f'(0)||z|} \quad \text{for } |z|<1.$$

LEMMA 2. *If* $p(z)$ *is a polynomial of degree* $n\geqslant 1$ *having no zero in* $|z|<1$ *and* $q(z)=z^n\overline{p(\frac{1}{\bar{z}})}$, *then for* $R\geqslant 1$, $0\leqslant s<n$, *and for all* z *on* $|z|=1$,

$$|p^{(s)}(Rz)|\leqslant \frac{|a_0|+R|a_n|}{|a_0|R+|a_n|}|q^{(s)}(Rz)|.$$

Proof. Since $p(z)$ has no zero in $|z|<1$, its conjugate reciprocal polynomial $q(z)$ has all its zeros in $|z|\leqslant 1$. Then $\frac{zq(z)}{p(z)}$ satisfies the hypothesis of Lemma 1, and hence we have for $|z|<1$,

$$|zq(z)|\leqslant |z|\frac{|z||a_0|+|a_n|}{|a_0|+|a_n||z|}|p(z)|,$$

which gives

$$|q(z)|\leqslant \frac{|z||a_0|+|a_n|}{|a_0|+|a_n||z|}|p(z)|. \tag{15}$$

Replacing z by $1/z$ in inequality (15), we have for $|z|>1$,

$$|p(z)|\leqslant \frac{|a_0|+|z||a_n|}{|a_0||z|+|a_n|}|q(z)|. \tag{16}$$

Note that inequality (16) is true for all z on $|z|=1$ also, and therefore for $R\geqslant 1$ and $0\leqslant\theta\leqslant 2\pi$,

$$\left|p(Re^{i\theta})\right|\leqslant \frac{|a_0|+R|a_n|}{|a_0|R+|a_n|}\left|q(Re^{i\theta})\right|.$$

Thus for all z on $|z|=1$, we have

$$|p(Rz)|\leqslant \frac{|a_0|+R|a_n|}{|a_0|R+|a_n|}|q(Rz)|. \tag{17}$$

Now, $q(Rz)$ has all its zeros in $|z|\leqslant\frac{1}{R}$, $\frac{1}{R}\leqslant 1$, and we consider the polynomial $p(Rz)-\gamma\frac{|a_0|+R|a_n|}{|a_0|R+|a_n|}q(Rz)$, where γ is a complex number with $|\gamma|>1$.

Then, on $|z|=1$,

$$|p(Rz)|\leqslant \frac{|a_0|+R|a_n|}{|a_0|R+|a_n|}|q(Rz)|<\left|\gamma\frac{|a_0|+R|a_n|}{|a_0|R+|a_n|}q(Rz)\right|.$$

By Rouche's theorem, it follows that

$$p(Rz)-\gamma\frac{|a_0|+R|a_n|}{|a_0|R+|a_n|}q(Rz)$$

has all its zeros in $|z|\leqslant\frac{1}{R}$, $\frac{1}{R}\leqslant 1$.

Gauss-Lucas theorem will then imply that the polynomial

$$R^s p^{(s)}(Rz)-\gamma\frac{|a_0|+R|a_n|}{|a_0|R+|a_n|}R^s q^{(s)}(Rz),\ \ 1\leqslant s<n,$$

has all its zeros in $|z|\leqslant\frac{1}{R}$, $\frac{1}{R}\leqslant 1$.

Therefore for all z on $|z|=1$,

$$|p^{(s)}(Rz)|\leqslant\frac{|a_0|+R|a_n|}{|a_0|R+|a_n|}|q^{(s)}(Rz)|,\ \ 1\leqslant s<n,$$

which, on being combined with (17), completes the proof of Lemma 2. □

The next lemma is due to Jain [14].

LEMMA 3. *If $p(z)$ is a polynomial of degree n and $q(z)=z^n\overline{p(\frac{1}{\bar z})}$, then for $0\leqslant s<n$,*

$$|p^{(s)}(z)|+|q^{(s)}(z)|\leqslant\left\{\left|\frac{d^s}{dz^s}(1)\right|+\left|\frac{d^s}{dz^s}(z^n)\right|\right\}\max_{|z|=1}|p(z)|\quad\text{for}\ \ |z|\geqslant 1.$$

LEMMA 4. *If $p(z)$ is a polynomial of degree n having all its zeros on $|z|=1$, then*

$$q(z)=up(z),$$

where $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$ and $|u|=1$.

