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CONVEX FUNCTIONS AND GENERALIZED BERNOULLI INEQUALITY

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Abstract. In this article we give an other proof (and stronger theorem) of the conditional inequalities for convex functions. As a result, we present new inequalities in the spirit the well-known Bernoulli inequality.

1. Introduction

Throughout this article, $I \subseteq \mathbb{R}$ stands for an interval of the form $[0,\vartheta)$ or $[0,\vartheta]$, with $\vartheta > 0$, or $I = [0,+\infty)$. The below result was proved in [1].

THEOREM 1. [1, Theorem 2.1, Remark 2.4] *Let $f,g\colon I \to [0,+\infty)$ be convex functions such that $f(0) = 1$ and $g(0) = 0$. Then, for any number $\alpha \in (1,+\infty)$ the following conditions are equivalent:*

(a) $1 \leqslant f(t) + g(t), \quad t \in I;$

(b) $1 \leqslant f(t)^{\alpha} + \alpha g(t), \quad t \in I.$

The applications of Theorem 1 were presented in the paper [1]. In particular, the aforementioned result allowed to investigate the well-known notion of approximate Birkhoff-James orthogonality (see [1, Theorem 3.1]), and it gave new results in geometry of operator spaces – see also [1, Theorem 4.2]. In order to prove Theorem 1, those authors of [1] applied the Bernoulli inequality $1+\alpha b \leqslant (1+b)^{\alpha}$ with $b \in (-1,0]$.

The statement of Theorem 1 is so simple, and its existing proof so extremely long, that one is easily seduced into an effort to find another, similar but stronger theorem. The present paper is the result of our attempt. In particular, our proof of Theorem 2 uses relatively simple tool: a right derivative of convex function. It is worth mentioning that we obtain stronger result than Theorem 1, but we do not apply the Bernoulli inequality. What's more, as an immediate consequence of our result we will prove new inequalities, which are similar to the classical Bernoulli inequality.

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2. Main result: the conditional inequalities for convex functions

It is known (see [2]) that if $f\colon I \to \mathbb{R}$ is a convex function, then f is continuous on the interior $\mathrm{int} I$ and f has a right derivative f'_+ at each point of $\mathrm{int} I$. Moreover, $f'_+\colon \mathrm{int} I \to \mathbb{R}$ is nondecreasing. And so, we show that there is another way to obtain Theorem 1. In fact, we are in position to extend Theorem 1 as follows.

THEOREM 2. *Let $f, b, g\colon I \to [0, +\infty)$ be convex functions such that $f(0) = 1$ and $b(0) = 1$ and $g(0) = 0$. Then, for any number $\alpha \in (1, +\infty)$ the following conditions are equivalent:*

(a) $2 \leqslant f(t) + b(t) + g(t), \quad t \in I;$

(b) $2 \leqslant f(t)^\alpha + b(t)^\alpha + \alpha g(t), \quad t \in I.$

Proof. Clearly $\lim\limits_{t\to 0^+} g(t) = g(0)$. Moreover, assuming the condition (a) or (b) we have $\lim\limits_{t\to 0^+} f(t) = f(0)$ and $\lim\limits_{t\to 0^+} b(t) = b(0)$. Thus f, b and g are continuous on $\{0\} \cup \mathrm{int} I$ and g is nondecreasing. Hence $g'_+ \geqslant 0$. Now, let us define two functions $\gamma, \varphi\colon [0, +\infty) \to [0, +\infty)$ by the following formulas:

$$\gamma(t) := f(t) + b(t) + g(t) \quad \text{and} \quad \varphi(t) := f(t)^\alpha + b(t)^\alpha + \alpha g(t).$$

First we prove (a)$\Rightarrow$(b). Assume (a). Since f and b are convex, f'_+ and b'_+ are nondecreasing. Therefore, we have three possibilities.

Possibility 1: the inequalities $0 \leqslant f'_+(t)$ and $0 \leqslant b'_+(t)$ hold for $t \in I$. Then we get $1 \leqslant f(t)$ and $1 \leqslant b(t)$. So, in this case the conditions (b) holds.

