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CESÀRO–LIKE OPERATOR ACTING BETWEEN BLOCH TYPE SPACES

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Abstract. Let μ be a finite positive Borel measure on the interval $[0,1)$ and $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$. The Cesàro-like operator is defined by

$$\mathscr{C}_\mu(f)(z)=\sum_{n=0}^{\infty}\left(\mu_n\sum_{k=0}^{n}a_k\right)z^n,\ z\in\mathbb{D},$$

where, for $n\geqslant 0$, μ_n denotes the n-th moment of the measure μ, that is, $\mu_n=\int_{[0,1)}t^n d\mu(t)$.

In this paper we investigate the action of the operators $\mathscr{C}_\mu$ from one Bloch type spaces $\mathscr{B}^\alpha$ into another one $\mathscr{B}^\beta$.

1. Introduction

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$.

For $0<\alpha<\infty$, the Bloch-type space, denoted by $\mathscr{B}^\alpha$, is defined as

$$\mathscr{B}^\alpha=\{f\in H(\mathbb{D}):||f||_{\mathscr{B}^\alpha}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|<\infty\}.$$

When $\alpha=1$, $\mathscr{B}^\alpha$ is just the classic Bloch space $\mathscr{B}$.

For $0<\alpha<1$, the analytic Lipschitz space Λ_α consists of the functions $f\in H(\mathbb{D})$ for which

$$||f||_{\Lambda_\alpha}=\sup\left\{\frac{|f(z)-f(w)|}{|z-w|^\alpha}:z,w\in\mathbb{D},z\neq w\right\}<\infty.$$

It is known that (see [13]) $\mathscr{B}^\alpha\cong\Lambda_{1-\alpha}$ for $0<\alpha<1$ and Λ_α is contained in the disc algebra.

For $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$, the Cesàro operator $\mathscr{C}$ is defined by

$$\mathscr{C}(f)(z)=\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}a_k\right)z^n,\ \ z\in\mathbb{D}.$$

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The boundedness of the Cesàro operator has been studied by several authors on certain spaces of analytic functions. See, e.g., [12, 14, 25, 26, 28, 29, 30, 39] and the references therein. The Cesàro operator $\mathscr{C}$ has also been generalized to various forms and its generalization has been widely studied on the space of holomorphic functions. For instance, Stević [32] studied the generalized Cesàro operator on the polydisc. Hu [20] studied the extended Cesàro operators on the Bloch space in the unit ball of $\mathbb{C}^n$. Stević also considered the generalized Cesàro operator on weighted-type spaces in [35] and studied the generalized Cesàro operators acting between Bloch type spaces in [34]. For more information on some generalizations of the Cesàro operator on spaces of holomorphic functions, the reader is referred to [1, 2, 10, 11, 23, 31, 33].

Recently, Galanopoulos, Girela and Merchán [16] introduced a Cesàro-like operator $\mathscr{C}_\mu$ on $H(\mathbb{D})$, which is a natural generalization of the classical Cesàro operator $\mathscr{C}$. They systemically studied the operators $\mathscr{C}_\mu$ acting on distinct spaces of analytic functions, such as Hardy space, Bergman space, Bloch space.

Let μ be a positive finite Borel measure on $[0,1)$ and $f(z)=\sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$. The Cesàro-like operator $\mathscr{C}_\mu$ is defined as follows:

$$\mathscr{C}_\mu(f)(z)=\sum_{n=0}^{\infty}\left(\mu_n \sum_{k=0}^{n} a_k\right) z^n=\int_0^1 \frac{f(tz)}{1-tz} d\mu(t), \quad z \in \mathbb{D}.$$

where μ_n denote the moment of order n of μ, that is, $\mu_n=\int_0^1 t^n d\mu(t)$. If μ is the Lebesgue measure on $[0,1)$, the operator $\mathscr{C}_\mu$ reduces to the classical Cesàro operator $\mathscr{C}$.

The Cesàro-like operator $\mathscr{C}_\mu$ can be regarded as an operator induced by the matrix

$$\mathscr{C}_\mu=\begin{pmatrix} \mu_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & \mu_1 & 0 & 0 & \cdots \\ \mu_2 & \mu_2 & \mu_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$$\begin{pmatrix} \mu_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & \mu_1 & 0 & 0 & \cdots \\ \mu_2 & \mu_2 & \mu_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}=\begin{pmatrix} \mu_0 a_0 \\ \mu_1 \sum_{k=0}^{1} a_k \\ \mu_2 \sum_{k=0}^{2} a_k \\ \vdots \end{pmatrix}.$$