The proof of this lemma is straightforward and here we omit the details.

The next lemma is another generalization of the well-known Schwarz Lemma due to Govil et al. [13].

LEMMA 5. *If $f(z)$ is analytic and $|f(z)|\leqslant 1$ in $|z|<1$, then*

$$|f(z)|\leqslant\frac{(1-|a|)|z|^2+|bz|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|bz|+(1-|a|)}\quad\text{for}\ \ |z|<1,$$

where $a=f(0)$ and $b=f'(0)$. The example

$$f(z)=\frac{a+\frac{b}{1+a}z-z^2}{1-\frac{b}{1+a}z-az^2}$$

shows that the estimate is sharp.

LEMMA 6. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\geqslant 1$, then for $1\leqslant s<n$,*

$$\max_{|z|=1}|p^{(s)}(z)|\leqslant n(n-1)\dots(n-s+1)$$
$$\times\left\{\frac{(1-|\lambda_s|)(1+k^2|\lambda_s|)+k(n-s)|\mu_s-\Lambda_s|}{(1-|\lambda_s|)(1+k^{s+2}+k^s|\lambda_s|+k^2|\lambda_s|)+k(1+k^s)(n-s)|\mu_s-\Lambda_s|}\right\}\max_{|z|=1}|p(z)|,$$

where

$$\lambda_s=\frac{k^s}{C(n,s)}\frac{a_s}{a_0},\ \ \mu_s=\frac{k^{s+1}}{C(n,s+1)}\frac{a_{s+1}}{a_0}\ \text{ and }\ \Lambda_s=\frac{k^{s+1}}{C(n,s)n}\frac{a_sa_1}{a_0^2}.$$

Proof. Since $p(z)$ has no zero in $|z|<k$, $k\geqslant 1$, the polynomial $M(z)=p(kz)$ has no zero in $|z|<1$. Let $N(z)=z^n\overline{M\left(\frac{1}{\bar z}\right)}$, then $N(z)$ has all its zeros in $|z|\leqslant 1$, and

$$|M(z)|=|N(z)|\quad\text{for}\quad |z|=1. \tag{18}$$

By Rouche's theorem, it follows that for every complex number β with $|\beta|>1$, the polynomial $L(z)=M(z)+\beta N(z)$ has all its zeros in $|z|\leqslant 1$. If all the zeros of $N(z)$ lie on $|z|=1$, then by Lemma 4, we have $M(z)=uN(z)$, where $|u|=1$, i.e., $L(z)=(u+\beta)N(z)$ has all its zeros on $|z|=1$, and hence in $|z|\leqslant 1$. Now, assume that $N(z)$ has m number of zeros in $|z|<1$, where $1\leqslant m\leqslant n$, and the remaining $(n-m)$ zeros on $|z|=1$. If $z_1,z_2,\dots,z_{n-m}$ are the zeros of $N(z)$ which lie on $|z|=1$, then we can write

$$N(z)=N_1(z)(z-z_1)(z-z_2)\dots(z-z_{n-m})=N_1(z)N_2(z)\quad\text{(say)}, \tag{19}$$

where all the zeros of $N_1(z)$ of degree $m\geqslant 1$ lie in $|z|<1$, and all the zeros of $N_2(z)$ of degree $(n-m)$ lie on $|z|=1$, so that

$$M(z)=z^n\overline{N\left(\frac{1}{\bar z}\right)}=z^m\overline{N_1\left(\frac{1}{\bar z}\right)}z^{n-m}\overline{N_2\left(\frac{1}{\bar z}\right)}=M_1(z)M_2(z)\quad\text{(say)}, \tag{20}$$

where $M_1(z)$ has no zero in $|z|\leqslant 1$, and all the zeros of $M_2(z)$ lie on $|z|=1$. Applying Lemma 4 to $M_2(z)$, from (20) we get

$$M(z)=uM_1(z)N_2(z),\quad\text{where}\ |u|=1. \tag{21}$$

Using (19) and (21) in (18), we get

$$|M_1(z)|=|N_1(z)|\quad\text{for}\quad |z|=1.$$

Since $M_1(z)$ has no zero in $|z|\leqslant 1$, the function $\frac{N_1(z)}{M_1(z)}$ is analytic in $|z|\leqslant 1$, and

