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COVERING THE UNIT BALL OF ℓ_p^n WITH SMALLER BALLS AND RELATED INEQUALITIES

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Abstract. Let B_p^n ($p \geqslant 1$) be the unit ball of ℓ_p^n and $\Gamma_m(B_p^n)$ be the smallest positive number γ such that B_p^n can be covered by m translates of γB_p^n. By using different configurations of translates of γB_p^n, we obtain a universal upper bound of $\Gamma_{2^n}(B_p^n)$ for fixed $p \in [1,\infty]$, a nontrivial upper bound for $\Gamma_{2^n}(B_p^n)$ for all $p \in [1,\infty]$ when n is small, and a useful upper bound of $\Gamma_{2^n}(B_p^n)$ when n and p are both large. It is still not clear whether there exists a constant $c \in (0,1)$ such that $\Gamma_{2^n}(B_p^n) \leqslant c$ holds whenever $p \geqslant 1$ and $n \geqslant 2$.

1. Introduction

Let $n \geqslant 2$ be an integer and $p \in [1,\infty)$. We denote by ℓ_p^n the space $(\mathbb{R}^n, \|\cdot\|_p)$, where, for each point $(\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n$,

$$\|(\alpha_1,\ldots,\alpha_n)\|_p = \left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p}.$$

We denote by ℓ_∞^n the space $(\mathbb{R}^n, \|\cdot\|_\infty)$, where

$$\|(\alpha_1,\ldots,\alpha_n)\|_\infty = \max_{1 \leqslant i \leqslant n} |\alpha_i|.$$

For each $p \in [1,\infty]$, let B_p^n and S_p^n be the *unit ball* and the *unit sphere* of ℓ_p^n, respectively, and let $\Gamma_m(B_p^n)$ be the smallest positive number γ such that B_p^n can be covered by m translates of γB_p^n. In general, for a *convex body* (a compact convex set having interior points) K in $\mathbb{R}^n$, we denote by $\Gamma_m(K)$ the smallest positive number γ such that K can be covered by m translates of γK. Estimating $\Gamma_m(K)$ plays an important role in Chuanming Zong's program to attack Hadwiger's covering conjecture, cf. [16]. For more information on Hadwiger's covering conjecture we refer to [3], [4], [2], and [5].

In the terminology of [14], $\Gamma_{2^n}(B_p^n)$ is the n-th *entropy number* of B_p^n in ℓ_p^n. By [7, Theorem 2], $\Gamma_{2^n}(B_p^n) \sim 1/2$, and the constants of equivalence may depend on p

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and independent on n. In fact, for each integer $n \geqslant 2$, $\Gamma_{2^n}\left(B_p^n\right) \geqslant 1/2$, $\forall p \in [1,\infty]$ and the equality holds only when $p = \infty$ (cf. [6, Theorem 1]).

Recently, Xue et al. [15] proved that

$$\limsup_{n\to\infty} \Gamma_{2^n}(B_p^n) \leqslant \left(\frac{1}{1+c_2}\right)^{\frac{1}{p}},$$

where c_2 is a constant in the interval $[0.2056, 0.2271]$. This estimate is not optimal when p is too large. We shall determine c_2 in Corollary 2.8.

Remark 6 in [9] shows that

$$\Gamma_{2^n}(B_p^n) \leqslant \left(\frac{n}{n+\lfloor b(2)n\rfloor}\right)^{\frac{1}{p(n)}}, \ \forall p \geqslant 1,$$

where $b(2) \approx 0.205597$ is the solution to the equation $2^x(1+x)^{1+x}/x^x = 2$, and $p(n)$ is the unique solution to

$$\left(\frac{n}{n+\lfloor b(2)n\rfloor}\right)^{\frac{1}{p}} = n^{\frac{1}{p}} - \frac{1}{2}.$$

Moreover

$$\frac{\ln n}{\ln\left(\frac{3}{2}\right)} \leqslant p(n) \leqslant \frac{\ln n}{\ln\left(\frac{1}{2}+\frac{1}{1+b(2)}\right)}.$$

It is not clear whether there exists a constant $c \in (0,1)$ such that $\Gamma_{2^n}(B_p^n) \leqslant c$ holds for all $p \in [1,\infty]$ and all $n \geqslant 2$, since it is difficult to dominate $\Gamma_{2^n}(B_p^n)$ when p and n are both large.

For each $p \in [1,\infty]$, we present a universal upper bound of $\Gamma_{2^n}(B_p^n)$, which can be used to dominate $\Gamma_{2^n}(B_p^n)$ when p is relatively small.

THEOREM 1.1. *For each $n \geqslant 2$, we have*

$$\Gamma_{2^n}(B_p^n) \leqslant \left(\frac{1}{b(2)+0.98}\right)^{\frac{1}{p}}.$$

The following result generalizes [16, Theorem 2] and provides an acceptable upper bound of $\Gamma_{2^n}(B_p^n)$ when n is not too large.

THEOREM 1.2. *Let $n \geqslant 2$. For each $p \in [1,\infty]$, we have*

$$\Gamma_{2^n}(B_p^n) \leqslant \left(1-\frac{1}{n}\right)^{\frac{1}{2}}.$$

When n and p are both large, we can use the following result to dominate $\Gamma_{2^n}(B_p^n)$.

THEOREM 1.3. *Suppose that* $n \geqslant 10$, $p \geqslant 2$, *and* $p_1(n)$ *is determined by* (3.3) *below. Then*

$$\Gamma_{2^n}(B_p^n) \leqslant \left(\frac{n}{n+\lfloor b(2)n\rfloor}\right)^{\frac{1}{p_1(n)}}.$$

Moreover,

$$\lim_{n\to\infty} \frac{\ln n}{\ln(\ln n)} \cdot \frac{1}{p_1(n)} = 1.$$

Throughout this paper, the dimension n of the underlying space is at least 2. For each positive integer m, we use the shorthand notation

$$[m] := \left\{i \in \mathbb{Z}^+ \mid 1 \leqslant i \leqslant m\right\}.$$

The cardinality of a set A will be denoted by $\#A$.

2. A lattice point based covering of B_p^n

For $n,k \in \mathbb{Z}^+$ satisfying $k \leqslant n$, set

$$m(n,k) = \sum_{i=1}^{k} 2^i \binom{n}{i}\binom{k-1}{i-1} + 1.$$

We shall use the convention that

$$\binom{0}{0} = \binom{n}{0} = 1 \quad \text{and} \quad m(n,0) = 1.$$

LEMMA 2.1. *Suppose that* $n,k \in \mathbb{Z}^+$, $n \geqslant 3$, *and* $p \geqslant 1$. *If* $k \leqslant n/2$, *then*

$$(n+k)^{\frac{1}{p}} B_p^n \subseteq n^{\frac{1}{p}} B_p^n + L_p^k$$

where

$$L_p^k = \left\{(\alpha_1, \cdots, \alpha_n) \mid \sum_{i\in[n]} |\alpha_i|^p = k,\ |\alpha_i|^p \in \mathbb{N},\ \forall i \in [n]\right\} \cup \{o\}.$$

Moreover, $\#L_p^k = m(n,k)$.

