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APPROXIMATION IN GENERALIZED MORREY SPACES USING THE SECOND–ORDER DIFFERENCE

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Abstract. The issue of the denseness property of smooth functions has been considered in many studies. To allow approximation by smooth functions in generalized Morrey spaces, generalized Zorko spaces and vanishing-type spaces are defined. While the generalized Zorko space employs the first-order difference, we construct a subspace of the generalized Morrey space utilizing the second-order difference. We investigate its properties concerning approximation by smooth functions and its relation to the generalized Zorko space.

1. Introduction

Morrey spaces introduced by C. B. Morrey in [13] play an important role in the study of elliptic partial differential equations. The classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$, $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant n$ consists of all p-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{p,\lambda} := \sup_{x \in \mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty. \tag{1}$$

The notation $B(x,r)$ denotes the ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. For $\lambda = 0$ the space $L^{p,0}(\mathbb{R}^n)$ reduces to the Lebesgue space $L^p(\mathbb{R}^n)$. Meanwhile, when $\lambda = \infty$, we have $L^{p,\infty}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. The function $r \mapsto r^\lambda$ in (1) can be replaced by a suitable function $\varphi : (0,\infty) \to (0,\infty)$ to define a generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$.

We can say that Morrey spaces include Lebesgue spaces as a special case. However, the denseness property of smooth functions in Lebesgue spaces is not carried over to Morrey spaces. Zorko [18] constructs a function in the Morrey space that cannot be approximated by smooth functions nor even by continuous functions. In the paper, Zorko proposes a closed subset of a Morrey space in which the translation of each function is continuous and provides approximation results in such subset by compactly supported smooth functions. We call such subset a Zorko space. Later, the Zorko space plays a significant part in approximation of functions in Morrey spaces and their variants, for instance [1, 2, 5, 6, 12].

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The translation property which is possessed by functions in Zorko spaces employs the difference of the first order of the form $f(.+h)-f(.)$. In [9], the authors modify the difference into the second order to introduce the modified Zorko space. Functions in the modified subspace have the property that the norm of the difference of order two is vanishing, i.e. $||f(.+y)+f(.-y)-2f(.)||_{p,\lambda} \to 0$ as $|y| \to 0$. This modification is motivated by the well-known fact that Besov spaces (a type of smoothness spaces) can be described using higher order differences with sufficiently large smoothness exponents (see [3], [4], and [11]). An approximation result by smooth functions in the modified Zorko space is obtained in [9]. An interesting result concerning the close relation between Zorko space and the modified one is also acquired by utilizing the relation between the Zokro space and the diamond space provided in [8].

In this paper, we extend the approximation result in Morrey spaces to the generalized Morrey spaces. We propose a subspace in a generalized Morrey space employing the second-order difference. This subspace can be considered as a modification of the generalized Zorko space introduced in [1]. By the triangle inequality, we can see easily that functions in the generalized Zorko space are always in the modified one. We investigate whether the converse is true or not. We study some characterizations of the modified generalized Zorko space in terms of the approximability by smooth functions and its relation to the generalized Zorko space. In this study, the diamond space is built over a generalized Morrey space. We prove the inclusion between the generalized Zorko space and the diamond space to find the relation between the generalized Zorko space and the modified one.

2. Preliminaries

2.1. Generalized Morrey spaces

Generalized Morrey spaces were introduced by Nakai in [15] in studying Hardy-Littlewood maximal operators, singular integral operators, and the Riesz potentials in the spaces. The generalized Morrey space involves a general function $\varphi : (0,\infty) \to (0,\infty)$ to replace r^λ in the definition of the classical Morrey space.

DEFINITION 2.1. Let $1 \leqslant p < \infty$ and let $\varphi : (0,\infty) \to (0,\infty)$ be a measurable function. The generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$ is the collection of all locally p-integrable functions f on $\mathbb{R}^n$ such that

$$||f||_{p,\varphi} := \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{\varphi(r)} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

The generalized Morrey space covers the classical Morrey space by letting $\varphi(r) = r^\lambda$, $0 \leqslant \lambda \leqslant n$, that is $L^{p,\varphi}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$. If $\varphi(r) = 1$, then $L^{p,\varphi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. On the other hand, taking $\varphi(r) = r^n$ will imply $L^{p,\varphi}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. To establish nice properties for the generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$, some assumptions on the function parameter φ are needed [1].

DEFINITION 2.2. The class Φ is the collection of all measurable functions $\varphi : (0,\infty) \to (0,\infty)$ such that

1. φ is almost increasing;
2. $\varphi(t)/t^n$ is almost decreasing;
3. $\inf_{t>\delta} \varphi(t) > 0$ for every $\delta > 0$.