$$\frac{|N_1(z)|}{|M_1(z)|}=1\quad\text{for}\quad |z|=1.$$

Also, since $\frac{N_1(z)}{M_1(z)}$ is not a constant, it follows by the maximum modulus principle that

$$|N_1(z)| < |M_1(z)| \quad \text{for} \quad |z| < 1.$$

Replacing z by $\frac{1}{z}$, we get

$$|M_1(z)| < |N_1(z)| \quad \text{for} \quad |z| > 1.$$

By Rouche's theorem, it follows that for every complex number β, u with $|\beta| > 1$ and $|u| = 1$, the polynomial $uM_1(z) + \beta N_1(z)$ of degree m has all its zeros in $|z| < 1$. Therefore, the polynomial

$$L(z) = uM_1(z)N_2(z) + \beta N_1(z)N_2(z) = (uM_1(z) + \beta N_1(z))\,N_2(z)$$

has all its zeros in $|z| \leqslant 1$ for every β with $|\beta| > 1$. Gauss-Lucas theorem will then imply that

$$L^{(s)}(z) = M^{(s)}(z) + \beta N^{(s)}(z), \;\; 1 \leqslant s < n,$$

has all its zeros in $|z| \leqslant 1$ for every β with $|\beta| > 1$, and therefore

$$|M^{(s)}(z)| \leqslant |N^{(s)}(z)| \quad \text{for} \;\; |z| \geqslant 1 \;\; \text{and} \;\; 1 \leqslant s < n. \tag{22}$$

Also, by the Gauss-Lucas theorem, all the zeros of $N^{(s)}(z)$ lie in $|z| \leqslant 1$. Let $H(z) = z^{n-s}\overline{N^{(s)}\left(\frac{1}{\bar{z}}\right)}$, then $H(z)$ has no zero in $|z| < 1$. Moreover, $M^{(s)}(z) = k^s p^{(s)}(kz)$, from (22) we get

$$k^s|p^{(s)}(kz)| \leqslant |N^{(s)}(z)| = \left| z^{n-s}\overline{N^{(s)}\left(\frac{1}{\bar{z}}\right)} \right| = |H(z)| \quad \text{for} \;\; |z| = 1.$$

Thus the function

$$R(z) = \frac{k^s p^{(s)}(kz)}{H(z)}$$

is analytic in $|z| < 1$, and $|R(z)| \leqslant 1$ for $|z| = 1$. Hence, it follows by the maximum modulus principle that

$$|R(z)| \leqslant 1 \quad \text{for} \;\; |z| \leqslant 1.$$

Also,

$$R(0) = \frac{k^s}{C(n,s)} \frac{a_s}{a_0} = \lambda_s$$

and

$$R'(0) = (n-s)\left\{ \frac{k^{s+1}}{C(n,s+1)} \frac{a_{s+1}}{a_0} - \frac{k^{s+1}}{C(n,s)n} \frac{a_s a_1}{a_0^2} \right\} = (n-s)(\mu_s - \Lambda_s).$$

Applying Lemma 5 to $R(z)$, we have

$$|R(z)| \leqslant \frac{(1-|\lambda_s|)|z|^2 + (n-s)|\mu_s - \Lambda_s||z| + |\lambda_s|(1-|\lambda_s|)}{|\lambda_s|(1-|\lambda_s|)|z|^2 + (n-s)|\mu_s - \Lambda_s||z| + (1-|\lambda_s|)} \quad \text{for} \;\; |z| \leqslant 1.$$

Equivalently,

$$\left|\frac{k^s p^{(s)}(kz)}{H(z)}\right| \leqslant \frac{(1-|\lambda_s|)|z|^2+(n-s)|\mu_s-\Lambda_s||z|+|\lambda_s|(1-|\lambda_s|)}{|\lambda_s|(1-|\lambda_s|)|z|^2+(n-s)|\mu_s-\Lambda_s||z|+(1-|\lambda_s|)} \quad \text{for } |z|\leqslant 1.$$