Proof. Let $(\alpha_1, \cdots, \alpha_n)$ be an arbitrary point in $(n+k)^{\frac{1}{p}} B_p^n$. Then

$$\sum_{i\in[n]} |\alpha_i|^p \leqslant (n+k).$$

If $\sum_{i\in[n]} |\alpha_i|^p \leqslant n$, then

$$(\alpha_1, \cdots, \alpha_n) \in n^{\frac{1}{p}} B_p^n \subseteq n^{\frac{1}{p}} B_p^n + L_p^k.$$

Otherwise, there exists $m \in [k]$ such that

$$n+m-1 < \sum_{i\in[n]} |\alpha_i|^p \leqslant n+m.$$

On the one hand, since $\sum_{i\in[n]}(|\alpha_i|^p - \lfloor|\alpha_i|^p\rfloor) < n$, we have

$$\sum_{i\in[n]} \lfloor|\alpha_i|^p\rfloor \geqslant m.$$

Then there exist $\beta_1,\cdots,\beta_n \in \mathbb{N}$ such that

$$\beta_i \leqslant |\alpha_i|^p,\ \forall i\in[n] \ \text{ and } \ \sum_{i\in[n]} \beta_i = m.$$

By Lemma 2.4 below, we have

$$\sum_{i\in[n]} \left|\alpha_i - \operatorname{sgn}(\alpha_i)\cdot\beta_i^{\frac{1}{p}}\right|^p \leqslant \sum_{i\in[n]} (|\alpha_i|^p - \beta_i) \leqslant n.$$

Therefore,

$$\begin{aligned}(\alpha_1,\cdots,\alpha_n) =& \left(\alpha_1 - \operatorname{sgn}(\alpha_1)\cdot\beta_1^{\frac{1}{p}},\cdots,\alpha_n - \operatorname{sgn}(\alpha_n)\cdot\beta_n^{\frac{1}{p}}\right)\\ &+\left(\operatorname{sgn}(\alpha_1)\cdot\beta_1^{\frac{1}{p}},\cdots,\operatorname{sgn}(\alpha_n)\cdot\beta_n^{\frac{1}{p}}\right)\\ \in& n^{\frac{1}{p}}B_p^n + L_p^m.\end{aligned}$$

On the other hand, set

$$m_i = \begin{cases}\lfloor|\alpha_i|^p\rfloor, & \text{if } |\alpha_i|^p - \lfloor|\alpha_i|^p\rfloor < \frac{1}{2},\\ \lfloor|\alpha_i|^p\rfloor + 1, & \text{if } |\alpha_i|^p - \lfloor|\alpha_i|^p\rfloor \geqslant \frac{1}{2},\end{cases} \quad \forall i\in[n]. \tag{2.1}$$

We have

$$\begin{aligned}(\alpha_1,\cdots,\alpha_n) =& \left(\alpha_1 - \operatorname{sgn}(\alpha_1)\cdot m_1^{\frac{1}{p}},\cdots,\alpha_n - \operatorname{sgn}(\alpha_n)\cdot m_n^{\frac{1}{p}}\right)\\ &+\left(\operatorname{sgn}(\alpha_1)\cdot m_1^{\frac{1}{p}},\cdots,\operatorname{sgn}(\alpha_n)\cdot m_n^{\frac{1}{p}}\right).\end{aligned}$$

By the triangle inequality, we have

$$n - \sum_{i\in[n]} m_i < \sum_{i\in[n]} |\alpha_i|^p - \sum_{i\in[n]} m_i \leqslant \sum_{i\in[n]} ||\alpha_i|^p - m_i| \leqslant \frac{n}{2}.$$

Thus,

$$\sum_{i\in[n]} m_i > \frac{n}{2} \geqslant k.$$

By (2.1), we have

$$\beta_i \leqslant \lfloor |\alpha_i|^p \rfloor \leqslant m_i \leqslant \lceil |\alpha_i|^p \rceil, \ \forall i \in [n].$$

Then there exist $m_1', \cdots, m_n' \in \mathbb{N}$ such that

$$\beta_i \leqslant m_i' \leqslant m_i, \ \forall i \in [n] \ \text{ and } \sum_{i \in [n]} m_i' = k.$$

Set

$$\phi_i(\lambda) = \left| |\alpha_i| - \lambda^{\frac{1}{p}} \right|^p, \ \ \forall i \in [n].$$

Then, ϕ_i is decreasing on $[\beta_i, \lfloor |\alpha_i|^p \rfloor]$. We claim that

$$\phi_i(\beta_i) \geqslant \phi_i(m_i'), \ \forall i \in [n]. \tag{2.2}$$

The case when $m_i' \in [\beta_i, \lfloor |\alpha_i|^p \rfloor]$ is clear. If $m_i' > \lfloor |\alpha_i|^p \rfloor$, then

$$m_i' = m_i = \lfloor |\alpha_i|^p \rfloor + 1 \quad \text{and} \quad 1/2 \leqslant |\alpha_i|^p - \lfloor |\alpha_i|^p \rfloor < 1.$$

Since $\psi(x) = x^{\frac{1}{p}}$ is strictly increasing and concave on $(0, \infty)$, we have

$$2\,|\alpha_i| \geqslant 2\left(\lfloor |\alpha_i|^p \rfloor + \frac{1}{2} \right)^{\frac{1}{p}} \geqslant (\lfloor |\alpha_i|^p \rfloor)^{\frac{1}{p}} + (\lfloor |\alpha_i|^p \rfloor + 1)^{\frac{1}{p}}.$$

Then,

$$\left(|\alpha_i| - \lfloor |\alpha_i|^p \rfloor^{\frac{1}{p}} \right)^p \geqslant \left((\lfloor |\alpha_i|^p \rfloor + 1)^{\frac{1}{p}} - |\alpha_i| \right)^p.$$

Thus,

$$\phi_i(\beta_i) \geqslant \phi_i(\lfloor |\alpha_i|^p \rfloor) \geqslant \phi_i(\lfloor |\alpha_i|^p \rfloor + 1) = \phi_i(m_i').$$

Hence (2.2) holds as claimed. It follows that

$$\sum_{i \in [n]} \left| |\alpha_i| - (m_i')^{\frac{1}{p}} \right|^p = \sum_{i \in [n]} \phi_i(m_i') \leqslant \sum_{i \in [n]} \phi_i(\beta_i) \leqslant n.$$

Therefore,

$$\begin{aligned}(\alpha_1, \cdots, \alpha_n) = & \left(\alpha_1 - \operatorname{sgn}(\alpha_1) \cdot (m_1')^{\frac{1}{p}}, \cdots, \alpha_n - \operatorname{sgn}(\alpha_n) \cdot (m_n')^{\frac{1}{p}} \right) \\ & + \left(\operatorname{sgn}(\alpha_1) \cdot (m_1')^{\frac{1}{p}}, \cdots, \operatorname{sgn}(\alpha_n) \cdot (m_n')^{\frac{1}{p}} \right) \\ \in & n^{\frac{1}{p}} B_p^n + L_p^k.\end{aligned}$$