Note that a function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if $\varphi(s) \lesssim \varphi(t)$ (resp. $\varphi(s) \gtrsim \varphi(t)$) for $s \leqslant t$. The expression $f \lesssim g$ means that $f \leqslant cg$ for some independent positive constant c.

The conditions above defining the class Φ are widely used in papers on generalized Morrey spaces such as in [7, 16, 17]. As stated in [14], taking φ in Φ ensures that the space $L^{p,\varphi}(\mathbb{R}^n)$ is non-trivial. Under such assumptions on φ, the characteristic function on a ball belongs to $L^{p,\varphi}(\mathbb{R}^n)$. Later, we show that the characteristic function on a ball is also a member of the proposed subspace.

2.2. Modified generalized Zorko spaces

Almeida & Samko [1] introduce the generalized Zorko space to give an approximation property by smooth functions. The space is defined using the first order of difference as follows.

DEFINITION 2.3. For $1 \leqslant p < \infty$ and $\varphi \in \Phi$, the generalized Zorko subspace, denoted by $\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ is given by

$$\mathbb{L}^{p,\varphi}(\mathbb{R}^n) := \{f \in L^{p,\varphi}(\mathbb{R}^n) : ||\tau_y f - f||_{p,\varphi} \to 0 \text{ as } |y| \to 0\},$$

where $\tau_y f := f(.-y), y \in \mathbb{R}^n$.

The generalized Zorko space is proposed to have the approximation property by C^∞-functions as stated in [1] in the following.

THEOREM 2.1. *Let $\varphi \in \Phi$ and $1 \leqslant p < \infty$. Then every function with Zorko property can be approximated in Morrey norm by C^∞-functions. Moreover, we have*

$$\overline{\mathbb{L}^{p,\varphi}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)} = \mathbb{L}^{p,\varphi}(\mathbb{R}^n).$$

Differences of functions play a significant role in function spaces. Hovemann [10] states that differences are suitable tools to describe smoothness Morrey spaces, such as Besov spaces and Triebel-Lizorkin spaces. Hovemann & Sickel in [11] study under which conditions the Besov-type spaces can be described by using higher-order differences. In this paper, we construct a subset of the generalized Morrey space utilizing the second-order difference to investigate its properties concerning approximation by smooth functions and its relation to the generalized Zorko space.

DEFINITION 2.4. Let $1 \leqslant p < \infty$ and $\varphi \in \Phi$. The modified generalized Zorko subspace, denoted by $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$, is given by

$$S\mathbb{L}^{p,\varphi}(\mathbb{R}^n) := \{f \in L^{p,\varphi}(\mathbb{R}^n) : ||f(.+y)+f(.-y)-2f||_{p,\varphi} \to 0 \text{ as } |y| \to 0\}.$$

We give an example of functions in the modified generalized Zorko space $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ as follows.

EXAMPLE 1. Let $a \in \mathbb{R}^n$ and $R > 0$. The characteristic function on the ball $B(a,R)$, $f(t) = \chi_{B(a,R)}(t)$, is in $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ if $\varphi \in \Phi$ and $\lim_{r\to 0} \frac{r^n}{\varphi(r)} = 0$.

Let $|y| < R$. We consider the value of following quantity.

$$|f(t+y)+f(t-y)-2f(t)| = \begin{cases} 0,\ t \in B(a-y,R) \cap B(a+y,R) \\ 1,\ t \in B(a-y,R) \triangle B(a+y,R) \\ 2,\ t \in B(a,R) \setminus [B(a-y,R) \cup B(a+y,R)]. \end{cases}$$

Let $x \in \mathbb{R}^n$ and $r > 0$. We have

$$\begin{aligned} \int_{B(x,r)} |f(t+y)+f(t-y)-2f(t)|^p dt &= \int_{B(x,r)\cap(B_1 \triangle B_2)} 1 dt + \int_{B(x,r)\cap A} 2^p dt \\ &\leqslant 2^p \left(\int_{B(x,r)\cap(B\triangle B_1)} 1 dt + \int_{B(x,r)\cap(B\triangle B_2)} 1 dt \right) \\ &\lesssim |B(x,r) \cap (B \triangle B_1)| + |B(x,r) \cap (B \triangle B_2)| \end{aligned}$$

with $B = B(a,R)$, $B_1 = B(a-y,R)$, $B_2 = B(a+y,R)$, and $A = B(a,R) \setminus [B(a-y,R) \cup B(a+y,R)]$.