The Cesàro-like operator $\mathscr{C}_\mu$ defined above has attracted the interest of many mathematicians. Jin and Tang [21] studied the boundedness and compactness of $\mathscr{C}_\mu$ from one Dirichlet-type space $\mathscr{D}_\alpha$ into another one $\mathscr{D}_\beta$. Bao, Sun and Wulan [3] studied the range of $\mathscr{C}_\mu$ acting on H^∞. Blasco [8, 9] investigated the operators $\mathscr{C}_\mu$ on Hardy spaces and on weighted Dirichlet spaces induce by complex Borel measures on $[0,1)$. Galanopoulos, Girela et al. [15] studied the behaviour of the operators $\mathscr{C}_\mu$ on the Dirichlet space and on the analytic Besov spaces. Recently, Sun, Ye et al. [36] studied the operator $\mathscr{C}_\mu$ from Besov spaces to X, where X is a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^s_{\frac{1}{s}} \subseteq X \subseteq \mathscr{B}$. Bao, Guo et al. [6] completely characterized the measures μ such that $\mathscr{C}_\mu$ is bounded (compact) on Dirichlet space. In [19], the authors of this paper also considered the boundedness and compactness of $\mathscr{C}_\mu$ between Bergman space and Bloch space. Beltrán-Meneu, Bonet and Jordá [7] systematically investigated the operator $\mathscr{C}_\mu$ on weighted Banach spaces of analytic function. The operators $\mathscr{C}_\mu$ associated to arbitrary complex Borel measures on $\mathbb{D}$ the reader is referred to [17, 42].

The Bolch type spaces $\mathscr{B}^\alpha$ are closely connected to many analytic function spaces, such as Bergman space, Korenblum space, Lipschitz space, $F(p,q,s)$ space, mixed norm space et al. Therefore, the operator $\mathscr{C}_\mu$ acting between Bloch type spaces can serve as a good model when we study the operator $\mathscr{C}_\mu$ on the spaces. In this paper we study the action of the operator $\mathscr{C}_\mu$ between Bloch type spaces. The operator $\mathscr{C}_\mu$ on such spaces do not seem to have been studied extensively in the past, so we have attempted to collect here the consequences of applying to them various standard techniques of analysis.

The Carleson-type measures play a basic role in the studies of $\mathscr{C}_\mu$. Let $I \subset \partial\mathbb{D}$ be an arc, and $|I|$ denote the length of I. The Carleson square $S(I)$ is defined as

$$S(I) = \left\{ re^{i\vartheta} : e^{i\vartheta} \in I,\ 1 - \frac{|I|}{2\pi} \leqslant r < 1 \right\}.$$

Let μ be a positive Borel measure on $\mathbb{D}$. For $0 \leqslant \beta < \infty$ and $0 < t < \infty$, we say that μ is a β-logarithmic t-Carleson measure (resp. a vanishing β-logarthmic t-Carleson measure) if

$$\sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))(\log \frac{2\pi}{|I|})^\beta}{|I|^t} < \infty, \quad \left(\text{resp. } \lim_{|I| \to 0} \frac{\mu(S(I))(\log \frac{2\pi}{|I|})^\beta}{|I|^t} = 0 \right).$$

If $\beta = 0$ and $t = 1$, we say that μ is a Carleson measure. See [41] for more about logarithmic type Carleson measure.

A positive Borel measure μ on $[0,1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\overline{\mu}$ defined by

$$\overline{\mu}(E) = \mu(E \cap [0,1)), \text{ for any Borel subset } E \text{ of } \mathbb{D}.$$

In this way, a positive Borel measure μ on $[0,1)$ is a β-logarithmic t-Carleson measure if and only if there exists a constant $M > 0$ such that

$$\mu([s,1)) \log^\beta \frac{e}{1-s} \leqslant M(1-s)^t, \quad 0 \leqslant s < 1.$$

Throughout the paper, the letter C will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation "$P \lesssim Q$" if there exists a constant $C = C(\cdot)$ such that "$P \leqslant CQ$", and "$P \gtrsim Q$" is understood in an analogous manner. In particular, if "$P \lesssim Q$" and "$P \gtrsim Q$", then we will write "$P \asymp Q$".

2. Preliminaries

In this section, we present some preliminary results needed for the rest of the paper. We start with the following lemma which can be found, for example, in [43].

LEMMA 1. *Let* $0<\alpha<\infty$ *and* $f\in\mathscr{B}^{\alpha}$. *Then for each* $z\in\mathbb{D}$, *we have the following inequalities:*

$$|f(z)|\lesssim\begin{cases}||f||_{\mathscr{B}^{\alpha}},\ \text{if}\ 0<\alpha<1;\\ ||f||_{\mathscr{B}^{\alpha}}\log\frac{2}{1-|z|},\ \text{if}\ \alpha=1;\\ \frac{||f||_{\mathscr{B}^{\alpha}}}{(1-|z|)^{\alpha-1}},\ \text{if}\ \alpha>1.\end{cases}$$

The following result follows from Corollary 3.2 in [37] or Theorem 2.26 in [38].

LEMMA 2. *Let* $\alpha>0$ *and* $f\in H(\mathbb{D})$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $a_n\geqslant 0$ *for all* $n\geqslant 0$. *Then* $f\in\mathscr{B}^{\alpha}$ *if and only if*

$$\sup_{n\geqslant 1}n^{-\alpha}\sum_{k=1}^{n}ka_k<\infty.$$

The following result follows from Theorem 2.1 and Theorem 2.4 in [4].

LEMMA 3. *Let* $0<s<\infty$ *and* μ *be a finite positive Borel measure on the interval* $[0,1)$. *Then the following statements hold:*
(1) μ *is an* s*-Carleson measure if and only if* $\mu_n=O(\frac{1}{n^s})$.
(2) μ *is a vanishing* s*-Carleson measure if and only if* $\mu_n=o(\frac{1}{n^s})$.