Clearly, the map

$$\begin{aligned} T:\ & L^k \to L_p^k, \\ (\alpha_1, \cdots, \alpha_n) \mapsto & \left(\operatorname{sgn}(\alpha_1) \cdot |\alpha_1|^{\frac{1}{p}}, \cdots, \operatorname{sgn}(\alpha_n) \cdot |\alpha_n|^{\frac{1}{p}} \right) \end{aligned}$$

is a bijection, where

$$L^k = \left\{ (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n \mid \sum_{i \in [n]} |\alpha_i| = k \right\} \cup \{o\}.$$

Clearly, $\#L^k$ equals the number of integer points in $(kB_1^n) \setminus ((k-1)B_1^n)$. By [12, Problem 29 and its solution] or [1], we have $\#L_p^k = \#L^k = m(n,k)$. $\square$

Let $k_1(n)$ be the nonnegative integer satisfying

$$m(n, k_1(n)) \leqslant 2^n < m(n, k_1(n)+1).$$

COROLLARY 2.2. *Let $p \geqslant 1$. We have*

$$\Gamma_{2^n}(B_p^n) \leqslant \left(\frac{n}{n+k_1(n)} \right)^{\frac{1}{p}}.$$

In Section 2.1 we collect some technical lemmas showing that $k_1(n)$ is well defined (cf. Lemma 2.3), and providing estimates of $n/(n+k_1(n))$.

2.1. Auxiliary Lemmas

We shall use the following Stirling's approximation (cf. [13]):

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{r_n},$$

where

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

LEMMA 2.3. *Let n and k be nonnegative integers with $n \geqslant 3$ and $k \leqslant n$. We have*

(a) $m(n,k)$ is strictly increasing with respect to k when $k \in [0, \lfloor n/2 \rfloor]$;

(b) $m(n,1) = 1+2n < 2^n$, $\forall n \geqslant 3$ and $m(3,2) = 19$;

(c) $m(n, \lfloor n/2 \rfloor) > 2^n$, $\forall n > 3$;

(d) $m(n,6) < 2^n$ holds for sufficiently large n;

(e) $m(n, \lfloor n/4 \rfloor) > 2^n$ holds for sufficiently large n.

Proof. (a) and (b) is obvious.

(c). If n is even, then $n = 2l$ for some integer $l \geqslant 2$. By

$$\begin{aligned}\binom{2l}{l} &= \binom{l}{l} \cdot \binom{l}{0} + \binom{l}{l-1} \cdot \binom{l}{1} + \cdots + \binom{l}{0} \cdot \binom{l}{l} \\ &\geqslant \binom{l}{l} + \binom{l}{l-1} + \cdots + \binom{l}{0} = 2^l,\end{aligned}$$

we have

$$m\left(n,\left\lfloor\frac{n}{2}\right\rfloor\right) > \binom{n}{\lfloor\frac{n}{2}\rfloor}\cdot 2^{\lfloor\frac{n}{2}\rfloor} \geqslant 2^n. \tag{2.3}$$

If n is odd, then there exists an integer $l \geqslant 2$ such that $n = 2l+1$. By

$$\begin{aligned}\binom{2l+1}{l} &= \binom{l}{l}\cdot\binom{l+1}{0} + \binom{l}{l-1}\cdot\binom{l+1}{1} + \cdots + \binom{l}{0}\cdot\binom{l+1}{l} \\ &\geqslant 2\left(\binom{l}{l}+\binom{l}{l-1}+\cdots+\binom{l}{0}\right) = 2^{l+1},\end{aligned}$$

we obtain (2.3) again.

(d). It is a consequence of the fact that $m(n,6)$ is a polynomial of degree 6 with respect to n.

(e). By Stirling's approximation, we have

$$\begin{aligned}\binom{n}{\lfloor\frac{n}{4}\rfloor} = \frac{n!}{\lfloor\frac{n}{4}\rfloor!\cdot\left(n-\lfloor\frac{n}{4}\rfloor\right)!} &= a_n\cdot\frac{n^n}{\lfloor\frac{n}{4}\rfloor^{\lfloor\frac{n}{4}\rfloor}\cdot\left(n-\lfloor\frac{n}{4}\rfloor\right)^{n-\lfloor\frac{n}{4}\rfloor}} \\ &= a_n\cdot\frac{1}{\left(\frac{\lfloor\frac{n}{4}\rfloor}{n}\right)^{\lfloor\frac{n}{4}\rfloor}\cdot\left(1-\frac{\lfloor\frac{n}{4}\rfloor}{n}\right)^{n-\lfloor\frac{n}{4}\rfloor}},\end{aligned}$$

where

$$a_n = \frac{\sqrt{2\pi\cdot n}}{\sqrt{2\pi\cdot\lfloor\frac{n}{4}\rfloor}\cdot\sqrt{2\pi\cdot\left(n-\lfloor\frac{n}{4}\rfloor\right)}}\cdot\frac{e^{r_n}}{e^{r_{\lfloor n/4\rfloor}}\cdot e^{r_{n-\lfloor n/4\rfloor}}}.$$

By

$$\lim_{n\to\infty}\frac{\lfloor\frac{n}{4}\rfloor}{n} = \frac{1}{4},$$

we have

$$\lim_{n\to\infty}\sqrt[n]{\frac{\binom{n}{\lfloor\frac{n}{4}\rfloor}\cdot 2^{\lfloor\frac{n}{4}\rfloor}}{2^n}} = \frac{1}{\frac{1}{4}^{\frac{1}{4}}\cdot(1-\frac{1}{4})^{1-\frac{1}{4}}}\cdot\frac{2^{\frac{1}{4}}}{2} = 2\sqrt[4]{\frac{2}{27}} > 1.$$

Therefore, when n is sufficiently large, we have

$$m\left(n,\left\lfloor\frac{n}{4}\right\rfloor\right) > \binom{n}{\lfloor\frac{n}{4}\rfloor}\cdot 2^{\lfloor\frac{n}{4}\rfloor} > 2^n. \quad \square$$

Clearly, $k_1(3) = 1 \leqslant \lfloor 3/2\rfloor$, which, together with (a), (b), and (c) in Lemma 2.3, shows that $k_1(n) \leqslant \lfloor n/2\rfloor$, $\forall n \geqslant 3$.