For $r \leqslant |y|^{\frac{1}{n}}$, we have

$$\frac{r^n}{\varphi(r)} \lesssim \frac{|y|}{\varphi(|y|^{\frac{1}{n}})}$$

as $\varphi(t)/t^n$ is almost decreasing. Thus,

$$\begin{aligned} K_1 &:= \sup_{x\in\mathbb{R}^n, 0<r\leqslant|y|^{\frac{1}{n}}} \left(\frac{1}{\varphi(r)} \int_{B(x,r)} |f(t+y)+f(t-y)-2f(t)|^p dt \right)^{\frac{1}{p}} \\ &\lesssim \sup_{x\in\mathbb{R}^n, 0<r\leqslant|y|^{\frac{1}{n}}} \left(\frac{1}{\varphi(r)} (|B(x,r) \cap (B \triangle B_1)| + |B(x,r) \cap (B \triangle B_2)|) \right)^{\frac{1}{p}} \\ &\lesssim \sup_{x\in\mathbb{R}^n, 0<r\leqslant|y|^{\frac{1}{n}}} \left(\frac{1}{\varphi(r)} |B(x,r)| \right)^{\frac{1}{p}} \\ &\lesssim \sup_{0<r\leqslant|y|^{\frac{1}{n}}} \left(\frac{r^n}{\varphi(r)} \right)^{\frac{1}{p}} \\ &\lesssim \left(\frac{|y|}{\varphi(|y|^{\frac{1}{n}})} \right)^{\frac{1}{p}}. \end{aligned}$$

Note that $\lim_{|y|\to 0} K_1 \lesssim \lim_{|y|\to 0}\left(\frac{|y|}{\varphi(|y|^{\frac{1}{n}})}\right)^{\frac{1}{p}} = 0$. On the other hand, for $r > |y|^{\frac{1}{n}}$ we obtain

$$\begin{aligned}
K_2 &:= \sup_{x\in\mathbb{R}^n, r>|y|^{\frac{1}{n}}} \left(\frac{1}{\varphi(r)}\int_{B(x,r)} |f(t+y)+f(t-y)-2f(t)|^p dt\right)^{\frac{1}{p}} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>|y|^{\frac{1}{n}}} \left(\frac{1}{\varphi(r)}(|B(x,r)\cap(B\triangle B_1)| + |B(x,r)\cap(B\triangle B_2)|)\right)^{\frac{1}{p}} \\
&\lesssim \sup_{r>|y|^{\frac{1}{n}}} \left(\frac{1}{\varphi(r)}(|B\triangle B_1| + |B\triangle B_2|)\right)^{\frac{1}{p}} \\
&\lesssim \sup_{r>|y|^{\frac{1}{n}}} \left(\frac{1}{\varphi(r)}|B\triangle B_1|\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\varphi(t)$ is almost increasing, we have $\frac{1}{\varphi(r)} \lesssim \frac{1}{\varphi(|y|^{\frac{1}{n}})}$. As a consequence,

$$\begin{aligned}
K_2 &\lesssim \sup_{r>|y|^{\frac{1}{n}}} \left(\frac{1}{\varphi(r)}|B\triangle B_1|\right)^{\frac{1}{p}} \\
&\lesssim \left(\frac{1}{\varphi(|y|^{\frac{1}{n}})}|y|(R+(k-\frac{1}{2})|y|)^{n-1}\right)^{\frac{1}{p}} \\
&= \left(\frac{|y|}{\varphi(|y|^{\frac{1}{n}})}(R+(k-\frac{1}{2})|y|)^{n-1}\right)^{\frac{1}{p}}.
\end{aligned}$$

Note that for $|y| \to 0$ we obtain $K_2 \to 0$ since $\frac{|y|}{\varphi(|y|^{\frac{1}{n}})} \to 0$. Based on the definition,

$$||f(.+y)+f(.-y)-2f||_{p,\varphi} \lesssim \max(K_1, K_2).$$

As $|y| \to 0$, we conclude that $||f(.+y)+f(.-y)-2f||_{p,\varphi} \to 0$. Hence, $f(t) = \chi_{B(a,R)}(t) \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$.

By using the triangle inequality and the limit property of the functions in the modified generalized Zorko space, the modified generalized Zorko space $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ is a closed subspace as stated in the lemma below.

LEMMA 2.2. *The modified generalized Zorko space* $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ *is closed in* $L^{p,\varphi}(\mathbb{R}^n)$.