The following integral estimates are practical. Although we only use a partial case in this article, we present a complete result here for the reader's reference.

LEMMA 4. *Let* $\delta>-1$, $c\geqslant 0$ *and* k *be a real number. Then the integral*

$$I_r=\int_0^1\frac{(1-t)^{\delta}}{(1-tr)^{\delta+c+1}}\log^k\frac{e}{1-t}dt,\ \ (0\leqslant r<1)$$

have the following asymptotic properties:
(1) If $c=0$ *and* $k<-1$, *then* $I_r\asymp 1$;
(2) If $c=0$ *and* $k=-1$, *then* $I_r\asymp\log\log\frac{e^2}{1-r}$;
(3) If $c=0$ *and* $k>-1$, *then* $I_r\asymp\log^{k+1}\frac{e}{1-r}$;
(4) If $c>0$, *then* $I_r\asymp\frac{1}{(1-r)^c}\log^k\frac{e}{1-r}$.

Proof. The proof of (3)–(4) is stated in [40, Lemma 2.2]. We just need to consider the case $c=0$ and $k\leqslant -1$.

Without loss of generality, we may assume that $1-\frac{e}{8}<r<1$. Let $x=\frac{r(1-t)}{1-tr}$, then

$$\begin{aligned}\int_0^1 \frac{(1-t)^\delta}{(1-tr)^{\delta+1}}\log^k\frac{e}{1-t}dt &= \int_0^r \frac{x^\delta}{r^{\delta+1}(1-x)}\log^k\frac{er(1-x)}{(1-r)x}dx\\ &\asymp \int_0^{\frac12} x^\delta\log^k\frac{e}{(1-r)x}dx+\int_{\frac12}^r\frac{1}{1-x}\log^k\frac{e(1-x)}{1-r}dx\\ &= \frac{1}{(1-r)^{\delta+1}}\int_0^{\frac{1-r}{2}} y^\delta\log^k\frac{e}{y}dy+\int_{2(1-r)}^1\frac{1}{y}\log^k\frac{e}{y}dy.\end{aligned}$$

It is clear that

$$\lim_{r\to 1^-}\frac{\int_0^{\frac{1-r}{2}} y^\delta\log^k\frac{e}{y}dy}{(1-r)^{\delta+1}\log^k\frac{e}{1-r}}=\frac{1}{(\delta+1)2^{\delta+1}}.$$

This implies that

$$\frac{1}{(1-r)^{\delta+1}}\int_0^{\frac{1-r}{2}} y^\delta\log^k\frac{e}{y}dy\asymp\log^k\frac{e}{1-r}\ (r\to 1^-). \tag{2.1}$$

At the same time,

$$\int_{2(1-r)}^1\frac{1}{y}\log^{-1}\frac{e}{y}dy=\log\log\frac{e}{2(1-r)}\asymp\log\log\frac{e^2}{1-r}\ (r\to 1^-). \tag{2.2}$$

When $k<-1$, we have

$$\int_{2(1-r)}^1\frac{1}{y}\log^k\frac{e}{y}dy\leqslant\int_0^1\frac{1}{y}\log^k\frac{e}{y}dy=\frac{-1}{k+1}. \tag{2.3}$$

By (2.1)–(2.3) we may obtain that (1) and (2) hold. □

The following lemma is a direct consequence of Theorem 3.1 in [24].

LEMMA 5. *Let $0<\alpha,\beta<\infty$. Suppose T is a bounded operator from $\mathscr{B}^\alpha$ into $\mathscr{B}^\beta$, then T is a compact operator from $\mathscr{B}^\alpha$ into $\mathscr{B}^\beta$ if and only if for any bounded sequence $\{h_n\}$ in $\mathscr{B}^\alpha$ which converges to 0 uniformly on every compact subset of $\mathbb{D}$, we have $\lim_{n\to\infty}||T(h_n)||_{\mathscr{B}^\beta}=0$.*

3. The boundedness of $\mathscr{C}_\mu$ acting between Bloch type spaces

We now study the boundedness of $\mathscr{C}_\mu$ acting between Bloch type spaces.

THEOREM 1. *Let μ be a finite positive Borel measure on the interval $[0,1)$. If $0<\alpha<1$ and $0<\beta<2$, then the following conditions are equivalent.*
(1) $\mathscr{C}_\mu:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is bounded.
(2) $\mathscr{C}_\mu:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is compact.
(3) The measure μ is a $2-\beta$ Carleson measure.

Proof. The implication of $(2) \Rightarrow (1)$ is obvious.