LEMMA 2.4. ([9]) *If either $x \geqslant a \geqslant 0$ or $x \leqslant a \leqslant 0$, then*

$$|x-a|^p \leqslant |x|^p - |a|^p,\ \forall p\in[1,\infty).$$

For each $x \in (0, 1/2]$, set

$$h(x) = \sqrt{1+x^2} - x, \quad g(x) = \sqrt{1+x^2} - 1,$$

and

$$f(x) = \frac{1}{(1-h(x))^{1-h(x)} \cdot h(x)^{h(x)}} \cdot \frac{x^x}{(1-h(x))^{1-h(x)} \cdot g(x)^{g(x)}} \cdot 2^{1-h(x)}.$$

LEMMA 2.5. *The function f defined above is strictly increasing on $(0, 1/2]$.*

Proof. Since $h(x)$ is strictly decreasing on $(0, 1/2]$, we have $h(x) < 1$, $\forall x \in (0, 1/2]$. Moreover,

$$\frac{x}{h(x)+x-1} > 1, \ \forall x \in \left(0, \frac{1}{2}\right].$$

Clearly, $g(x)$ is strictly increasing on $(0, 1/2]$. It can be verified that

$$\begin{aligned}\ln(f(x)) = & -2(1-h(x))\ln(1-h(x)) - h(x)\ln(h(x)) - g(x)\ln(g(x)) \\ & + x\ln x + \ln 2 \cdot (1-h(x)).\end{aligned}$$

Since

$$g(x) = h(x) + x - 1 \quad \text{and} \quad (1-h(x))^2 = 2 \cdot h(x)(h(x)+x-1),$$

we have

$$\begin{aligned}\frac{\mathrm{d}(\ln(f(x)))}{\mathrm{d}x} = & 2h'(x)(1+\ln(1-h(x))) - h'(x)(1+\ln(h(x))) \\ & - (h'(x)+1)(1+\ln(h(x)+x-1)) + (1+\ln x) - \ln 2 \cdot h'(x) \\ = & h'(x)[2\ln(1-h(x)) - \ln h(x) - \ln(h(x)+x-1) - \ln 2] \\ & + \ln x - \ln(h(x)+x-1) \\ = & h'(x) \cdot \ln\left(\frac{(1-h(x))^2}{2 \cdot h(x) \cdot (h(x)+x-1)}\right) + \ln\left(\frac{x}{h(x)+x-1}\right) \\ = & \ln\left(\frac{x}{h(x)+x-1}\right) > 0.\end{aligned}$$

This completes the proof. □

Let $a(2)$ be the solution to the equation $f(x) = 2$ on $(0, 1/2]$. Numerical calculation shows that $a(2) \approx 0.2140287$.

For each $n \geqslant 12$, $k \in [\lfloor n/2 \rfloor] \setminus [5]$, and $j \in [k]$, set

$$b(n,k,j) = \binom{n}{j} \cdot \binom{k-1}{j-1} \cdot 2^j.$$

For each $j \in [k-1]$, we have

$$\frac{b(n,k,j+1)}{b(n,k,j)} = 2 \cdot \frac{\binom{n}{j+1} \cdot \binom{k-1}{j}}{\binom{n}{j} \cdot \binom{k-1}{j-1}} = 2 \cdot \frac{(n-j)(k-j)}{(j+1)j} \leqslant nk.$$

Clearly

$$j(n,k) = n+k+\frac{1}{2} - \sqrt{n^2+k^2+n+k+\frac{1}{4}}. \tag{2.4}$$

is the root of the equation $b(n,k,j+1)/b(n,k,j) = 1$ that is strictly less than k (the other one is strictly greater than k). Therefore,

$$b(n,k,1) < \cdots < b(n,k,\lfloor j(n,k) \rfloor)$$

and

$$b(n,k,\lfloor j(n,k) \rfloor + 1) > \cdots > b(n,k,k).$$

Moreover,

$$\begin{aligned} j(n,k) &= n+k+\frac{1}{2} - \sqrt{n^2+k^2} - \left(\sqrt{n^2+k^2+n+k+\frac{1}{4}} - \sqrt{n^2+k^2}\right) \\ &= n+k+\frac{1}{2} - \sqrt{n^2+k^2} - \frac{n+k+\frac{1}{4}}{\sqrt{n^2+k^2+n+k+\frac{1}{4}} + \sqrt{n^2+k^2}}. \end{aligned}$$

By

$$0 < \frac{n+k+\frac{1}{4}}{\sqrt{n^2+k^2+n+k+\frac{1}{4}} + \sqrt{n^2+k^2}} < \frac{n+k+\frac{1}{4}}{\left(n+\frac{1}{2}\right)+k} < 1,$$

we have

$$n+k-\sqrt{n^2+k^2} - \frac{3}{2} < \lfloor j(n,k) \rfloor < n+k-\sqrt{n^2+k^2} + \frac{1}{2}. \tag{2.5}$$

Since $k \geqslant 6$, it can be verified that

$$\lfloor j(n,k) \rfloor > n+k-\sqrt{n^2+k^2} - \frac{3}{2} > \frac{k}{2}.$$

By (2.5), there exists $\theta_1(n,k) \in \left(-\frac{3}{2}, \frac{1}{2}\right)$ such that

$$\lfloor j(n,k) \rfloor = n+k-\sqrt{n^2+k^2} + \theta_1(n,k).$$

By Stirling's approximation, we have

$$\begin{aligned} b(n,k,j) &= \binom{n}{j} \cdot \binom{k-1}{j-1} \cdot 2^j \\ &= \frac{j}{k} \cdot \binom{n}{j} \cdot \binom{k}{j} \cdot 2^j \\ &= \theta_2(n,k,j) \cdot 2^j \cdot \frac{n^n}{j^j \cdot (n-j)^{n-j}} \cdot \frac{k^k}{j^j \cdot (k-j)^{k-j}}, \end{aligned} \tag{2.6}$$

where

$$\theta_2(n,k,j) = \frac{j}{k} \cdot \frac{\sqrt{2\pi \cdot n}}{\sqrt{2\pi \cdot j} \cdot \sqrt{2\pi \cdot (n-j)}} \cdot \frac{e^{r_n}}{e^{r_j} \cdot e^{r_{n-j}}} \cdot \frac{\sqrt{2\pi \cdot k}}{\sqrt{2\pi \cdot j} \cdot \sqrt{2\pi \cdot (k-j)}} \cdot \frac{e^{r_k}}{e^{r_j} \cdot e^{r_{k-j}}}.$$

For each sequence $\{(n,k_n,j_n)\}_{n=12}^{\infty}$ of triples satisfying $k_n \in [\lfloor n/2 \rfloor] \setminus [5]$ and $j_n \in [k_n - 1]$, we have

$$\lim_{n\to\infty} \sqrt[n]{\theta_2(n,k_n,j_n)} = 1.$$

LEMMA 2.6. *Let $n \geqslant 12$, $k_n \in [\lfloor n/2 \rfloor] \setminus [5]$, and $j(n,k_n)$ be defined as (2.4), and*

$$\theta(n,k_n) := b(n,k_n,\lfloor j(n,k_n) \rfloor) \cdot \left(f\left(\frac{k_n}{n}\right)\right)^{-n}.$$

Then

$$\lim_{n\to\infty} \sqrt[n]{\theta(n,k_n)} = 1.$$

Proof. Set $k = k_n$, $j = \lfloor j(n,k) \rfloor$,

$$A(n,k) = n + k - \sqrt{n^2+k^2},\ B(n,k) = \sqrt{n^2+k^2} - k,\ C(n,k) = \sqrt{n^2+k^2} - n,$$

and

$$\theta_3(n,k) = \frac{(A(n,k)+\theta_1(n,k))^{A(n,k)}}{(A(n,k))^{A(n,k)}} \cdot (A(n,k)+\theta_1(n,k))^{\theta_1(n,k)}.$$