3. Approximation results in modified generalized Zorko spaces

In this section, we provide some properties regarding approximation by smooth functions in the modified generalized Zorko space $S\mathbb{L}^{p,\varphi}$. In the beginning we show that the space $S\mathbb{L}^{p,\varphi}$ is closed under the operator of convolution with integrable kernels. This result is used later to form smooth functions that approximate a function in the modified generalized Zorko space. We also present an approximation result in the modified space by continuous functions with compact support.

The lemma below shows that the convolution result of a function in a generalized Morrey space with an integrable function is in the generalized Morrey space.

LEMMA 3.1. ([1]) *Let $1 \leqslant p < \infty$ and $\varphi \in \Phi$. If $\phi \in L^1(\mathbb{R}^n)$ and f is locally p-integrable on $\mathbb{R}^n$, then*

$$\frac{1}{\varphi(r)}\int_{B(x,r)} |(f*\phi)(t)|^p dt \leqslant ||\phi||_{L_1}^p \sup_{z\in\mathbb{R}^n} \frac{1}{\varphi(r)}\int_{B(z,r)} |f(s)|^p ds$$

for every $x \in \mathbb{R}^n$ and $r > 0$. Consequently, for $f \in L^{p,\varphi}$ we have

$$||f*\phi||_{p,\varphi} \leqslant ||\phi||_1 ||f||_{p,\varphi}.$$

In line with the above result, the modified generalized Zorko space $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ is also closed under the convolution operator with an integrable kernel as presented in the following lemma.

LEMMA 3.2. *Let $1 \leqslant p < \infty$ and $\varphi \in \Phi$. The modified generalized Zorko space $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ is closed under the convolution operator with a kernel $\phi \in L^1(\mathbb{R}^n)$.*

Proof. Let $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ and $\phi \in L^1(\mathbb{R}^n)$. For $x, y \in \mathbb{R}^n$ and $r > 0$, the Minkowski's integral inequality implies

$$\begin{aligned}
&\left[\int_{B(x,r)} |(f*\phi)(t+y) + (f*\phi)(t-y) - 2(f*\phi)(t)|^p dt\right]^{1/p} \\
&\leqslant \int_{\mathbb{R}^n} \left[\int_{B(x,r)} |f(t-s+y) + f(t-s-y) - 2f(t-s)|^p\, dt\right]^{1/p} |\phi(s)| ds \\
&= \int_{\mathbb{R}^n} \left[\int_{B(x-s,r)} |f(u+y) + f(u-y) - 2f(u)|^p\, du\right]^{1/p} |\phi(s)| ds.
\end{aligned}$$

Thus, by the definition of the generalized Morrey norm we have

$$||(f*\phi)(\cdot+y) + (f*\phi)(\cdot-y) - 2(f*\phi)||_{p,\varphi} \leqslant ||f(\cdot+y) + f(\cdot-y) - 2f||_{p,\varphi} ||\phi||_1.$$

Since $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$, we come to the result that

$$\lim_{|y|\to 0} ||(f*\phi)(.+y) + (f*\phi)(.-y) - 2(f*\phi)||_{p,\varphi} = 0.$$

This implies that $f*\phi \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ for any $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$. $\square$

If the kernel of the convolution in Lemma 3.1 is a smooth function, then the result of the convolution is also a smooth function as stated in the following lemma.

LEMMA 3.3. *Let $1 \leqslant p < \infty$ and $\varphi \in \Phi$. Let $\phi \in C_c^\infty$ with $||\phi||_1 = 1$. If $f \in L^{p,\varphi}(\mathbb{R}^n)$, then $f*\phi \in C^\infty$.*

Now we ready to establish our main theorem discussing an approximation result in the modified generalized Zorko space $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$.

THEOREM 3.4. *Let $1 \leqslant p < \infty$ and $\varphi \in \Phi$. If $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$, then f can be approximated by smooth functions in the generalized Morrey norm. In other words,*

$$\overline{S\mathbb{L}^{p,\varphi}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)} = S\mathbb{L}^{p,\varphi}(\mathbb{R}^n).$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ with $||\phi||_1 = 1$. We define $\tilde{\phi}(x) = \phi(-x)$ and $\phi_j(x) := j^n\phi(jx)$ for all $j \in \mathbb{N}$. By virtue of Lemma 3.2 and Lemma 3.3, we know that $f*\phi_j$ and $f*\tilde{\phi}_j$ is in $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, $r > 0$, and $j \in \mathbb{N}$ we observe that using the Minkowski's inequality yields

$$\begin{aligned}
&\left[\int_{B(x,r)} \left|\frac{f*\phi_j(u) + f*\tilde{\phi}_j(u)}{2} - f(u)\right|^p du\right]^{\frac{1}{p}} \\
&\lesssim \left[\int_{B(x,r)} \left|\int_{\mathbb{R}^n} (f(u-s) + f(u+s) - 2f(u))\phi_j(s)ds\right|^p du\right]^{\frac{1}{p}} \\
&\lesssim \int_{\mathbb{R}^n} \left(\int_{B(x,r)} |f(u-s) + f(u+s) - 2f(u)|^p du\right)^{\frac{1}{p}} |\phi_j(s)|ds.
\end{aligned}$$

