Mathematical
Inequalities
& Applications
Volume 28, Number 1 (2025), 95–98 doi:10.7153/mia-2025-28-06

NEWTON–LIKE INEQUALITIES FOR LINEAR COMBINATIONS

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(*Communicated by I. Perić*)

Abstract. We provide conditions under which linear combinations of normalized elementary symmetric polynomials satisfy some Newton-like inequalities. Namely, the log-concavity of the coefficients and nonnegativity of the arguments. We prove that such conditions are essential. That is, dropping any of these conditions leads to counterexamples. This settled a conjecture of Ren [C. Ren, *A generalization of Newton-Maclaurin's inequalities*, Int. Math. Res. Not. IMRN **5**, (2024), 3799–3822].

1. Introduction

The so-called *Newton's inequalities* were first stated by Newton [7] and later proved by Maclaurin [4]. These state that

$$e_k^2(Z) \geqslant \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) e_{k-1}(Z)e_{k+1}(Z) \tag{1}$$

for $1 \leqslant k \leqslant n-1$, where Z is an arbitrary multiset of real numbers with cardinality n, and $e_0,\dots,e_n$ are the *elementary symmetric polynomials in n variables*. These polynomials are defined by the following equation in the ring of polynomials with complex coefficients and indeterminate x:

$$\prod_{z\in Z}(x+z) = \sum_{k=0}^{n} e_{n-k}(Z)x^k.$$

Using the following identity between binomial coefficients

$$\binom{n}{k}^2 = \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right)\binom{n}{k-1}\binom{n}{k+1},$$

we can show that (1) is equivalent to

$$E_k^2(Z) \geqslant E_{k-1}(Z)E_{k+1}(Z) \tag{2}$$

Mathematics subject classification (2020): 26D15, 05E05, 05A20, 11B83.
Keywords and phrases: Newton's inequalities, Newton-like inequalities, log-concave sequence, elementary symmetric polynomials, normalized elementary symmetric polynomials.
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for $1 \leqslant k \leqslant n-1$, where each $E_k(Z)$ is the *k-th normalized elementary symmetric polynomial*, i.e., $E_k(Z) = e_k(Z)/\binom{n}{k}$. A proof of Newton's inequalities in the form of (2) can be found in [2, §4.3, p. 104].

Over the years, several generalizations of Newton's inequalities have been found [1, 6, 8–10, 12, 13]. All of these, along with similar ones, are known as *Newton-like inequalities*. Specifically, in [9], Ren found Newton-like inequalities involving linear combinations of two normalized elementary symmetric polynomials. Ren then conjectured about conditions that would allow more general linear combinations to satisfy similar inequalities. This conjecture was motivated by the following result:

THEOREM 1. (Ren) *Let $n \geqslant 3$, $1 \leqslant k \leqslant n-2$. For any real number $\alpha \in \mathbb{R}$ and any multiset Z of real numbers with cardinality n, we have*

$$\big(\alpha E_k(Z) + E_{k+1}(Z)\big)^2 \geqslant \big(\alpha E_{k-1}(Z) + E_k(Z)\big)\big(\alpha E_{k+1}(Z) + E_{k+2}(Z)\big).$$

A natural generalization of this result would be

$$\left(\sum_{i=1}^{n} \alpha_i E_i(Z)\right)^2 \geqslant \left(\sum_{i=1}^{n} \alpha_i E_{i-1}(Z)\right)\left(\sum_{i=1}^{n} \alpha_i E_{i+1}(Z)\right). \tag{3}$$

Ren proved that (3) cannot hold true for any arbitrary sequence $(\alpha_1, \ldots, \alpha_n)$ of real numbers. He then conjectured that (3) might be valid under certain structural conditions satisfied by the sequence of coefficients $(\alpha_1, \ldots, \alpha_n)$. In this paper we show that such a structural condition is log-concavity, while the multiset Z must be formed by nonnegative real numbers. Furthermore, we show that both conditions are essential.

2. Log-concavity

DEFINITION 1. A sequence $(\alpha_k)_{1 \leqslant k \leqslant n}$ of nonnegative real numbers:

(i) *has no internal zeros* if the set $\{k \in \mathbb{N} : \alpha_k \neq 0\}$ is an interval of integer numbers;

(ii) is *log-concave* if it has no internal zeros and $\alpha_k^2 \geqslant \alpha_{k-1}\alpha_{k+1}$ for all $2 \leqslant k \leqslant n-1$.