$(1) \Rightarrow (3)$. Suppose $\mathscr{C}_\mu : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. Let $f(z) = \sum_{n=1}^{\infty} n^{\alpha-2} z^n$, then

$$|f'(z)| = \left| \sum_{n=0}^{\infty} (n+1)^{\alpha-1} z^n \right| \leqslant \sum_{n=0}^{\infty} (n+1)^{\alpha-1} |z|^n \lesssim \frac{1}{(1-|z|)^\alpha}.$$

This means that $f \in \mathscr{B}^\alpha$. Since

$$\mathscr{C}_\mu(f)(z) = \sum_{n=1}^{\infty} \mu_n \left(\sum_{k=1}^{n} k^{\alpha-2} \right) z^n \in \mathscr{B}^\beta$$

and the sequence $\left\{ \mu_n \left(\sum_{k=1}^{n} k^{\alpha-2} \right) \right\}_{n=1}^{\infty}$ is a nonnegative sequence, it follows from Lemma 2 that

$$\sup_{n \geqslant 1} n^{-\beta} \sum_{k=1}^{n} k \mu_k \left(\sum_{j=1}^{k} j^{\alpha-2} \right) < \infty.$$

Since $\alpha \in (0,1)$, for each $n \geqslant 1$, it follows that

$$\begin{aligned} 1 &\gtrsim n^{-\beta} \sum_{k=1}^{n} k \mu_k \left(\sum_{j=1}^{k} j^{\alpha-2} \right) \\ &\gtrsim \mu_n n^{-\beta} \sum_{k=1}^{n} k \asymp \mu_n n^{2-\beta}. \end{aligned}$$

Lemma 2.3 shows that μ is a $2-\beta$ Carleson measure.

$(3) \Rightarrow (1)$. Assume μ is a $2-\beta$ Carleson measure. Since $0 < \alpha < 1$, the integral $\int_0^1 \frac{dt}{(1-t)^\alpha} < \infty$. Thus, for any given $\varepsilon > 0$, there exists $0 < t_0 < 1$ such that

$$\int_{t_0}^{1} \frac{dt}{(1-t)^\alpha} < \varepsilon. \tag{3.1}$$

This also yields that

$$1 - t_0 < \varepsilon^{\frac{1}{1-\alpha}}. \tag{3.2}$$

Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathscr{B}^\alpha$ which converges to 0 uniformly on every compact subset of $\mathbb{D}$. Without loss of generality, we may assume that $\sup_{n \geqslant 1} ||f_n||_{\mathscr{B}^\alpha} \leqslant 1$. By the integral representation of $\mathscr{C}_\mu$ we see that

$$\begin{aligned} \mathscr{C}_\mu(f_n)(z) &= \int_0^1 \frac{f_n(tz) - f_n(t)}{1-tz} d\mu(t) + \int_0^1 \frac{f_n(t)}{1-tz} d\mu(t) \\ &:= \mathscr{J}_\mu(f_n)(z) + \mathscr{H}_\mu(f_n)(z). \end{aligned}$$

It follows that

$$||\mathscr{C}_\mu(f_n)||_{\mathscr{B}^\beta} \leqslant ||\mathscr{J}_\mu(f_n)||_{\mathscr{B}^\beta} + ||\mathscr{H}_\mu(f_n)||_{\mathscr{B}^\beta}.$$

Note that the second part of the right-hand side is the integral type Hilbert operator (see e.g. [18, 37] for the definition). Therefore, Corollary 5.4 in [37] shows that the integral type Hilbert operator $\mathscr{H}_\mu$ is compact from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$ whenever $0<\alpha<1$ and $0<\beta<2$. This implies that

$$\lim_{n\to\infty} ||\mathscr{H}_\mu(f_n)||_{\mathscr{B}^\beta} = 0.$$

To complete the proof, it is suffices to prove that $\lim_{n\to\infty} ||\mathscr{J}_\mu(f_n)||_{\mathscr{B}^\beta} = 0$ by Lemma 2.5. It is easy to see that

$$|\mathscr{J}_\mu(f_n)'(z)| \leqslant \int_0^1 G_n^z(t)d\mu(t)$$

where

$$G_n^z(t) = \frac{|f_n'(tz)|}{|1-tz|} + \frac{|f_n(tz)-f_n(t)|}{|1-tz|^2}, \quad z\in\mathbb{D}.$$

The Cauchy's integral theorem implies that $\{f_n'\}_{n=1}^\infty$ converges to 0 uniformly on every compact subset of $\mathbb{D}$. Hence

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \int_0^{t_0} G_n^z(t)d\mu(t) \lesssim \sup_{|w|\leqslant t_0}(|f_n'(w)|+|f_n(w)|) \to 0, \quad \text{as } n\to\infty.$$

Since $\mathscr{B}^\alpha \cong \Lambda_{1-\alpha}$, we have

$$|f_n(tz)-f_n(t)| \leqslant t|1-z|^{1-\alpha}||f_n||_{\Lambda_{1-\alpha}} \lesssim |1-z|^{1-\alpha}.$$

For $0<t<1$ and $z\in\mathbb{D}$, the inequalities

$$\frac{|1-z|}{|1-tz|} \leqslant \frac{1-t}{|1-tz|} + \frac{|t-z|}{|1-tz|} \leqslant 2$$

imply that

$$\frac{|f_n(tz)-f_n(t)|}{|1-tz|^2} \lesssim \frac{1}{|1-tz|^{1+\alpha}}. \tag{3.3}$$