Taking $z=\frac{e^{i\theta}}{k}$, $0\leqslant\theta<2\pi$, so that $|z|=\frac{1}{k}\leqslant 1$, we get

$$\left|\frac{k^s p^{(s)}(e^{i\theta})}{H(\frac{e^{i\theta}}{k})}\right| \leqslant \frac{(1-|\lambda_s|)+(n-s)|\mu_s-\Lambda_s|k+|\lambda_s|(1-|\lambda_s|)k^2}{|\lambda_s|(1-|\lambda_s|)+(n-s)|\mu_s-\Lambda_s|k+(1-|\lambda_s|)k^2}.$$

Since $H(\frac{z}{k})=z^{n-s}\overline{q^s\left(\frac{1}{\bar z}\right)}$, it follows that for all z on $|z|=1$,

$$k^s\frac{|\lambda_s|(1-|\lambda_s|)+(n-s)|\mu_s-\Lambda_s|k+(1-|\lambda_s|)k^2}{(1-|\lambda_s|)+(n-s)|\mu_s-\Lambda_s|k+|\lambda_s|(1-|\lambda_s|)k^2}|p^{(s)}(z)|\leqslant|q^{(s)}(z)|. \tag{23}$$

Now by Lemma 3, we have for $1\leqslant s<n$, and for all z on $|z|=1$,

$$|p^{(s)}(z)|+|q^{(s)}(z)|\leqslant n(n-1)\dots(n-s+1)\max_{|z|=1}|p(z)|. \tag{24}$$

Using (23) in (24), we get

$$\left\{1+k^s\frac{|\lambda_s|(1-|\lambda_s|)+(n-s)|\mu_s-\Lambda_s|k+(1-|\lambda_s|)k^2}{(1-|\lambda_s|)+(n-s)|\mu_s-\Lambda_s|k+|\lambda_s|(1-|\lambda_s|)k^2}\right\}|p^{(s)}(z)| \leqslant n(n-1)\dots(n-s+1)\max_{|z|=1}|p(z)|,$$

i.e.,

$$|p^{(s)}(z)|\leqslant n(n-1)\dots(n-s+1) \times\left\{\frac{(1-|\lambda_s|)(1+k^2|\lambda_s|)+k(n-s)|\mu_s-\Lambda_s|}{(1-|\lambda_s|)(1+k^{s+2}+k^s|\lambda_s|+k^2|\lambda_s|)+k(1+k^s)(n-s)|\mu_s-\Lambda_s|}\right\}\max_{|z|=1}|p(z)|.$$

This completes the proof of Lemma 6. □

From Lemma 6, we easily get the following result which is the s^{th} derivative generalization of inequality (4) due to Govil et al. [13].

LEMMA 7. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\geqslant 1$, then for $0\leqslant s<n$,*

$$\max_{|z|=1}|p^{(s)}(z)|\leqslant\left\{\frac{(1-|\lambda_s|)(1+k^2|\lambda_s|)+k(n-s)|\mu_s-\Lambda_s|}{(1-|\lambda_s|)(1+k^{s+2}+k^s|\lambda_s|+k^2|\lambda_s|)+k(1+k^s)(n-s)|\mu_s-\Lambda_s|}\right\} \times\max_{|z|=1}|p(z)|\left[\left\{\frac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right].$$

where λ_s, μ_s and Λ_s are as defined in Lemma 6.

The below result is due to Aziz and Rather [3].

LEMMA 8. *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z|<k,\ k\geqslant 1$, *then for* $0\leqslant s<n$,

$$\frac{1}{C(n,s)}\left|\frac{a_s}{a_0}\right|k^s\leqslant 1.$$

The next lemma is due to Aziz and Mohammad [2].

LEMMA 9. *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z|<k,\ k\geqslant 1$, *then for* $1\leqslant R\leqslant k^2$,

$$\max_{|z|=R}|p(z)|\leqslant\left(\frac{R+k}{1+k}\right)^n\max_{|z|=1}|p(z)|.$$

The last lemma is due to Gardner et al. [11].

LEMMA 10. *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z|<k,\ k>0$, *then*

$$|p(z)|\geqslant m\quad\text{for}\quad |z|\leqslant k,$$

where $m=\min_{|z|=k}|p(z)|$.

4. Proofs of the main results

We first prove Theorem 4.

Proof of Theorem 4. By hypothesis, $p(z)$ has all its zeros in $|z|\geqslant k$, $k\geqslant 1$. In case $m=\min_{|z|=k}|p(z)|\neq 0$, consider the polynomial $G(z)=p(z)-\gamma m$, where γ is a complex number with $|\gamma|<1$.