We have

$$\begin{aligned} j^j =& (A(n,k)+\theta_1(n,k))^{A(n,k)+\theta_1(n,k)} \\ =& (A(n,k))^{A(n,k)} \cdot \frac{(A(n,k)+\theta_1(n,k))^{A(n,k)+\theta_1(n,k)}}{(A(n,k))^{A(n,k)}} \\ =& (A(n,k))^{A(n,k)} \cdot \theta_3(n,k). \end{aligned}$$

It can be verified that

$$\left| \frac{\theta_1(n,k)}{A(n,k)} \right| \leqslant \frac{3}{k+3} \leqslant \frac{1}{3}.$$

Set

$$f_1(x) = \begin{cases} (1+x)^{\frac{1}{x}}, & x \in (-1,+\infty) \setminus \{0\}, \\ e, & x = 0. \end{cases}$$

Then f_1 is strictly decreasing on $(-1,+\infty)$. Thus there exist positive numbers δ and Δ which are universal lower and upper bound of

$$\frac{(A(n,k)+\theta_1(n,k))^{A(n,k)}}{(A(n,k))^{A(n,k)}} = \left(f_1\left(\frac{\theta_1(n,k)}{A(n,k)}\right)\right)^{\theta_1(n,k)},$$

respectively. Since $k/2 < j < k$, we have

$$k^{-\frac{3}{2}} \leqslant (A(n,k)+\theta_1(n,k))^{\theta_1(n,k)} \leqslant k^{\frac{1}{2}}.$$

Then it is clear that $\lim_{n\to\infty} \sqrt[n]{\theta_3(n,k)} = 1$. Similarly, there exist $\theta_4(n,k)$ and $\theta_5(n,k)$ satisfying

$$\begin{aligned}(n-j)^{n-j} &= (B(n,k))^{B(n,k)} \cdot \theta_4(n,k),\\ (k-j)^{k-j} &= (C(n,k))^{C(n,k)} \cdot \theta_5(n,k),\\ \lim_{n\to\infty} \sqrt[n]{\theta_4(n,k)} &= \lim_{n\to\infty} \sqrt[n]{\theta_5(n,k)} = 1.\end{aligned}$$

Then, by (2.6), we have

$$\begin{aligned}b(n,k,j) =&\theta_2(n,k,j)\cdot 2^{\theta_1(n,k)}\cdot 2^{A(n,k)}\\ &\cdot \frac{n^n}{\left[(A(n,k))^{A(n,k)}\cdot\theta_3(n,k)\right]\cdot\left[(B(n,k))^{B(n,k)}\cdot\theta_4(n,k)\right]}\\ &\cdot \frac{k^k}{\left[(A(n,k))^{A(n,k)}\cdot\theta_3(n,k)\right]\cdot\left[(C(n,k))^{C(n,k)}\cdot\theta_5(n,k)\right]}\\ =&\theta_2(n,k,j)\cdot 2^{\theta_1(n,k)}\cdot\frac{1}{(\theta_3(n,k))^2\cdot\theta_4(n,k)\cdot\theta_5(n,k)}\cdot\left(f\left(\frac{k}{n}\right)\right)^n.\end{aligned}$$

It follows that

$$\theta(n,k) = \theta_2(n,k,j)\cdot 2^{\theta_1(n,k)}\cdot\frac{1}{(\theta_3(n,k))^2\cdot\theta_4(n,k)\cdot\theta_5(n,k)}.$$

Therefore, $\lim_{n\to\infty} \sqrt[n]{\theta(n,k)} = 1$. □

LEMMA 2.7. *We have*

$$\lim_{n\to\infty}\frac{k_1(n)}{n} = a(2) \approx 0.2140287.$$

Proof. By (d) and (e) in Lemma 2.3, $6 \leqslant k_1(n) < \lfloor n/4 \rfloor$ holds for large n. On the one hand, by $b(n,k_1(n),\lfloor j(n,k_1(n))\rfloor) \leqslant 2^n$, we have

$$\sqrt[n]{\theta(n,k_1(n))}\cdot f\left(\frac{k_1(n)}{n}\right) \leqslant 2,$$

or, equivalently,

$$\frac{k_1(n)}{n} \leqslant f^{-1}\left(\frac{2}{\sqrt[n]{\theta(n,k_1(n))}}\right).$$

Therefore,

$$\limsup_{n\to\infty}\frac{k_1(n)}{n} \leqslant a(2).$$

On the other hand, by

$$\begin{aligned} m(n,k) &= 1+\binom{n}{1}\cdot 2+\binom{n}{2}\cdot\binom{k-1}{1}\cdot 2^2+\cdots+\binom{n}{k}\cdot\binom{k-1}{k-1}\cdot 2^k \\ &\leqslant (k+1)(b(n,k,\lfloor j(n,k)\rfloor)+b(n,k,\lfloor j(n,k)\rfloor+1)) \\ &\leqslant (k+1)(nk+1)b(n,k,\lfloor j(n,k)\rfloor) \\ &< n^3 b(n,k,\lfloor j(n,k)\rfloor), \end{aligned}$$

we have

$$\begin{aligned} 2^n &< n^3\cdot b(n,k_1(n)+1,\lfloor j(n,k_1(n)+1)\rfloor) \\ &= n^3\cdot\theta(n,k_1(n)+1)\cdot\left(f\left(\frac{k_1(n)+1}{n}\right)\right)^n. \end{aligned}$$

Hence

$$\frac{k_1(n)+1}{n} > f^{-1}\left(\frac{2}{\sqrt[n]{n^3\cdot\theta(n,k_1(n)+1)}}\right).$$

Therefore,

$$\liminf_{n\to\infty}\frac{k_1(n)}{n}\geqslant a(2).$$

This completes the proof. □

COROLLARY 2.8. *We have*

$$\limsup_{n\to\infty}\Gamma_{2^n}(B_p^n)\leqslant\left(\frac{1}{1+a(2)}\right)^{\frac{1}{p}}.$$

Proof of Theorem 1.1. We claim that, for each integer $n\in[2,49]$, we have

$$\Gamma_{2^n}(B_p^n)\leqslant\left(\frac{5}{6}\right)^{\frac{1}{p}}.$$

If $n=2$, then, since each planar convex body K can be covered by 4 translates of $(\sqrt{2}/2)K$ (cf. [8]),

$$\Gamma_{2^n}(B_p^n)\leqslant\frac{\sqrt{2}}{2}<\frac{5}{6}\leqslant\left(\frac{5}{6}\right)^{\frac{1}{p}},\ \forall p\geqslant 1.$$