By the definition of $\mathscr{B}^\alpha$ and (3.3) we have

$$\begin{aligned}\int_{t_0}^1 G_n^z(t)d\mu(t) &\lesssim \int_{t_0}^1 \left(\frac{1}{(1-t|z|)^\alpha|1-tz|} + \frac{1}{|1-tz|^{1+\alpha}}\right)d\mu(t)\\ &\lesssim \int_{t_0}^1 \frac{d\mu(t)}{(1-t|z|)^{\alpha+1}}.\end{aligned}$$

Bearing in the mind that μ is a $2-\beta$ Carleson measure and that there exists $0<\delta<1$ such that $(1-|z|^2)^\beta<\varepsilon$ for all $\delta<|z|<1$, by integrating by parts (see [16, Theorem

5]) and using (3.1)–(3.2) we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \int_{t_0}^1 G_n^z(t)d\mu(t)\\
&\lesssim \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \int_{t_0}^1 \frac{d\mu(t)}{(1-t|z|)^{\alpha+1}}\\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \left(\frac{\mu([t_0,1))}{(1-t_0|z|)^{\alpha+1}} + (\alpha+1)|z|\int_{t_0}^1 \frac{\mu([t,1))}{(1-t|z|)^{\alpha+2}}dt\right)\\
&\lesssim \left(\sup_{|z|\leqslant\delta} + \sup_{\delta<|z|<1}\right)(1-|z|^2)^\beta \frac{\mu([t_0,1))}{(1-t_0|z|)^{\alpha+1}}\\
&\quad + \sup_{z\in\mathbb{D}}\int_{t_0}^1 \frac{(1-t)^{2-\beta}(1-|z|^2)^\beta}{(1-t|z|)^{\alpha+2}}dt\\
&\lesssim (1-t_0)^{2-\beta} + \varepsilon + \int_{t_0}^1 \frac{dt}{(1-t)^\alpha}\\
&\lesssim \varepsilon^{\frac{2-\beta}{1-\alpha}} + \varepsilon.
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} ||\mathscr{J}_\mu(f_n)||_{\mathscr{B}^\beta} = 0.$$

This implies that $\mathscr{C}_\mu : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact. $\square$

THEOREM 2. *Let μ be a finite positive Borel measure on the interval $[0,1)$. If $\alpha > 1$ and $0 < \beta < \alpha+1$, then $\mathscr{C}_\mu : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded if and only if μ is an $\alpha+1-\beta$ Carleson measure.*

Proof. Suppose $\mathscr{C}_\mu : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. For $0 < a < 1$, let

$$f_a(z) = \frac{(1-a)}{(1-az)^\alpha}.$$

Then it is easy to check that $\sup_{0<a<1} ||f_a||_{\mathscr{B}^\alpha} \lesssim 1$. By the integral form of $\mathscr{C}_\mu$ we get

$$\mathscr{C}_\mu(f_a)'(z) = \int_0^1 \frac{tf_a'(tz)}{1-tz}d\mu(t) + \int_0^1 \frac{tf_a(tz)}{(1-tz)^2}d\mu(t).$$

The boundedness of $\mathscr{C}_\mu$ and Lemma 2.1 imply that

$$|\mathscr{C}_\mu(f_a)'(z)| \lesssim \frac{||f_a||_{\mathscr{B}^\alpha}}{(1-|z|)^\beta}.$$

Therefore, for any $\frac{1}{2} < a < 1$ we have

$$\begin{aligned}\frac{1}{(1-a)^{\beta}} &\gtrsim |\mathscr{C}_\mu(f_a)'(a)| = \mathscr{C}_\mu(f_a)'(a)\\ &\geqslant \int_0^1 \frac{tf_a(ta)}{(1-ta)^2}d\mu(t)\\ &= (1-a)\int_0^1 \frac{td\mu(t)}{(1-ta)^2(1-a^2t)^{\alpha}}\\ &\gtrsim (1-a)\int_a^1 \frac{d\mu(t)}{(1-ta)^2(1-a^2t)^{\alpha}}\\ &\gtrsim \frac{\mu([a,1))}{(1-a)^{\alpha+1}}.\end{aligned}$$

This gives that

$$\mu([a,1)) \lesssim (1-a)^{\alpha+1-\beta} \text{ for all } \frac{1}{2} < a < 1.$$

Hence μ is an $\alpha+1-\beta$ Carleson measure.

Conversely, suppose μ is an $\alpha+1-\beta$ Carleson measure and $f \in \mathscr{B}^{\alpha}$. Using the integral form of $\mathscr{C}_\mu(f)$ and Lemma 1 we deduce that

$$\begin{aligned}|\mathscr{C}_\mu(f)'(z)| &\leqslant \int_0^1 \frac{|f'(tz)|}{|1-tz|}d\mu(t) + \int_0^1 \frac{|f(tz)|}{|1-tz|^2}d\mu(t)\\ &\lesssim ||f||_{\mathscr{B}^{\alpha}} \int_0^1 \frac{d\mu(t)}{(1-t|z|)^{\alpha+1}}.\end{aligned} \tag{3.4}$$