Now, on $|z|=k$,

$$|\gamma m|<m\leqslant|p(z)|.$$

Then by Rouche's theorem, it follows that $G(z)$ has all its zeros in $|z|>k$, and in case $m=0$, $G(z)=p(z)$. Thus, in any case, $G(z)$ has all its zeros in $|z|\geqslant k$. And so, the polynomial $P(z)=G(kz)$ has all its zeros in $|z|\geqslant 1$.

Applying Lemma 2 to $P(z)$, we get for $\frac{R}{k}\geqslant 1$, $0\leqslant s<n$, and for all z on $|z|=1$,

$$|P^{(s)}\left(\frac{R}{k}z\right)|\leqslant\frac{|a_0-\gamma m|k+|a_n|Rk^n}{|a_0-\gamma m|R+|a_n|k^{n+1}}|Q^{(s)}\left(\frac{R}{k}z\right)|,\tag{25}$$

where $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$.

By Lemma 3, we have for $\frac{R}{k}\geqslant 1$, $0\leqslant s<n$, and for all z on $|z|=1$,

$$|P^{(s)}\left(\frac{R}{k}z\right)|+|Q^{(s)}\left(\frac{R}{k}z\right)|\leqslant k^s\left\{\frac{d^s}{dR^s}\left(\left(\frac{R}{k}\right)^n+1\right)\right\}\max_{|z|=1}|P(z)|.\tag{26}$$

Using (25) in (26), we get

$$\left\{1+\frac{|a_0-\gamma m|R+|a_n|k^{n+1}}{|a_0-\gamma m|k+|a_n|Rk^n}\right\}|P^{(s)}\left(\frac{R}{k}z\right)| \leqslant k^{s-n}\left\{\frac{d^s}{dR^s}(R^n+k^n)\right\}\max_{|z|=1}|P(z)|,$$

i.e.,

$$|p^{(s)}(Rz)| \leqslant \frac{1}{k^n}\left\{\frac{|a_0-\gamma m|k+|a_n|Rk^n}{(R+k)(|a_0-\gamma m|+|a_n|k^n)}\right\}\left\{\frac{d^s}{dR^s}(R^n+k^n)\right\}\max_{|z|=k}|p(z)-\gamma m|. \tag{27}$$

Then applying Lemma 9 to $G(z)$ for $R=k$, we get

$$\max_{|z|=k}|p(z)-\gamma m| \leqslant \left(\frac{2k}{1+k}\right)^n \max_{|z|=1}|p(z)-\gamma m|. \tag{28}$$

Combining (27) and (28), we get

$$\begin{aligned}|p^{(s)}(Rz)| \leqslant &\left\{\frac{|a_0-\gamma m|k+|a_n|Rk^n}{(R+k)(|a_0-\gamma m|+|a_n|k^n)}\right\}\left\{\frac{d^s}{dR^s}(R^n+k^n)\right\}\\ &\times\left(\frac{2}{1+k}\right)^n \max_{|z|=1}|p(z)-\gamma m|, \quad R \geqslant k.\end{aligned} \tag{29}$$

Now, applying Lemma 7 to the polynomial $T(z)=G(Rz)$, $1 \leqslant R \leqslant k$, having all its zeros in $|z| \geqslant \frac{k}{R}$, we have for $0 \leqslant s < n$,

$$\begin{aligned}&\max_{|z|=1}|T^{(s)}(z)|\\ &\leqslant \left\{\frac{(1-|\lambda_s'|)(1+(\frac{k}{R})^2|\lambda_s'|)+\frac{k}{R}(n-s)|\mu_s'-\Lambda_s'|}{(1-|\lambda_s'|)(1+(\frac{k}{R})^{s+2}+(\frac{k}{R})^s|\lambda_s'|+(\frac{k}{R})^2|\lambda_s'|)+\frac{k}{R}(1+(\frac{k}{R})^s)(n-s)|\mu_s'-\Lambda_s'|}\right\}\\ &\quad\times \max_{|z|=1}|T(z)|\left[\left\{\frac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right],\end{aligned} \tag{30}$$

where

$$\lambda_s' = \frac{\left(\frac{k}{R}\right)^s}{C(n,s)}\frac{a_sR^s}{(a_0-\gamma m)} = \frac{k^s}{C(n,s)}\frac{a_s}{(a_0-\gamma m)},$$