Possibility 2: there is $t_1 > 0$ such that the inequalities $f'_+(t) < 0$ and $0 \leqslant b'_+(t)$ hold for $t \in (0, t_1)$; or conversely, the inequalities $0 \leqslant f'_+(t)$ and $b'_+(t) < 0$ hold for $t \in (0, t_1)$. Without loss of generality we assume that

$$f'_+(t) < 0 \quad \text{and} \quad 0 \leqslant b'_+(t) \quad \text{for all} \quad t \in (0, t_1). \tag{1}$$

These inequalities and convexity imply

$$f(t) < 1 \quad \text{and} \quad 1 \leqslant b(t) \quad \text{for all} \quad t \in (0, t_1). \tag{2}$$

Then for all $t \in (0, t_1)$ we have

$$\begin{aligned} \varphi'_+(t) &= \alpha\Big(f(t)^{\alpha-1} f'_+(t) + b(t)^{\alpha-1} b'_+(t) + g'_+(t)\Big) \\ &\overset{(1),(2)}{\geqslant} \alpha\Big(f(t)^{\alpha-1} f'_+(t) + f(t)^{\alpha-1} b'_+(t) + g'_+(t)\Big) \\ &= \alpha\Big(f(t)^{\alpha-1} \gamma'_+(t) + g'_+(t)\big(1 - f(t)^{\alpha-1}\big)\Big) \overset{(2)}{\geqslant} 0. \end{aligned}$$

Thus we get $\varphi'_+(t) \geqslant 0$ and this implies that φ is nondecreasing on $(0, t_1)$. But since φ is convex, φ is nondecreasing on the whole half-line $[0, +\infty)$. So, in particular, $\varphi(0) \leqslant \varphi(t)$ for all $t \geqslant 0$, i.e. $2 \leqslant f(t)^\alpha + b(t)^\alpha + \alpha g(t)$.

Possibility 3: there is $t_1 > 0$ such that the inequalities

$$f'_+(t) < 0 \quad \text{and} \quad b'_+(t) < 0 \quad \text{hold for} \quad t \in (0,t_1). \tag{3}$$

These inequalities and convexity imply

$$f(t) < 1 \quad \text{and} \quad b(t) < 1 \quad \text{for all} \quad t \in (0,t_1). \tag{4}$$

Put $M(t) := \max\left\{f(t)^{\alpha-1}, b(t)^{\alpha-1}\right\}$. Now, it follows that for all $t \in (0,t_1)$ we have

$$\begin{aligned}
\varphi'_+(t) &= \alpha\Big(f(t)^{\alpha-1}f'_+(t) + b(t)^{\alpha-1}b'_+(t) + g'_+(t)\Big) \\
&\overset{(3),(4)}{\geqslant} \alpha\Big(M(t)f'_+(t) + M(t)b'_+(t) + g'_+(t)\Big) \\
&= \alpha\Big(M(t)\gamma'_+(t) + g'_+(t)\big(1 - M(t)\big)\Big) \overset{(4)}{\geqslant} 0.
\end{aligned}$$

Hence we obtain $\varphi'_+(t) \geqslant 0$, and again, in a similar way as in *Possibility 1* we prove that φ is nondecreasing on the whole half-line $[0,+\infty)$. The proof of (a)$\Rightarrow$(b) is complete.

In order to prove (b)$\Rightarrow$(a), assume that the condition (b) holds. This implies an inequality $\varphi'_+(t) \geqslant 0$ for all $t \in \operatorname{int} I$. Hence

$$\begin{aligned}
0 \leqslant \frac{1}{\alpha}\varphi'_+(t) &= f(t)^{\alpha-1}f'_+(t) + b(t)^{\alpha-1}b'_+(t) + g'_+(t) \\
&= f(t)^{\alpha-1}\gamma'_+(t) + \big(1 - f(t)^{\alpha-1}\big)g'_+(t) + \big(b(t)^{\alpha-1} - f(t)^{\alpha-1}\big)b'_+(t)
\end{aligned} \tag{5}$$

for all $t \in \operatorname{int} I$. Now, it suffices to prove the inequality $\gamma'_+(t) \geqslant 0$ on the set $\operatorname{int} I$. Suppose, for a contradiction, that there are $t_z \in \operatorname{int} I$ and $\eta < 0$ such that $\gamma'_+(t_z) < \eta$. Next, we may consider a sequence $t_n := \frac{1}{n}t_z \xrightarrow{n} 0^+$. Since b, g, γ are convex, b'_+, g'_+, γ'_+ are nondecreasing. Note that $t_n < t_z$. Thus we obtain

$$\gamma'_+(t_n) \leqslant \gamma'_+(t_z) < \eta. \tag{6}$$

Recall that $\varphi'_+ \geqslant 0$. This inequalities implies that

$$0 \leqslant \frac{1}{\alpha}\varphi'_+(t_n) = f(t_n)^{\alpha-1}f'_+(t_n) + b(t_n)^{\alpha-1}b'_+(t_n) + g'_+(t_n). \tag{7}$$