Given a polynomial $p(x) \in \mathbb{R}[x]$, we call $p(x)$ *log-concave* if and only if the sequence of its coefficients is log-concave.

THEOREM 2. *Let $(\alpha_k)_{1 \leqslant k \leqslant n}$ be a sequence of nonnegative real numbers such that it has not internal zeros. Then, $(\alpha_k)_{1 \leqslant k \leqslant n}$ is log-concave if and only if for any $q(x)$ log-concave polynomial, it holds*

$$\left(\sum_{k=1}^{n} \alpha_k b_k\right)^2 \geqslant \left(\sum_{k=1}^{n} \alpha_k b_{k-1}\right)\left(\sum_{k=1}^{n} \alpha_k b_{k+1}\right), \tag{4}$$

where $q(x) = \sum_k b_k x^k$.

Proof. Assume that $(\alpha_k)_{1\leqslant k\leqslant n}$ is a log-concave sequence. Let

$$p(x)=\sum_{k=1}^{n}\alpha_{n+1-k}x^k.$$

From the well-known fact that the product of two log-concave polynomials is log-concave (see [3, p. 394], [5], [11, Proposition 2]), it follows that the product $p(x)q(x)$ is log-concave, which means that

$$\left(\sum_{i=0}^{k+1}\alpha_{n+1-i}b_{k+1-i}\right)^2\geqslant\left(\sum_{i=0}^{k}\alpha_{n+1-i}b_{k-i}\right)\left(\sum_{i=0}^{k+2}\alpha_{n+1-i}b_{k+2-i}\right),\quad k\geqslant 0. \tag{5}$$

By substituting $k=n$, $\alpha_{n+1}=0$, $\alpha_0=0$, and $\alpha_{-1}=0$ in (5) we get

$$\left(\sum_{i=1}^{n}\alpha_{n+1-i}b_{n+1-i}\right)^2\geqslant\left(\sum_{i=1}^{n}\alpha_{n+1-i}b_{n-i}\right)\left(\sum_{i=1}^{n}\alpha_{n+1-i}b_{n+2-i}\right),$$

which is equivalent to (4). Conversely, if the sequence $(\alpha_k)_{1\leqslant k\leqslant n}$ holds (4) for any $q(x)$ log-concave polynomial, then for any $2\leqslant i\leqslant n-1$, (4) must hold for the log-concave polynomial $q_i(x)=x^i$, which leads to $\alpha_i^2\geqslant\alpha_{i+1}\alpha_{i-1}$. □

COROLLARY 1. *Let $(\alpha_k)_{1\leqslant k\leqslant n}$ be a log-concave sequence and E_k, $0\leqslant k\leqslant n+1$, the normalized elementary symmetric polynomials in $n+1$ variables. If Z is a multiset of nonnegative real numbers with cardinality $n+1$, then (3) holds.*

Proof. Newton's inequalities in the form (2) imply that the polynomial $q(x)$, defined by $q(x)=\sum_{k=0}^{n+1}E_k(Z)x^k$ is log-concave. The result then follows from Theorem 2. □

Corollary 1 allows us to assert that log-concavity and nonnegativity are precisely the structural conditions sought by Ren, as these conditions are essential for real numbers. Indeed, the assumption of nonnegativity for the elements of Z is crucial. For example, when $n=3$, the log-concave sequence $(1,3/2,1)$ and $Z=[-1,-1,1,1]$ show that:

$$\left(E_1(Z)+\frac{3}{2}E_2(Z)+E_3(Z)\right)^2 \\ -\left(E_0(Z)+\frac{3}{2}E_1(Z)+E_2(Z)\right)\left(E_2(Z)+\frac{3}{2}E_3(Z)+E_4(Z)\right)=-\frac{7}{36}.$$

That is, the conclusion of Corollary 1 no longer holds. Additionally, the assumption that the sequence of coefficients $(\alpha_1,\ldots,\alpha_n)$ is log-concave is essential. For instance, consider $Z=[4,4,1/4,1/4]$ and the non-log-concave sequence of coefficients $(100,1,50)$. In this case we have:

$$\left(100E_1(Z)+E_2(Z)+50E_3(Z)\right)^2 \\ -\left(100E_0(Z)+E_1(Z)+50E_2(Z)\right)\left(100E_2(Z)+E_3(Z)+50E_4(Z)\right)=-\frac{353175}{1024}.$$

Once again, the conclusion of Corollary 1 fails.

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(Received July 6, 2024)

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