$$\mu_s' = \frac{\left(\frac{k}{R}\right)^{s+1}}{C(n,s+1)}\frac{a_{s+1}R^{s+1}}{(a_0-\gamma m)} = \frac{k^{s+1}}{C(n,s+1)}\frac{a_{s+1}}{(a_0-\gamma m)}$$

and

$$\Lambda_s' = \frac{\left(\frac{k}{R}\right)^{s+1}}{C(n,s)n}\frac{a_sR^sa_1R}{(a_0-\gamma m)^2} = \frac{k^{s+1}}{C(n,s)n}\frac{a_sa_1}{(a_0-\gamma m)^2}.$$

Inequality (30) on simplification becomes,

$$\max_{|z|=R}|p^{(s)}(z)| \leqslant \left\{\frac{(1-|\lambda_s'|)(R^2+k^2|\lambda_s'|)+Rk(n-s)|\mu_s'-\Lambda_s'|}{(1-|\lambda_s'|)(R^{s+2}+k^{s+2}+R^2k^s|\lambda_s'|+R^sk^2|\lambda_s'|)+Rk(R^s+k^s)(n-s)|\mu_s'-\Lambda_s'|}\right\} \times \max_{|z|=R}|p(z)-\gamma m|\left[\left\{\frac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right]. \tag{31}$$

Further, applying Lemma 9 to $G(z)$, we get

$$\max_{|z|=R}|p(z)-\gamma m| \leqslant \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1}|p(z)-\gamma m|. \tag{32}$$

Combining (31) and (32), we get

$$\max_{|z|=R}|p^{(s)}(z)| \leqslant \left\{\frac{(1-|\lambda_s'|)(R^2+k^2|\lambda_s'|)+Rk(n-s)|\mu_s'-\Lambda_s'|}{(1-|\lambda_s'|)(R^{s+2}+k^{s+2}+R^2k^s|\lambda_s'|+R^sk^2|\lambda_s'|)+Rk(R^s+k^s)(n-s)|\mu_s'-\Lambda_s'|}\right\} \times \left[\left\{\frac{d^s}{dx^s}(1+x^n)\right\}_{x=1}\right]\left(\frac{R+k}{1+k}\right)^n \max_{|z|=1}|p(z)-\gamma m|, \quad 1 \leqslant R \leqslant k. \tag{33}$$

Suppose z_0 on $|z|=1$ be such that

$$\max_{|z|=1}|p(z)-\gamma m| = |p(z_0)-\gamma m|. \tag{34}$$

Now, we can write

$$\begin{aligned} |p(z_0)-\gamma m| &= \left||p(z_0)|e^{i\theta_0}-|\gamma|e^{i\theta}m\right| \\ &= \left||p(z_0)|-|\gamma|e^{i(\theta-\theta_0)}m\right|. \end{aligned}$$

Choosing the argument of γ as $\theta=\theta_0$ gives

$$|p(z_0)-\gamma m| = ||p(z_0)|-|\gamma|m|. \tag{35}$$

By Lemma 10, we have for $|\gamma|<1$ and $|z|=1$,

$$|p(z_0)|-|\gamma|m \geqslant 0. \tag{36}$$

In view of (36), equality (35) becomes

$$|p(z_0)-\gamma m| = |p(z_0)|-|\gamma|m. \tag{37}$$

From (34) and (37), and using the fact that $|p(z_0)| \leqslant \max_{|z|=1}|p(z)|$, we get

$$\max_{|z|=1}|p(z)-\gamma m| \leqslant \max_{|z|=1}|p(z)|-|\gamma|m. \tag{38}$$

On combining (38) with inequalities (29) and (33), we get the required results and thus the proof of Theorem 4 is completed. □

Proof of Theorem 3. The proof of this theorem follows on the same lines as that of Theorem 4 but instead of applying Lemmas 2 and 3 to $P(z) = p(kz) - \gamma m$, we apply the same lemmas to the polynomial $p(kz)$ and then apply Lemma 9 to $p(z)$ for $R = k$ to obtain inequality (11), while obtaining inequality (12), we simply apply Lemma 7 to the polynomial $p(Rz)$ instead of $T(z) = p(Rz) - \gamma m$ and then Lemma 9 to $p(z)$. □

5. Numerical example and graphical representations

As an illustration of the obtained results, in this section, we consider the following example and compare the bounds obtained from our results with previously known results.