The case when $n\in[3,49]\setminus\{6\}$ can be seen from Table 1. Now suppose that $n=6$. If $p\in[1,2]$, by Lemma 3.1 below, we have

$$\Gamma_{2^n}(B_p^n)\leqslant\Gamma_{2n}(B_p^n)\leqslant\left(\frac{5}{6}\right)^{\frac{1}{p}};$$

n	$k_1(n)$	$\frac{n}{n+k_1(n)}$	n	$k_1(n)$	$\frac{n}{n+k_1(n)}$	n	$k_1(n)$	$\frac{n}{n+k_1(n)}$
3	1	0.75	4	1	0.8	5	1	0.83333
6	1	0.85714	7	2	0.77778	8	2	0.8
9	2	0.81818	10	2	0.83333	11	3	0.78571
12	3	0.80000	13	3	0.8125	14	3	0.82353
15	3	0.83333	16	4	0.80000	17	4	0.80952
18	4	0.81818	19	4	0.82609	20	5	0.80000
21	5	0.80769	22	5	0.81481	23	5	0.82143
24	5	0.82579	25	6	0.80645	26	6	0.81250
27	6	0.81818	28	6	0.82353	29	7	0.80556
30	7	0.81081	31	7	0.81579	32	7	0.82051
33	7	0.825	34	8	0.80952	35	8	0.81395
36	8	0.81818	37	8	0.82222	38	9	0.80851
39	9	0.8125	40	9	0.81633	41	9	0.82
42	9	0.82353	43	10	0.81132	44	10	0.81482
45	10	0.81818	46	10	0.82143	47	11	0.81035
48	11	0.81356	49	11	0.81667	50	11	0.81967

Table 1: *Estimates of $n/(n+k_1(n))$ in low dimensions.*

if $p \geqslant 2$, then, by Lemma 3.4 below, we have

$$\Gamma_{2^n}(B_p^n) \leqslant \left(\frac{5}{6}\right)^{\frac{1}{2}} \leqslant \left(\frac{5}{6}\right)^{\frac{1}{p}}.$$

This proves the claim.

Set $c = b(2) - 0.02$. For each $n \geqslant 50$, we have $cn \leqslant b(2)n - 1$, which shows that $(1+c)n \leqslant n + \lfloor b(2)n \rfloor$. By [9, Proposition 5], we have

$$\Gamma_{2^n}(B_p^n) \leqslant \left(\frac{n}{n+\lfloor b(2)n \rfloor}\right)^{\frac{1}{p}} \leqslant \left(\frac{n}{(1+c)n}\right)^{\frac{1}{p}} = \left(\frac{1}{b(2)+0.98}\right)^{\frac{1}{p}} \approx 0.8435^{\frac{1}{p}}.$$

Thus, for each $n \geqslant 3$, we have

$$\Gamma_{2^n}(B_p^n) \leqslant \max\left\{\left(\frac{5}{6}\right)^{\frac{1}{p}}, \left(\frac{1}{b(2)+0.98}\right)^{\frac{1}{p}}\right\} = \left(\frac{1}{b(2)+0.98}\right)^{\frac{1}{p}}. \quad \square$$

3. Further covering methods

In this section we present two elementary configurations of smaller balls to cover B_p^n. They yield good upper bounds for $\Gamma_{2^n}(B_p^n)$ when n is small.

LEMMA 3.1. *If $p \in [1,\infty]$ and $n \geqslant 2$, then*

$$\Gamma_{2n}(B_p^n) \leqslant \left(1-\frac{1}{n}\right)^{\frac{1}{p}}.$$

Proof. Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{R}^n$. For each $i \in [n]$, set

$$c_i^+ = \left(\frac{1}{n}\right)^{\frac{1}{p}} e_i, \quad c_i^- = -\left(\frac{1}{n}\right)^{\frac{1}{p}} e_i,$$

$$H_i^+ = \left\{ x \in \mathbb{R}^n \mid \langle x|e_i\rangle \geqslant \left(\frac{1}{n}\right)^{\frac{1}{p}} \right\},$$

and

$$H_i^- = \left\{ x \in \mathbb{R}^n \mid \langle x|e_i\rangle \leqslant -\left(\frac{1}{n}\right)^{\frac{1}{p}} \right\}.$$

Clearly,

$$S_p^n = \bigcup_{i\in[n]} \left[(S_p^n \cap H_i^+) \cup (S_p^n \cap H_i^-)\right].$$

Let x be an arbitrary point in B_p^n. If $x = o$, then

$$\|x-c_1^+\|_p = \|c_1^+\|_p = \left(\frac{1}{n}\right)^{\frac{1}{p}} \leqslant \left(1-\frac{1}{n}\right)^{\frac{1}{p}}.$$

Otherwise, assume without loss of generality that, $y = x/\|x\|_p \in S_n^p \cap H_1^+$. Then there exists $z \in S_p^n$ satisfying $\langle z|e_1\rangle = (1/n)^{1/p}$ such that

$$y \in \left\{ \frac{\lambda e_1 + (1-\lambda)z}{\|\lambda e_1 + (1-\lambda)z\|_p} \mid \lambda \in [0,1] \right\}.$$

By [11, Lemma 2.1], we have

$$\|y-c_1^+\|_p \leqslant \|z-c_1^+\|_p = \left(1-\frac{1}{n}\right)^{\frac{1}{p}}.$$

By [10, Lemma 5],

$$\begin{aligned}\|x-c_1^+\|_p &\leqslant \max\left\{\|y-c_1^+\|_p, \|c_1^+\|_p\right\}\\ &\leqslant \max\left\{\left(1-\frac{1}{n}\right)^{\frac{1}{p}}, \left(\frac{1}{n}\right)^{\frac{1}{p}}\right\} = \left(1-\frac{1}{n}\right)^{\frac{1}{p}}.\end{aligned}$$

Thus

$$B_p^n \subseteq \bigcup_{i\in[n]} \left[\left(c_i^+ + \left(1-\frac{1}{n}\right)^{\frac{1}{p}} B_p^n\right) \cup \left(c_i^- + \left(1-\frac{1}{n}\right)^{\frac{1}{p}} B_p^n\right)\right]. \quad \square$$

LEMMA 3.2. *Let* $p \geqslant 2$, $\lambda \in \left(0, (1/n)^{1/p}\right)$, $c_\lambda = (\lambda, \ldots, \lambda) \in \mathbb{R}^n$, *and*

$$D = \left\{(\alpha_1, \cdots, \alpha_n) \mid \sum_{i\in[n]} \alpha_i^p \leqslant 1;\ \alpha_i \geqslant 0,\ \forall i \in [n]\right\}.$$

Then

$$M(\lambda) := \max\left\{\|c_\lambda - x\|_p^p \mid x \in D\right\} = \max\left\{(1-\lambda)^p + (n-1)\lambda^p, n\lambda^p\right\}.$$

Proof. Let x be an arbitrary point in $D \setminus \{c_\lambda\}$, y be the point of intersection of the ray $[c_\lambda, x\rangle$ and the boundary of D. First suppose that $\|y\|_p < 1$. If y is the origin o, then

$$\|c_\lambda - x\|_p^p \leqslant \|c_\lambda - y\|_p^p = n\lambda^p.$$

Otherwise, let $z = y/\|y\|_p$. Then, by [10, Lemma 5], we have

$$\|c_\lambda - x\|_p^p \leqslant \|c_\lambda - y\|_p^p \leqslant \max\left\{\|c_\lambda - o\|_p^p, \|c_\lambda - z\|_p^p\right\} \leqslant \max\left\{\|c_\lambda - o\|_p^p, \gamma\right\},$$

where

$$\gamma := \max\left\{\|c_\lambda - w\|_p^p \mid w \in D \cap S_p^n\right\}.$$

Therefore, to complete the proof, we only need to show that $\gamma = (1-\lambda)^p + (n-1)\lambda^p$. Clearly,

$$\gamma \geqslant (1-\lambda)^p + (n-1)\lambda^p.$$

Assume that γ is attained at $w_0 = (\alpha_1^0, \ldots, \alpha_n^0) \in D \cap S_p^n$. Set

$$J = \left\{i \in [n] \mid \alpha_i^0 \neq 0\right\} \quad \text{and} \quad k := \#J.$$