Take $z \in \mathbb{D}$ and let $|z| = r$. Integrating by parts and using the fact that μ is an $\alpha+1-\beta$ Carleson measure and Lemma 2.4, we obtain

$$\begin{aligned}\int_0^1 \frac{d\mu(t)}{(1-tr)^{\alpha+1}} &= \mu([0,1)) + (\alpha+1)r\int_0^1 \frac{\mu([t,1))}{(1-tr)^{\alpha+2}}dt\\ &\lesssim \mu([0,1)) + \int_0^1 \frac{(1-t)^{\alpha+1-\beta}}{(1-tr)^{\alpha+2}}dt\\ &\lesssim \mu([0,1)) + \frac{1}{(1-r)^{\beta}}.\end{aligned}$$

This together with (3.4) imply that $\mathscr{C}_\mu : \mathscr{B}^{\alpha} \to \mathscr{B}^{\beta}$ is bounded. $\square$

THEOREM 3. *Let μ be a finite positive Borel measure on the interval $[0,1)$. If $0 < \beta \leqslant 2$, then $\mathscr{C}_\mu : \mathscr{B} \to \mathscr{B}^{\beta}$ is bounded if and only if*

$$\sup_{0<t<1} \frac{\mu([t,1))\log\frac{e}{1-t}}{(1-t)^{2-\beta}} < \infty. \tag{3.5}$$

Proof. Suppose $\mathscr{C}_\mu : \mathscr{B} \to \mathscr{B}^\beta$ is bounded. Let $f(z) = \log \frac{1}{1-z}$, it is clear that $f \in \mathscr{B}$ and

$$\mathscr{C}_\mu(f)(z) = \sum_{n=1}^{\infty} \mu_n \left(\sum_{k=1}^{n} \frac{1}{k} \right) z^n, \quad z \in \mathbb{D},$$

Since $\mathscr{C}_\mu(f) \in \mathscr{B}^\beta$, by the definition of $\mathscr{B}^\beta$ we have that

$$\sum_{n=1}^{\infty} n\mu_n \left(\sum_{k=1}^{n} \frac{1}{k} \right) r^{n-1} \lesssim \frac{1}{(1-r)^\beta}, \quad 0 < r < 1.$$

For $N \geqslant 2$ take $r_N = 1 - \frac{1}{N}$. Since the sequence $\{\mu_k\}$ is decreasing, simple estimations lead us to the following

$$\begin{aligned} N^\beta &\gtrsim \sum_{n=1}^{\infty} n\mu_n \left(\sum_{k=1}^{n} \frac{1}{k} \right) r_N^{n-1} \\ &\gtrsim \sum_{n=1}^{N} n\mu_n \left(\sum_{k=1}^{n} \frac{1}{k} \right) r_N^{N-1} \\ &\gtrsim \mu_N \sum_{n=1}^{N} n \log(n+1) \\ &\gtrsim \mu_N N^2 \log(N+1) \end{aligned}$$

This implies that $\mu_N = O\left(\frac{1}{N^{2-\beta} \log(N+1)} \right)$. The desired result follows from the inequalities

$$\mu\left(\left[1 - \frac{1}{N}, 1 \right) \right) \lesssim \int_{1-\frac{1}{N}}^{1} t^N d\mu(t) \leqslant \int_0^1 t^N d\mu(t) \lesssim \frac{1}{N^{2-\beta} \log(N+1)}.$$

Conversely, suppose (3.5) holds. Integrating by parts we have

$$\begin{aligned} \int_0^1 t^n d\mu(t) &= n \int_0^1 t^{n-1} \mu([t,1)) dt \\ &\lesssim n \int_0^1 t^{n-1} (1-t)^{2-\beta} \log^{-1} \frac{e}{1-t} dt. \end{aligned}$$

Let $\phi(t) = (1-t)^{2-\beta} \log^{-1} \frac{e}{1-t}$, then $\phi(t)$ is regular in the sense of Peláez and Rättyä [27]. Then, using Lemma 1.3 and (1.1) in [27], we have

$$n \int_0^1 t^{n-1} \phi(t) dt \asymp \phi\left(1 - \frac{1}{n} \right) \asymp \frac{1}{n^{2-\beta} \log(n+1)}.$$

This implies that

$$\mu_n \lesssim \frac{1}{n^{2-\beta} \log(n+1)} \quad \text{for all } n \geqslant 1. \tag{3.6}$$

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathscr{B}$, it follows from Corollary D in [22] that

$$\left|\sum_{k=1}^{n}a_k\right|\lesssim||f||_{\mathscr{B}}\log(n+1).$$

By (3.6) and above inequality we get

$$\begin{aligned}(1-|z|^2)^{\beta}|\mathscr{C}_\mu(f)'(z)|&=(1-|z|^2)^{\beta}\left|\sum_{n=1}^{\infty}n\mu_n\left(\sum_{k=1}^{n}a_k\right)z^{n-1}\right|\\&\lesssim(1-|z|^2)^{\beta}\sum_{n=1}^{\infty}n\mu_n\left|\sum_{k=1}^{n}a_k\right||z|^{n-1}\\&\lesssim||f||_{\mathscr{B}}(1-|z|^2)^{\beta}\sum_{n=1}^{\infty}n\mu_n\log(n+1)|z|^{n-1}\\&\lesssim||f||_{\mathscr{B}}(1-|z|^2)^{\beta}\sum_{n=1}^{\infty}n^{\beta-1}|z|^{n-1}\\&\lesssim||f||_{\mathscr{B}}.\end{aligned}$$