We now have the following expression by employing the change of variable method to the above inequality,

$$\left\|\frac{f*\phi_j + f(u)*\tilde{\phi}_j(u)}{2} - f\right\|_{p,\varphi} \lesssim \int_{\mathbb{R}^n} \|f(\cdot + t/j) + f(\cdot - t/j) - 2f(\cdot)\|_{p,\varphi} |\phi(t)|dt.$$

Since $\|f(\cdot + t/j) + f(\cdot - t/j) - 2f(\cdot)\|_{p,\varphi} \leqslant 4\|f\|_{p,\varphi}$, applying the Lebesgue's dominated convergence theorem and using the fact that $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ results in

$$\begin{aligned}
&\lim_{j\to\infty} \left\|\frac{f*\phi_j + f*\tilde{\phi}_j}{2} - f\right\|_{p,\varphi} \\
&\leqslant \lim_{j\to\infty} \frac{1}{2}\int_{\mathbb{R}^n} \left\|f(\cdot + \frac{t}{j}) + f(\cdot - \frac{t}{j}) - 2f(\cdot)\right\|_{p,\varphi} |\phi(t)|dt \\
&= \frac{1}{2}\int_{\mathbb{R}^n} \lim_{j\to\infty} \left\|f(\cdot + \frac{t}{j}) + f(\cdot - \frac{t}{j}) - 2f(\cdot)\right\|_{p,\varphi} |\phi(t)|dt \\
&= 0.
\end{aligned}$$

Thus $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ can be approximated in $L^{p,\varphi}(\mathbb{R}^n)$ norm by smooth functions. By virtue of Lemma 2.2, we can conclude that

$$\overline{S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)} = S\mathbb{L}^{p,\varphi}(\mathbb{R}^n). \quad \square$$

The next result shows another approximation property in the modified generalized Zorko space $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$. The theorem below gives a sufficient condition for functions in the modified generalized Zorko space in terms of approximability by continuous functions.

THEOREM 3.5. *Let* $1 \leqslant p < \infty$ *and* $\varphi \in \Phi$. *If* $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ *can be approximated in generalized Morrey norm by continuously differentiable functions with compact support, then* $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$.

Proof. Let $\{g_j\} \subseteq C_c^1(\mathbb{R}^n)$ such that $\lim_{j\to\infty} ||g_j - f||_{p,\varphi} = 0$. By the triangle inequality,

$$\begin{aligned}||f(\cdot+y)+f(\cdot-y)-2f||_{p,\varphi} \leqslant\; & ||f(\cdot+y)-g_j(\cdot+y)||_{p,\varphi}+||g_j(\cdot+y)-g_j||_{p,\varphi}\\ &+||f(\cdot-y)-g_j(\cdot-y)||_{p,\varphi}+||g_j(\cdot-y)-g_j||_{p,\varphi}\\ &+2||f-g_j||_{p,\varphi}.\end{aligned}$$

Suppose that $\operatorname{supp}(g_j) \subseteq B(0,R_j-1)$ for some $R_j > 1$. If we take $|y| < 1$, then by the mean value theorem we obtain

$$\begin{aligned}|g_j(x+y)-g_j(x)| &= \left|\int_0^1 y.\nabla g_j(x+ty)dt\right|\\ &\leqslant \int_0^1 |y||\nabla g_j(x+ty)|dt\\ &\leqslant \int_0^1 |y|||\nabla g_j||_\infty dt\\ &= |y|||\nabla g_j||_\infty.\end{aligned}$$

Let $M = \sup_j\{R_j\}$. As a consequence, for any $x \in \mathbb{R}^n$ and $r > 0$ we have

$$\begin{aligned}\int_{B(x,r)} |g_j(x+y)-g_j(x)|^p dx &= \int_{B(x,r)\cap B(0,M)} |g_j(x+y)-g_j(x)|^p dx\\ &\leqslant \int_{B(x,r)\cap B(0,M)} |y|^p||\nabla g_j||_\infty^p dx\\ &= |y|^p||\nabla g_j||_\infty^p|B(x,r)\cap B(0,M)|.\end{aligned}$$