Clearly, $k > 0$. Without loss of generality, we may assume that $J = [k]$. Then $(\alpha_1^0, \ldots, \alpha_n^0)$ is a solution to the optimization problem

$$\begin{aligned} \max \quad & |\alpha_1 - \lambda|^p + \cdots + |\alpha_k - \lambda|^p + (n-k)\lambda^p \\ \text{s.t.} \quad & \sum_{i\in[k]} \alpha_i^p = 1, \\ & \alpha_i \geqslant 0,\ \forall i \in [k]. \end{aligned}$$

Let

$$L(\alpha_1, \ldots, \alpha_k) = |\alpha_1 - \lambda|^p + \cdots + |\alpha_k - \lambda|^p + (n-k)\lambda^p + \mu(\alpha_1^p + \cdots + \alpha_k^p - 1).$$

By the Lagrange multiplier method, $\left(\alpha_1^0, \cdots, \alpha_k^0\right)$ is a solution to the following system of equations

$$\frac{\partial L}{\partial \alpha_i} = p \cdot |\alpha_i - \lambda|^{p-1} \cdot \operatorname{sgn}(\alpha_i - \lambda) + \mu \cdot p \cdot \alpha_i^{p-1} = 0, \ \forall i \in [k], \tag{3.1}$$

$$\alpha_1^p + \cdots + \alpha_k^p - 1 = 0. \tag{3.2}$$

By (3.1), we have

$$\left|\frac{\alpha_i^0 - \lambda}{\alpha_i^0}\right|^{p-1} \cdot \operatorname{sgn}(\alpha_i^0 - \lambda) = -\mu, \forall i \in [k].$$

Then

$$\frac{\alpha_1^0 - \lambda}{\alpha_1^0} = \cdots = \frac{\alpha_k^0 - \lambda}{\alpha_k^0} \neq 1.$$

Therefore $\alpha_1^0 = \cdots = \alpha_k^0$. By (3.2), we have

$$\alpha_1^0 = \cdots = \alpha_k^0 = \left(\frac{1}{k}\right)^{\frac{1}{p}}.$$

Note that

$$\left(\frac{1}{k}\right)^{\frac{1}{p}} \geqslant \left(\frac{1}{n}\right)^{\frac{1}{p}} > \lambda.$$

Thus

$$\begin{aligned} \gamma = \|c_\lambda - w_0\|_p &= k\left(\left(\frac{1}{k}\right)^{\frac{1}{p}} - \lambda\right)^p + (n-k)\lambda^p \\ &= (1 - \lambda \cdot k^{\frac{1}{p}})^p + (n-k)\lambda^p \\ &\leqslant (1-\lambda)^p + (n-1)\lambda^p. \quad \square \end{aligned}$$

LEMMA 3.3. *If $n \geqslant 3$ and $p \geqslant 2$, then*

$$1 + (n-1)^{\frac{1}{p-1}} > n^{\frac{1}{p}}.$$

Proof. Let

$$g(x) = 1 + (x-1)^{\frac{1}{p-1}} - x^{\frac{1}{p}}, \ \forall x \in [3, \infty).$$

Then,

$$g'(x) = \frac{1}{p-1} \cdot \frac{1}{(x-1)^{\frac{p-2}{p-1}}} - \frac{1}{p} \cdot \frac{1}{x^{\frac{p-1}{p}}}.$$

Since

$$\frac{1}{p-1} > \frac{1}{p} > 0 \quad \text{and} \quad 0 \leqslant \frac{p-2}{p-1} < \frac{p-1}{p},$$

we have

$$(x-1)^{\frac{p-2}{p-1}} < (x-1)^{\frac{p-1}{p}} < x^{\frac{p-1}{p}}.$$

Thus $g'(x) > 0$. Consequently,

$$g(n) \geqslant g(3) = 1 + 2^{\frac{1}{p-1}} - 3^{\frac{1}{p}} > 1 + 2^{\frac{1}{p}} - 3^{\frac{1}{p}}.$$

Set $h(t) = 1 + 2^t - 3^t$, $\forall t \in (0, 1/2]$. Since

$$h'(t) = 2^t \cdot \ln 2 - 3^t \cdot \ln 3 = 2^t \cdot \ln 3 \cdot \left(\frac{\ln 2}{\ln 3} - \left(\frac{3}{2}\right)^t \right) < 0,$$

we have $h(t) \geqslant h(1/2) = 1 + \sqrt{2} - \sqrt{3} > 0$. This completes the proof. □

LEMMA 3.4. *If $p \geqslant 2$ and $n \geqslant 3$, then*

$$\Gamma_{2^n}(B_p^n) \leqslant \gamma(n,p) := \frac{(n-1)^{\frac{1}{p}}}{\left(1 + (n-1)^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}}.$$

Moreover, $\gamma(n,p)$ is strictly decreasing on $[2,\infty)$ with respect to p.

Proof. Take

$$\lambda = \frac{1}{1 + (n-1)^{\frac{1}{p-1}}}$$

in Lemma 3.2. Then $\lambda < 1/2$, which implies that $(1-\lambda)^p > \lambda^p$. It follows that

$$(1-\lambda)^p + (n-1)\lambda^p > n\lambda^p.$$

Thus

$$\begin{aligned} M(\lambda) =& (1-\lambda)^p + (n-1)\lambda^p \\ =& \left(\frac{(n-1)^{\frac{1}{p-1}}}{1 + (n-1)^{\frac{1}{p-1}}} \right)^p + \left(\frac{1}{1 + (n-1)^{\frac{1}{p-1}}} \right)^p \cdot (n-1) \\ =& \frac{(n-1)\left(1 + (n-1)^{\frac{1}{p-1}}\right)}{\left(1 + (n-1)^{\frac{1}{p-1}}\right)^p} = \gamma(n,p)^p. \end{aligned}$$

This means that the portion of B_p^n in the nonnegative orthant can be covered by a ball (with respect to $\|\cdot\|_p$) having radius $\gamma(n,p)$. Therefore, $\Gamma_{2^n}(B_p^n) \leqslant \gamma(n,p)$.