Note that the sequences $f'_+(t_n)$, $b'_+(t_n)$ and $g'_+(t_n)$ are decreasing and therefore

$$f'_+(t_n) \leqslant f'_+(t_z), \quad b'_+(t_n) \leqslant b'_+(t_z) \quad \text{and} \quad 0 \leqslant g'_+(t_n) \leqslant g'_+(t_z). \tag{8}$$

Moreover, we have

$$f(t_n)^{\alpha-1} \xrightarrow{n} 1 \quad \text{and} \quad b(t_n)^{\alpha-1} \xrightarrow{n} 1. \tag{9}$$

The last part of (8) implies that the sequence $\left|g'_+(t_n)\right|$ is bounded. Now we show that the sequence $b'_+(t_n)$ is bounded below. Suppose, for a contradiction, that

$$b'_+(t_n) \to -\infty. \tag{10}$$

Since (9) holds, it follows from (10) that

$$0 \overset{(7)}{\leqslant} \frac{1}{\alpha}\varphi'_+(t_n) = f(t_n)^{\alpha-1}f'_+(t_n) + b(t_n)^{\alpha-1}b'_+(t_n) + g'_+(t_n) \to -\infty,$$

and this is a contradiction. Thus $b'_+(t_n)$ is bounded below. Hence by (8), the sequence $\left|b'_+(t_n)\right|$ is bounded.

We have already observed that both $\left|b'_+(t_n)\right|$ and $\left|g'_+(t_n)\right|$ are bounded. Therefore, putting t_n in place of t in the inequality (5) we get

$$0 \leqslant f(t_n)^{\alpha-1}\gamma'_+(t_n) + \left(1 - f(t_n)^{\alpha-1}\right)g'_+(t_n) + \left(b(t_n)^{\alpha-1} - f(t_n)^{\alpha-1}\right)b'_+(t_n). \tag{11}$$

The condition (9) implies that the right side of (11) tends to η, whence $0 \leqslant \eta$. But this is in contradiction to $\eta < 0$, and we are done. $\square$

REMARK 1. The anonymous reviewer showed in the report that the implication (a)$\Rightarrow$(b) in Theorem 2 follows from that the implication (a)$\Rightarrow$(b) in Theorem 1. The proof given the anonymous reviewer clever and quicker. Note that, the main tool in the proof of Theorem 1 was the classical Bernoulli inequality. However, the aim of our paper was to prove Theorem 2 and we did not want to apply the Bernoulli inequality. In fact, we wanted to give another way to obtian this well known inequality.

3. The applications of Theorem 2

Unlike [1], the above proof of Theorem 2 does not rely on the aforementioned version of Bernoulli inequality. But curiously, this well-known inequality can be derived from Theorem 2. Indeed, we present now a new proof of the celebrated Bernoulli inequality.

THEOREM 3. *For $b \in (-1,0]$ and $\alpha > 1$ we have $1 + \alpha b \leqslant (1+b)^\alpha$.*

Proof. Define $f,b,g\colon [0,1] \to [0,+\infty)$ by $f(t) := 1 + bt$, $b(t) := 1$ and $g(t) := -bt$. Clearly $2 \leqslant f(\cdot) + b(\cdot) + g(\cdot)$. From Theorem 2 we get $2 \leqslant f(t)^\alpha + b(t)^\alpha + \alpha g(t)$ for $t \in [0,1]$. Putting $t := 1$ we obtain the Bernoulli inequality. $\square$

Motivated by the proof of Theorem 3, we present results concerning similar inequality. But first, we need some auxiliary result.