Thus we have come to the following expression,

$$||g_j(.+y)-g_j(.)|| \leqslant |y|||\nabla g_j||_\infty \sup_{x\in\mathbb{R}^n, r>0}\left[\frac{1}{\varphi(r)}|B(x,r)\cap B(0,M)|\right]^{\frac{1}{p}}.$$

For $r < M$, since $\varphi(t)/t^n$ is almost decreasing, we obtain

$$\frac{r^n}{\varphi(r)} \lesssim \frac{M^n}{\varphi(M)}.$$

Thus,

$$\begin{aligned}\sup_{x\in\mathbb{R}^n, r<M}\left[\frac{1}{\varphi(r)}|B(x,r)\cap B(0,R)|\right]^{\frac{1}{p}} &\leqslant \sup_{r<M}\left[\frac{1}{\varphi(r)}|B(x,r)|\right]^{\frac{1}{p}}\\ &\lesssim \sup_{r<M}\left[\frac{r^n}{\varphi(r)}\right]^{\frac{1}{p}}\\ &\lesssim \left[\frac{M^n}{\varphi(M)}\right]^{\frac{1}{p}}.\end{aligned}$$

On the other hand, for $r \geqslant M$ we have the following relation as $\varphi(t)$ is an almost increasing function

$$\frac{1}{\varphi(r)} \lesssim \frac{1}{\varphi(M)}.$$

So,

$$\begin{aligned}\sup_{x\in\mathbb{R}^n, r\geqslant M}\left[\frac{1}{\varphi(r)}|B(x,r)\cap B(0,M)|\right]^{\frac{1}{p}} &\leqslant \sup_{r\geqslant M}\left[\frac{1}{\varphi(r)}|B(0,M)|\right]^{\frac{1}{p}}\\ &\lesssim \sup_{r\geqslant M}\left[\frac{R^n}{\varphi(r)}\right]^{\frac{1}{p}}\\ &\lesssim \left[\frac{M^n}{\varphi(M)}\right]^{\frac{1}{p}}.\end{aligned}$$

As a consequence,

$$||g_j(.+y)-g_j(.)||_{p,\varphi} \lesssim |y|||\nabla g_j||_\infty \left[\frac{M^n}{\varphi(M)}\right]^{\frac{1}{p}}.$$

Hence, we obtain

$$||f(.+y)+f(.-y)-2f||_{p,\varphi} \lesssim 4||f-g_j||_{p,\varphi} + 2|y|||\nabla g_j||_\infty \left[\frac{M^n}{\varphi(M)}\right]^{\frac{1}{p}}.$$

Taking $y \to 0$ yields

$$\begin{aligned}&\limsup_{y\to 0}||f(.+y)+f(.-y)-2f||_{p,\varphi}\\ &\lesssim \limsup_{y\to 0} 4||f-g_j||_{p,\varphi} + \limsup_{y\to 0} 2|y|||\nabla g_j||_\infty \left[\frac{M^n}{\varphi(M)}\right]^{\frac{1}{p}}\\ &= 4||f-g_j||_{p,\varphi}.\end{aligned}$$

Now let j gets larger and larger to attain

$$\limsup_{y\to 0}||f(.+y)+f(.-y)-2f||_{p,\varphi} \lesssim \lim_{j\to\infty} 4||f-g_j||_{p,\varphi} = 0.$$

We can conclude that $\lim_{y\to 0}||f(.+y)+f(.-y)-2f||_{p,\varphi} = 0$, or in other words, $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$. It completes the proof. □

4. Diamond spaces built in generalized Morrey spaces

In this section, we investigate a close relation between the generalized Zorko space and the modified one by employing properties of the diamond space. The notion of diamond space is introduced in [19] in studying interpolation properties. In this study, the diamond space is built over generalized Morrey spaces. The definition of the diamond space is given in the following.

DEFINITION 4.1. The diamond space $\diamond L^{p,\varphi}(\mathbb{R}^n)$ denotes the closure in $L^{p,\varphi}(\mathbb{R}^n)$ of the set of all infinitely differentiable functions f such that $\partial^\alpha f \in L^{p,\varphi}(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$.

By using the triangle inequality, it is easy to see the following inclusion.