EXAMPLE 1. Let $p(z) = z^4 + 3^4$ with all zeros $3e^{\frac{i\pi}{4}(1+2m)}$, $m = 0,1,2,3$ on $|z| = 3$, and $s = 3$, so that Theorems 2, 3, and 4 hold for $1 \leqslant k \leqslant 3$. Then on the circle $|z| = R$, we have

$$|p(Re^{i\theta})| = \sqrt{R^8 + 6561 + 162R^4 \cos 4\theta}$$

and their graphics for $0 \leqslant \theta < 2\pi$ and $R = 1,2,3$ are presented below in Figure 1.

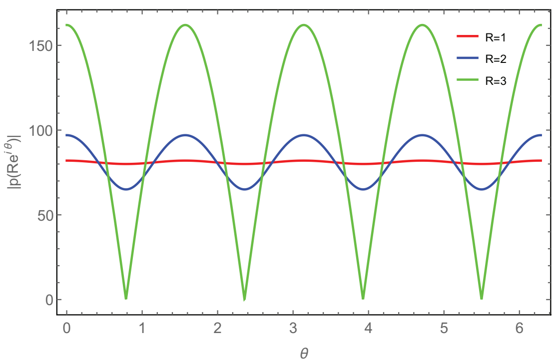

Figure 1: *Graphics of the periodic functions* $\theta \mapsto |p(Re^{i\theta})|$ *with period* $\frac{\pi}{2}$ *for* $0 \leqslant \theta < 2\pi$ *and* $R = 1,2,3$, *clearly showing the extremals.*

Clearly, we have

$$M_R = \max_{|z|=R} |p(z)| = \max_{0 \leqslant \theta < 2\pi} |p(Re^{i\theta})| = R^4 + 3^4$$

and $m = \min_{|z|=k} |p(z)| = p(k) = 3^4 - k^4$, $1 \leqslant k \leqslant 3$.

We intend to illustrate that the bounds of Theorem 4 improve most for $|\gamma| = 1$ (for γ with $|\gamma| = 1$, the result follows by continuity) by considering level graphs for $|\gamma|$, followed by comparisons of the bounds of Theorem 2, Theorem 3, and Theorem 4 for $|\gamma| = 1$.

Case 1. For $R \geqslant k$, let $R = 3$. Then

$$M_3''' = \max_{|z|=3} |p'''(z)| = \max_{0\leqslant\theta<2\pi} |p'''(3e^{i\theta})| = 72.$$

We can consider the difference between the right and the left-hand sides in (13) of Theorem 4 as

$$\beta_1(\gamma,k) = 72\left\{\frac{3k^4 - |\gamma|mk + 81k}{(k+3)(k^4 - |\gamma|m + 81)}\right\}\left(\frac{2}{1+k}\right)^4 (M_1 - |\gamma|m) - M_3'''.$$

Graphics of the function $k \mapsto \beta_1(\gamma,k)$ for $|\gamma| = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ are presented below in Figure 2 (left). In the same figure (right), we show the difference $k \mapsto \Delta(k)$ between the right and the left-hand sides in the inequalities (9) of Theorem 2 due to Jain [14], (11) of Theorem 3, and (13) of Theorem 4 for $|\gamma| = 1$.

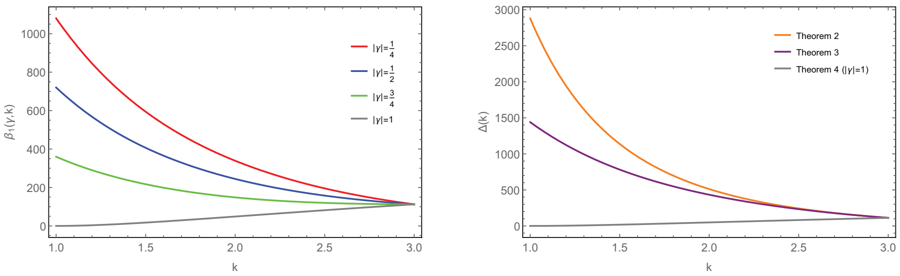

Figure 2: *(Left) The function* $k \mapsto \beta_1(\gamma,k)$, $1 \leqslant k \leqslant 3$ *for* $|\gamma| = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$*; (right) comparison of the differences* $k \mapsto \Delta(k)$ *in the inequalities* (9), (11), *and* (13) *for* $|\gamma| = 1$.