THEOREM 4. *Suppose that $a,b,c \in (0,2)$ and $\alpha > 1$. If $2 \leqslant a+b+c$, then $2 \leqslant a^\alpha + b^\alpha + \alpha c$.*

Proof. Suppose that $2 \leqslant a+b+c$. Define three functions $f,b,g\colon [0,1] \to \mathbb{R}$ by $f(t) := 1-(1-a)t$, $b(t) := 1-(1-b)t$ and $g(t) := ct$. It is easy to check that $2 \leqslant f(t)+b(t)+g(t)$. It follows from Theorem 2 that $2 \leqslant f(t)^\alpha + b(t)^\alpha + g(t)^\alpha$. Putting $t := 1$, we get $2 \leqslant a^\alpha + b^\alpha + \alpha c$. $\square$

Now, we are in position to prove the three results in the spirit the Bernoulli inequality.

THEOREM 5. *If $a,b,c \in (0,+\infty)$ and $\alpha > 1$, then*

$$(a+b+c)^{\alpha} \leqslant 2^{\alpha-1}\left(a^{\alpha}+b^{\alpha}\right)+\alpha c(a+b+c)^{\alpha-1}. \tag{12}$$

Proof. Put $d := \frac{a+b+c}{2}$. Then $\frac{a}{d},\frac{b}{d},\frac{c}{d} \in (0,2)$ and $2 \leqslant \frac{a}{d}+\frac{b}{d}+\frac{c}{d}$. Theorem 4 implies that $2 \leqslant \frac{a^{\alpha}}{d^{\alpha}}+\frac{b^{\alpha}}{d^{\alpha}}+\frac{\alpha c}{d}$. Now, simple calculation shows that (12) holds. □

THEOREM 6. *If $u,w \in (0,+\infty)$ and $\alpha \in (1,2)$, then*

$$(2-\alpha)(u+w)^{\alpha} \leqslant \left(u^{\alpha}+w^{\alpha}\right). \tag{13}$$

Proof. It is clear that $2 \leqslant \frac{u}{u+w}+\frac{w}{u+w}+1$. It follows from Theorem 4 that $2 \leqslant \left(\frac{u}{u+w}\right)^{\alpha}+\left(\frac{w}{u+w}\right)^{\alpha}+\alpha$. and from this inequality we obtain (13). □

Now, suppose that an equality $a=b=\frac{x}{2}$ holds. Then, as an immediate consequence of the above inequality (12), we deduce that the below result is true.

COROLLARY 1. *If $x,c \in (0,+\infty)$, then $(x+c)^{\alpha} \leqslant x^{\alpha}+\alpha c(x+c)^{\alpha-1}$.*

To end this article we show that Theorem 2 can be extended in a straightforward manner to the case of convex functions defined on an arbitrary real vector spaces. More precisely, if a domain of functions is balanced (i.e. $[-1,1]\cdot U \subseteq U$) subset, then Theorem 2 works “on rays”.

THEOREM 7. *Let X be a real vector space and let $U \subseteq X$ be a convex balanced subset. Let $f,b,g\colon U \to [0,+\infty)$ be convex functions with $f(0)=1=b(0)$ and $g(0)=0$. Then, for $\alpha \in (1,+\infty)$ the following conditions are equivalent:*

(A) $2 \leqslant f(t)+b(t)+g(t), \quad t \in U;$

(B) $2 \leqslant f(t)^{\alpha}+b(t)^{\alpha}+\alpha g(t), \quad t \in U.$

Proof. Assume (A). Fix $x \in U\setminus\{0\}$ and define $\widehat{f},\widehat{b},\widehat{g}\colon [0,1] \to [0,+\infty)$ by $\widehat{f}(t) := f(tx)$, $\widehat{b}(t) := b(tx)$ and $\widehat{g}(t) := g(tx)$. It is clear that $\widehat{f},\widehat{b},\widehat{g}$ satisfy the assumptions and condition (a) in Theorem 2. From Theorem 2 we have an inequality $1 \leqslant \widehat{f}(t)^{\alpha}+\widehat{b}(t)^{\alpha}+\alpha\widehat{g}(t)$ for $t \in [0,1]$. Put $t := 1$. From this we obtain inequality $1 \leqslant f(x)^{\alpha}+b(x)^{\alpha}+\alpha g(x)$ and we get (B). The proof of (B)⇒(A) runs similarly. □

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