LEMMA 4.1. *Let $1 \leqslant p < \infty$ and $\varphi \in \Phi$. We have the inclusion*

$$\mathbb{L}^{p,\varphi}(\mathbb{R}^n) \subseteq S\mathbb{L}^{p,\varphi}(\mathbb{R}^n).$$

The following theorem discusses the inclusion relation between the modified generalized Zorko space $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ and the diamond space $\diamond L^{p,\varphi}(\mathbb{R}^n)$.

THEOREM 4.2. *Let $1 \leqslant p < \infty$ and $\varphi \in \Phi$. If $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$, then $f \in \diamond L^{p,\varphi}(\mathbb{R}^n)$.*

Proof. Let $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$. Consider a non-negative function $\phi \in C_c^\infty(\mathbb{R}^n)$ and $||\phi||_1 = 1$. Suppose that $\mathrm{supp}(\phi) \subseteq B(0,R)$ for some $R > 0$. We set $\phi_j(x) := j^n\phi(jx)$ for each $j \in \mathbb{N}$ and $\tilde{\phi}(x) := \phi(-x)$. Define $f_j := \frac{1}{2}(f*\phi_j + f*\tilde{\phi}_j)$. For any $x \in \mathbb{R}^n$ and $r > 0$, by using the Minkowski's inequality, we have

$$\left[\int_{B(x,r)}\left|\frac{1}{2}(f*\phi_j(u)+f*\tilde{\phi}_j(u))-f(u)\right|^p du\right]^{\frac{1}{p}}$$
$$\lesssim \int_{\mathbb{R}^n}\left[\int_{B(x,r)}|f(u-s)+f(u+s)-2f(u)|^p du\right]^{\frac{1}{p}}|\phi_j(s)|ds.$$

This will lead to the expressions

$$\begin{aligned}||f_j-f||_{p,\varphi} &\lesssim \int_{\mathbb{R}^n}||f(\cdot-s)+f(\cdot+s)-2f(\cdot)||_{p,\varphi}|\phi_j(s)|ds\\ &= \int_{|s|<R/j}||f(\cdot-s)+f(\cdot+s)-2f(\cdot)||_{p,\varphi}|\phi_j(s)|ds.\end{aligned}$$

Taking supremum over the ball $B(0,R/j)$ for the value of $||f(\cdot-s)+f(\cdot+s)-2f(\cdot)||_{p,\varphi}$ implies

$$||f_j - f||_{p,\varphi} \lesssim \int_{|s|<R/j} \sup_{|t|<R/j} ||f(\cdot-t)+f(\cdot+t)-2f(\cdot)||_{p,\varphi}|\phi_j(s)|ds.$$

Since $||\phi||_1 = 1$, we obtain

$$||f_j - f||_{p,\varphi} \lesssim \sup_{|t|<R/j} ||f(\cdot-t)+f(\cdot+t)-2f(\cdot)||_{p,\varphi}.$$

As $f \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$, taking $j \to \infty$ will gain

$$\lim_{j\to\infty} ||f_j - f||_{p,\varphi} = 0.$$

Thus $f_j \to f$ in the norm of $L^{p,\varphi}(\mathbb{R}^n)$. By the property of convolution, we see that

$$\begin{aligned} \partial^\alpha f_j &= \partial^\alpha \left(\frac{1}{2}(f*\phi_j + f*\tilde{\phi}_j) \right) \\ &= \frac{1}{2}(f*\partial^\alpha \phi_j + (-1)^{|\alpha|}(f*\partial^\alpha \tilde{\phi}_j)). \end{aligned}$$

Since $\partial^\alpha \phi_j$ and $\partial^\alpha \tilde{\phi}_j$ are in $C^\infty(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ and $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ is invariant to convolution with an integrable kernel, we have $\partial^\alpha f_j \in S\mathbb{L}^{p,\varphi}(\mathbb{R}^n) \subseteq L^{p,\varphi}(\mathbb{R}^n)$. Now we have come to the conclusion that $f \in \diamond L^{p,\varphi}(\mathbb{R}^n)$. □

In the following theorem, we provide the inclusion relation between the diamond space and the generalized Zorko space.