This shows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|\mathscr{C}_\mu(f)'(z)|\lesssim||f||_{\mathscr{B}}.$$

Hence, we have that $\mathscr{C}_\mu:\mathscr{B}\to\mathscr{B}^{\beta}$ is bounded. $\square$

There are three cases left: (i) $\alpha=1$ and $\beta>2$; (ii) $0<\alpha<1$ and $\beta\geqslant 2$; (iii) $\alpha>1$ and $\beta\geqslant\alpha+1$. We show that the operator $\mathscr{C}_\mu$ is always a bounded operator from $\mathscr{B}^{\alpha}$ to $\mathscr{B}^{\beta}$ in these cases.

THEOREM 4. *Let μ be a finite positive Borel measure on the interval $[0,1)$. If α and β satisfies one of the conditions (i)–(iii), then $\mathscr{C}_\mu:\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}$ is bounded.*

Proof. We only prove the case of (i), since the proofs of other cases are similar. Let $f\in\mathscr{B}$, then

$$\begin{aligned}|\mathscr{C}_\mu(f)'(z)|&\leqslant\int_0^1\frac{|f'(tz)|}{|1-tz|}d\mu(t)+\int_0^1\frac{|f(tz)|}{|1-tz|^2}d\mu(t)\\&\lesssim||f||_{\mathscr{B}}\int_0^1\frac{d\mu(t)}{(1-t|z|)|1-tz|}+||f||_{\mathscr{B}}\int_0^1\frac{\log\frac{e}{1-t|z|}}{|1-tz|^2}d\mu(t)\\&\lesssim||f||_{\mathscr{B}}\int_0^1\frac{\log\frac{e}{1-t|z|}}{(1-t|z|)^2}d\mu(t).\end{aligned}$$

If $\beta>2$, it is clear that

$$\sup_{z\in\mathbb{D}}(1-|z|)^{\beta-2}\log\frac{e}{1-|z|}<\infty.$$

This implies that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|\mathscr{C}_\mu(f)'(z)| \lesssim ||f||_{\mathscr{B}}.$$

The proof is complete. □

Let $1 \leqslant p < \infty$ and $0 < \alpha \leqslant 1$, the mean Lipschitz space Λ^p_α consists of the functions $f \in H(\mathbb{D})$ having a non-tangential limit almost everywhere for which $\omega_p(t,f) = O(t^\alpha)$ as $t \to 0$. Here $\omega_p(\cdot,f)$ is the integral modulus of continuity of order p of the function $f(e^{i\theta})$. It is known that (see e.g., [13, Chapter 5]) Λ^p_α is a subset of H^p and Λ^p_α consists of those functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\Lambda^p_\alpha} = |f(0)| + \sup_{0<r<1}(1-r)^{1-\alpha}M_p(r,f') < \infty.$$

Theorem 4.2 in [3] and Theorem 1 lead to the following corollary.

COROLLARY 1. *Let μ be a finite positive Borel measure on the interval $[0,1)$, $0<\alpha<1$ and $1<p<\infty$. Let X and Y be two Banach subspaces of $H(\mathbb{D})$ with $\mathscr{B}^\alpha \subset X \subset H^\infty$ and $\Lambda^p_{\frac{1}{p}} \subset Y \subset \mathscr{B}$. Then the following conditions are equivalent.*

(1) The measure μ is a Carleson measure.

(2) The operator $\mathscr{C}_\mu$ is bounded from X into Y.

Proof. $(1) \Rightarrow (2)$. Assume that μ is a Carleson measure and take $f \in X$. Since $X \subset H^\infty$, we have that $f \in H^\infty$. Theorem 4.2 in [3] shows that $\mathscr{C}_\mu(H^\infty) \subset Y$ if and only if μ is a Carleson measure. This implies that $\mathscr{C}_\mu(f) \in Y$.

$(2) \Rightarrow (1)$. Suppose that $\mathscr{C}_\mu$ is bounded from X into Y. Then $\mathscr{C}_\mu$ is bounded from $\mathscr{B}^\alpha$ into $\mathscr{B}$. Now, Theorem 1 shows that μ is a Carleson measure. □

4. The compactness of $\mathscr{C}_\mu$ acting between Bloch type spaces

THEOREM 5. *Let μ be a finite positive Borel measure on the interval $[0,1)$. If $\alpha > 1$ and $0 < \beta < \alpha+1$, then $\mathscr{C}_\mu : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact if and only if μ is a vanishing $\alpha+1-\beta$ Carleson measure.*

Proof. Assume that $\mathscr{C}_\mu : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact. For $0<a<1$, set

$$f_a(z) = \frac{1-a}{(1-az)^\alpha},\ z\in\mathbb{D}.$$

Then it is clear that $f_a \in \mathscr{B}^\alpha$ for all $0<a<1$ and $\sup_{0<a<1}||f_a||_{\mathscr{B}^\alpha} \lesssim 1$. Moreover, $f_a \to 0$, as $a \to 1$, uniformly on compact subset of $\mathbb{D}$. Lemma 2.5 implies that

$$||\mathscr{C}_\mu(f_a)||_{\mathscr{B}^\beta} \to 0,\ \text{as } a \to 1. \tag{4.1}$$

Arguing as the proof of Theorem 2 we have

$$\mu([a,1)) \lesssim (1-a)^{\alpha+1-\beta}||\mathscr{C}_\mu(f_a)||_{\mathscr{B}^\beta}.$$

This and (4.1) show that μ is a vanishing $\alpha+1-\beta$ Carleson measure.