Case 2. For $1 \leqslant R \leqslant k$, let $R = 1$. Since,

$$M_1''' = \max_{|z|=1} |p'''(z)| = \max_{0\leqslant\theta<2\pi} |p'''(e^{i\theta})| = 24,$$

we can consider the difference between the right and the left-hand sides in (14) of Theorem 4 as

$$\beta_2(\gamma,k) = 24\left\{\frac{k^5 - |\gamma|m + 81}{(k^5+1)(81 - |\gamma|m) + k^5(k^3+1)}\right\}(M_1 - |\gamma|m) - M_1''',$$

and graphics of the function $k \mapsto \beta_2(\gamma,k)$ for $|\gamma| = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ are presented below in Figure 3 (left). In the same figure (right), we show the difference $k \mapsto \delta(k)$ between the right and the left-hand sides in the inequalities (10) of Theorem 2 due to Jain [14], (12) of Theorem 3, and (14) of Theorem 4 for $|\gamma| = 1$.

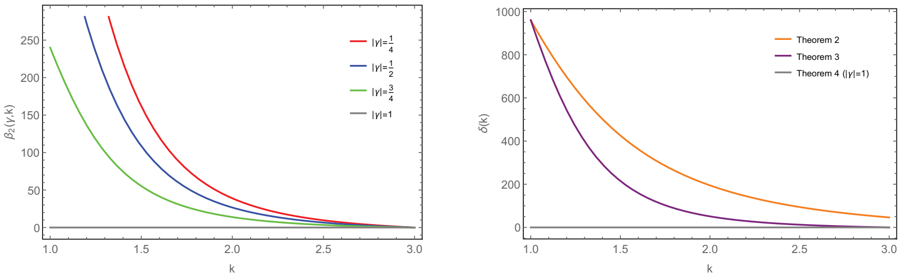

Figure 3: *(Left) The function* $k \mapsto \beta_2(\gamma,k)$, $1 \leqslant k \leqslant 3$ *for* $|\gamma| = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$; *(right) comparison of the differences* $k \mapsto \delta(k)$ *in the inequalities* (10), (12), *and* (14) *for* $|\gamma| = 1$.

In the tables given below, we have obtained the values of the upper bounds of the modulus of the third derivative of the considered polynomial and for $k = 2$ from our results after calculations and compared the improvements in percentage $(\%)$ with the bound values obtained from the previously known results.

Theorems	**Upper bound values of** $M(p''',3)$	**% of improvements over bound obtained from Theorem 2** (inequality (9))
Theorem 2 (inequality (9))	583.111	
Theorem 3 (inequality (11))	504.962	13.40
Theorem 4 ($|\gamma| = 1$) (inequality (13))	120.889	79.27

Table 1: *Upper bound values for the case* $R \geqslant k$ *obtained from various results for the considered polynomial.*

Theorems	**Upper bound values of** $M(p''',1)$	**% of improvements over bound obtained from Theorem 2** (inequality (10))
Theorem 2 (inequality (10))	218.667	
Theorem 3 (inequality (12))	75.104	65.65
Theorem 4 ($|\gamma| = 1$) (inequality (14))	24	89.02

Table 2: *Upper bound values for the case* $1 \leqslant R \leqslant k$ *obtained from various results for the considered polynomial.*

6. Conclusion

Studying extremal problems in functions of a complex variable and generalizing classical polynomial inequalities is a current topic in geometric function theory. This paper considers the well-known Ankeny-Rivlin type inequalities that give the relative growth of a polynomial concerning two circles $|z| = 1$ and $|z| = R \geqslant 1$. The authors establish for a certain class of polynomials some new bounds for the s^{th} derivative of a polynomial on $|z| = R$ while taking into account the placement of the coefficients of the underlying polynomial. The results obtained produce inequalities that are sharper than the previous ones known in very rich literature on this subject which the authors have also proven by considering a concrete numerical example and then illustrated graphically.

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Conflict of interest. The authors declare that they have no conflict of interest.

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