THEOREM 4.3. *Let $1 \leqslant p < \infty$ and $\varphi \in \Phi$. We have the inclusion*

$$\diamond L^{p,\varphi}(\mathbb{R}^n) \subseteq \mathbb{L}^{p,\varphi}(\mathbb{R}^n).$$

Proof. Let $f \in \diamond L^{p,\varphi}(\mathbb{R}^n)$. Then there exists a sequence of smooth functions $\{f_j\}_{j=1}^\infty \subseteq L^{p,\varphi}(\mathbb{R}^n)$ such that $\partial^\alpha f_j \in L^{p,\varphi}(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ and $j \in \mathbb{N}$, and $\lim_{j\to\infty} f_j = f$ in the norm of $L^{p,\varphi}(\mathbb{R}^n)$. Let $y \in \mathbb{R}^n$ with $|y| < 1$. By virtue of the triangle inequality we have

$$||f(\cdot+y) - f||_{p,\varphi} \leqslant ||f(\cdot+y) - f_j(\cdot+y)||_{p,\varphi} + ||f_j(\cdot+y) - f_j||_{p,\varphi} + ||f_j - f||_{p,\varphi}.$$

The function f_j is smooth in the ball $B(x, r+|y|)$. Thus by mean value theorem we obtain

$$f_j(s+y) - f_j(s) = \int_0^1 y.\nabla f_j(s+ty)dt.$$

Using Minkowski's inequality implies to the following expressions

$$\begin{aligned}
||f_j(\cdot+y)-f_j(\cdot)||_{p,\varphi} &= \sup_{x\in\mathbb{R}^n, r>0}\left(\frac{1}{\varphi(r)}\int_{B(x,r)}\left|\int_0^1 y.\nabla f_j(s+ty)dt\right|^p ds\right)^{\frac{1}{p}} \\
&\leqslant \sup_{x\in\mathbb{R}^n, r>0}\left(\frac{1}{\varphi(r)}\int_{B(x,r)}\left[\int_0^1 |\nabla f_j(s+ty)||y|dt\right]^p\right)^{\frac{1}{p}} \\
&\leqslant \sup_{x\in\mathbb{R}^n, r>0}\int_0^1\left(\frac{1}{\varphi(r)}\int_{B(x,r)}|\nabla f_j(s+ty)|^p|y|^p ds\right)^{\frac{1}{p}} dt \\
&\leqslant |y|\int_0^1 \sup_{x\in\mathbb{R}^n, r>0}\left(\frac{1}{\varphi(r)}\int_{B(x,r)}|\nabla f_j(s+ty)|^p ds\right)^{\frac{1}{p}} dt.
\end{aligned}$$

This process will lead to the expression

$$||f_j(\cdot+y)-f_j(\cdot)||_{p,\varphi} \leqslant |y|||\nabla f_j||_{p,\varphi}.$$

Consequently,

$$||f(\cdot+y)-f||_{p,\varphi} \leqslant |y|||\nabla f_j||_{p,\varphi}+2||f-f_j||_{p,\varphi}.$$

Taking $y\to 0$ will imply

$$\begin{aligned}
\limsup_{y\to 0}||f(\cdot+y)-f||_{p,\varphi} &\leqslant \limsup_{y\to 0}|y|.||\nabla f_j||_{p,\varphi}+\limsup_{y\to 0}2||f-f_j||_{p,\varphi} \\
&= 2||f-f_j||_{p,\varphi}.
\end{aligned}$$

Continuing the process by letting $j\to\infty$ yields

$$\limsup_{y\to 0}||f(\cdot+y)-f(\cdot)||_{p,\varphi} \leqslant 0.$$

We can conclude that $\lim_{y\to 0}||f(\cdot+y)-f(\cdot)||_{p,\varphi}$. In other words, f is in the Zorko space $\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$. □

By employing Lemma 4.1, Theorem 4.2, and Theorem 4.3, we come to the conclusion that the generalized Zorko space $\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ and the modified generalized Zorko space $S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ represent the same set as stated in the following corollary.

COROLLARY 4.4. *Let* $1\leqslant p<\infty$ *and* $\varphi\in\Phi$. *We have*

$$S\mathbb{L}^{p,\varphi}(\mathbb{R}^n)=\mathbb{L}^{p,\varphi}(\mathbb{R}^n).$$

This relation shows that modifying the order of difference from the first order to the second one in generalized Zorko spaces does not alter approximation properties by smooth functions in generalized Morrey spaces.

5. Conclusion

The set of smooth functions is not dense in the generalized Morrey spaces. To deal with the issue, some subspaces with specific properties are defined in generalized Morrey spaces. One of them is the generalized Zorko space in which its members' first order of difference is vanishing. Every function in the generalized Zorko space can be approximated by smooth functions. In this paper, the subspace of generalized Morrey space utilizing the second order of difference is defined. Using convolution with compactly supported smooth functions, approximation property by C^∞-functions is still satisfied in the new subspace. The inclusion between the diamond space and the generalized Zorko space suggests that modifying the difference of order one with order two does not alter the subspace. It is still in question whether the higher order of difference would impact the generalized Zorko space and its approximation properties.

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