On the other hand, suppose μ is a vanishing $\alpha+1-\beta$ Carleson measure. Then for any $\varepsilon>0$, there exists $0<t_0<1$ such that

$$\mu([t,1))<\varepsilon(1-t)^{\alpha+1-\beta} \text{ whenever } t_0\leqslant t<1. \tag{4.2}$$

Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in $\mathscr{B}^{\alpha}$ which converges to 0 uniformly on every compact subset of $\mathbb{D}$. It is sufficient to prove that

$$\lim_{k\to\infty}||\mathscr{C}_{\mu}(f_k)||_{\mathscr{B}^{\beta}}=0$$

by Lemma 2.5. By the integral form of $\mathscr{C}_{\mu}(f)$ we have that

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|\mathscr{C}_{\mu}(f_k)'(z)|\leqslant&\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\int_0^1\frac{|f_k'(tz)|}{|1-tz|}d\mu(t)\\&+\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\int_0^1\frac{|f_k(tz)|}{|1-tz|^2}d\mu(t).\end{aligned}$$

The Cauchy's integral theorem implies that $\{f_k'\}_{k=1}^{\infty}$ converges to 0 uniformly on every compact subset of $\mathbb{D}$. This gives

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\int_0^{t_0}\frac{|f_k'(tz)|}{|1-tz|}d\mu(t)\lesssim\sup_{|w|\leqslant t_0}|f_k'(w)|\to 0,\ \text{ as } k\to\infty.$$

Note that there exists $0<\delta<1$ such that $(1-|z|^2)^{\beta}<\varepsilon$ for all $\delta<|z|<1$, by integrating by parts and using (4.2) and Lemma 2.4 we have

$$\begin{aligned}&\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\int_{t_0}^1\frac{|f_k'(tz)|}{|1-tz|}d\mu(t)\\&\lesssim\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\int_{t_0}^1\frac{d\mu(t)}{(1-t|z|)^{\alpha+1}}\\&=\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\left(\frac{\mu([t_0,1))}{(1-t_0|z|)^{\alpha+1}}+(\alpha+1)|z|\int_{t_0}^1\frac{\mu([t,1))}{(1-t|z|)^{\alpha+2}}dt\right)\\&\lesssim\left(\sup_{|z|\leqslant\delta}+\sup_{\delta<|z|<1}\right)(1-|z|^2)^{\beta}\frac{\mu([t_0,1))}{(1-t_0|z|)^{\alpha+1}}+\varepsilon\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\int_{t_0}^1\frac{(1-t)^{\alpha+1-\beta}}{(1-t|z|)^{\alpha+2}}dt\\&\lesssim\varepsilon(1-t_0)^{\alpha+1-\beta}+\varepsilon+\varepsilon\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\int_0^1\frac{(1-t)^{\alpha+1-\beta}}{(1-t|z|)^{\alpha+2}}dt\\&\lesssim\varepsilon.\end{aligned}$$

Since ε is arbitrary, it follows that

$$\lim_{k\to\infty}\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\int_0^1\frac{|f_k'(tz)|}{|1-tz|}d\mu(t)=0.$$

Similarly, we can obtain that

$$\lim_{k\to\infty}\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\int_0^1\frac{|f_k(tz)|}{|1-tz|^2}d\mu(t)=0.$$

It is obvious that

$$\lim_{k\to\infty}|\mathscr{C}_\mu(f_k)(0)|=\mu([0,1))\lim_{k\to\infty}|f_k(0)|=0.$$

Therefore,

$$\lim_{k\to\infty}||\mathscr{C}_\mu(f_k)||_{\mathscr{B}^\beta}=\lim_{k\to\infty}\left(|\mathscr{C}_\mu(f_k)(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|\mathscr{C}_\mu(f_k)'(z)|\right)=0.$$

This means that $\mathscr{C}_\mu:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is compact. □

THEOREM 6. *Let μ be a finite positive Borel measure on the interval $[0,1)$. If $0<\beta\leqslant 2$, then $\mathscr{C}_\mu:\mathscr{B}\to\mathscr{B}^\beta$ is compact if and only if*

$$\lim_{t\to 1}\frac{\mu([t,1))\log\frac{e}{1-t}}{(1-t)^{2-\beta}}=0.$$

Proof. The proof of the sufficiency is similar to that of Theorem 5. Take the test functions

$$f_a(z)=\log^{-1}\frac{2}{1-a}\left(\log\frac{2}{1-az}\right)^2,\quad a\in(0,1),\quad z\in\mathbb{D}.$$

Then arguing as the proof of Theorem 5 we can obtain the necessity. □

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