Let

$$L = L(p) = \frac{\ln(n-1)}{p} - \frac{p-1}{p} \cdot \ln\left[1 + (n-1)^{\frac{1}{p-1}}\right].$$

Then

$$\frac{\mathrm{d}L}{\mathrm{d}p} = -\frac{1}{p^2}\left[\ln(1+(n-1)^{\frac{1}{p-1}}) - \ln(n-1)^{\left(\frac{p}{p-1}\cdot\frac{(n-1)^{\frac{1}{p-1}}}{1+(n-1)^{\frac{1}{p-1}}}-1\right)}\right].$$

It is not difficult to verify that

$$\frac{1}{p-1} > \frac{p}{p-1}\cdot\frac{(n-1)^{\frac{1}{p-1}}}{1+(n-1)^{\frac{1}{p-1}}} - 1.$$

Hence $\mathrm{d}L\big/\mathrm{d}p < 0$. This completes the proof. □

Proof of Theorem 1.2. The case when $n = 2$ follows from the fact that (as we have mentioned in the proof of Theorem 1.1)

$$\Gamma_4(B_p^2) \leqslant \frac{\sqrt{2}}{2},\ \forall p \in [1,\infty].$$

The case when $n \geqslant 3$ follows from Lemma 3.1 and Lemma 3.4. □

LEMMA 3.5. *If $\alpha > 1$, then $\phi_\alpha(t) = (\alpha^t - 1)^{\frac{1}{t}}$ is strictly increasing on $(0,\infty)$.*

Proof. Clearly,

$$\frac{\mathrm{d}\ln\phi_\alpha(t)}{\mathrm{d}t} = \frac{\frac{\alpha^t\cdot\ln\alpha}{\alpha^t-1}\cdot t - \ln(\alpha^t-1)}{t^2} = \frac{\alpha^t\cdot\ln\alpha\cdot t - (\alpha^t-1)\cdot\ln(\alpha^t-1)}{t^2\cdot(\alpha^t-1)}.$$

Set

$$\psi(t) = \alpha^t\cdot\ln\alpha\cdot t - (\alpha^t-1)\cdot\ln(\alpha^t-1),\ \forall t\in(0,+\infty).$$

Then $\psi(0^+) = 0$ and

$$\begin{aligned}\psi'(t) &= \ln\alpha\left[\alpha^t + t\cdot\alpha^t\cdot\ln\alpha\right] - \left(1+\ln(\alpha^t-1)\right)\cdot\alpha^t\cdot\ln\alpha\\ &= \ln\alpha\left[\alpha^t\ln\alpha^t - \alpha^t\cdot\ln(\alpha^t-1)\right] > 0.\end{aligned}$$

It follows that $\psi(t) > 0$, $\forall t > 0$. Therefore $\phi_\alpha(t)$ is strictly increasing on $(0,\infty)$ as claimed. □

LEMMA 3.6. *Let ϕ_α be defined as in Lemma 3.5. We have*

$$\lim_{n\to\infty}\frac{\ln n}{\ln(\ln n)}\cdot\phi_\alpha^{-1}\left(\frac{1}{n}\right) = 1.$$

Proof. Let $t(n) := \phi_\alpha^{-1}\left(\frac{1}{n}\right)$. Then $t(n) > 0$, $\forall n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} t(n) = 0$. Now,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln n}{\ln(\ln n)} \cdot \phi_\alpha^{-1}\left(\frac{1}{n}\right) &= \lim_{t\to 0^+} \frac{\ln \frac{1}{\phi_\alpha(t)}}{\ln\left[\ln \frac{1}{\phi_\alpha(t)}\right]} \cdot t \\
&= \lim_{t\to 0^+} \frac{-t \cdot \frac{\ln(\alpha^t - 1)}{t}}{\ln\left(-\frac{\ln(\alpha^t-1)}{t}\right)} \\
&= \lim_{t\to 0^+} \frac{-\ln(\alpha^t - 1)}{\ln[-\ln(\alpha^t - 1)] - \ln t}.
\end{aligned}$$

The desired equality follows directly from

$$\lim_{t\to 0^+} \frac{\ln(\alpha^t - 1)}{\ln t} = \lim_{t\to 0^+} \frac{\frac{1}{\alpha^t - 1} \cdot \alpha^t \cdot \ln\alpha}{\frac{1}{t}} = 1,$$

and

$$\lim_{t\to 0^+} \frac{\ln(-\ln(\alpha^t - 1))}{\ln t} = \lim_{t\to 0^+} \frac{\frac{1}{-\ln(\alpha^t-1)} \cdot \frac{-1}{\alpha^t - 1} \cdot \alpha^t \cdot \ln\alpha}{\frac{1}{t}} = 0. \qquad \square$$

Assume that $n \geqslant 10$. It is clear that $\left(\frac{n}{n+\lfloor b(2)n\rfloor}\right)^{\frac{1}{p}}$ is strictly increasing with respect to p. It can be verified that

$$\left(\frac{n}{n+\lfloor b(2)n\rfloor}\right)^{\frac{1}{2}} < \frac{(n-1)^{\frac{1}{2}}}{[1+(n-1)]^{\frac{1}{2}}} \iff \frac{n}{n-1} < \lfloor b(2)n\rfloor$$

and that

$$\lim_{p\to+\infty} \gamma(n,p) = \frac{1}{2}, \quad \lim_{p\to+\infty} \left(\frac{n}{n+\lfloor b(2)n\rfloor}\right)^{\frac{1}{p}} = 1.$$

Thus there is a unique number $p_1(n) \in (2, +\infty)$ satisfying

$$\gamma(n, p_1(n)) = \left(\frac{n}{n+\lfloor b(2)n\rfloor}\right)^{\frac{1}{p_1(n)}}. \tag{3.3}$$

Proof of Theorem 1.3. The first inequality follows directly from Lemma 3.4 and the definition of $p_1(n)$.

Set $p = p_1(n)$. Then

$$1 + \left(\frac{1}{n-1}\right)^{\frac{1}{p-1}} = \left(\frac{n+\lfloor b(2)n\rfloor}{n}\right)^{\frac{1}{p-1}}. \tag{3.4}$$

Since

$$\frac{n+b(2)n-1}{n} < \frac{n+\lfloor b(2)n\rfloor}{n} \leqslant \frac{n+b(2)n}{n},$$

we have

$$1+b(2)-\frac{1}{10}<\frac{n+\lfloor b(2)n\rfloor}{n}\leqslant 1+b(2),$$

or, equivalently,

$$(0.9+b(2))^{\frac{1}{p-1}}<\left(\frac{n+\lfloor b(2)n\rfloor}{n}\right)^{\frac{1}{p-1}}\leqslant (1+b(2))^{\frac{1}{p-1}}.$$

By (3.4), we have

$$(0.9+b(2))^{\frac{1}{p-1}}-1<\left(\frac{1}{n-1}\right)^{\frac{1}{p-1}}\leqslant (1+b(2))^{\frac{1}{p-1}}-1.$$

It follows that

$$\phi_{0.9+b(2)}\left(\frac{1}{p-1}\right)<\frac{1}{n-1}\leqslant \phi_{1+b(2)}\left(\frac{1}{p-1}\right).$$

Therefore,

$$\phi^{-1}_{1+b(2)}\left(\frac{1}{n-1}\right)<\frac{1}{p-1}\leqslant \phi^{-1}_{0.9+b(2)}\left(\frac{1}{n-1}\right).$$

By Lemma 3.6, the proof is complete. □

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