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ONE–SIDED MAXIMAL OPERATORS ON HERZ SPACES WITH VARIABLE EXPONENTS

KWOK-PUN HO

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Abstract. This paper extends the boundedness of the one-sided maximal operators from the Lebesgue spaces with variable exponents to the one-sided Herz spaces with variable exponents. The main result generalizes the boundedness of the one-sided maximal operators on the classical Herz spaces.

1. Introduction

In this paper, we establish the boundedness of the one-sided maximal operators on the one-sided Herz spaces with variable exponents.

The Herz spaces with variable exponents are extensions of the Herz spaces and the Lebesgue spaces with variable exponents. The Herz spaces were introduced by Herz in [9] to study the Fourier series. The Herz spaces also provide applications on summability of Fourier series, harmonic analysis and partial differential equations, see [6, 15, 22, 30, 31]. The Lebesgue spaces with variable exponent are extensions of the Lebesgue spaces. It provides applications on partial differential equation and fluid dynamics, see [4, Part III]. An important breakthrough for the studies of the Lebesgue spaces with variable exponent is on the boundedness of the Hardy-Littlewood maximal function, see [3, 4, 27, 28].

The one-sided maximal operators are the "original" maximal function studies by Hardy and Littlewood in [8]. They are ancestors of the nowadays well known Hardy-Littlewood maximal function. The study of the one-sided maximal operators has it own independent interest. For example, it offers applications on the ergodic theory. For the mapping properties of the one-sided maximal operators and theirs applications, see [23, 24, 25, 26, 34].

The boundedness of the one-sided maximal operators has been extended to the Lebesgue spaces with variable exponents in [5, 29]. It motivates us to investigate the boundedness of the one-sided maximal operators on the Herz spaces with variable exponents. We introduce the one-sided Herz spaces with variable exponents to study the boundedness of the one-sided maximal operators. We obtain our main result by using the notion of localized operators introduced in [14].

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This paper is organized as follows. Section 2 contains the definitions of the one-sided maximal operators, the Lebesgue spaces with variable exponents and the one-sided Herz spaces with variable exponents. The boundedness of the one-sided maximal operators on the Lebesgue spaces with variable exponents is also recalled in this section. The main result of this paper, the boundedness of the one-sided maximal operators on the one-sided Herz spaces with variable exponents, is established in Section 3.

2. Definitions and preliminaries

Let $\mathscr{M}$ and L^1_{loc} denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R}$, respectively.

For any $f \in L^1_{\text{loc}}$, the one-sided Hardy-Littlewood maximal operators $M^+ f$ and $M^- f$ are defined as

$$M^+ f(x) = \sup_{t>0} \frac{1}{t} \int_x^{x+t} |f(y)|dy, \quad x \in \mathbb{R},$$
$$M^- f(x) = \sup_{t>0} \frac{1}{t} \int_{x-t}^{x} |f(y)|dy, \quad x \in \mathbb{R},$$

respectively.

We now recall the definition of the Lebesgue spaces with variable exponents on $\mathbb{R}$ [3, 4].

DEFINITION 1. Let $p(\cdot) : \mathbb{R} \to (1,\infty)$ be a Lebesgue measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ so that

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : \rho(f/\lambda) \leqslant 1\} < \infty$$

where

$$\rho(f) = \int_{\mathbb{R}} |f(x)|^{p(x)} dx.$$

We call $p(\cdot)$ the exponent function of $L^{p(\cdot)}$.

The Lebesgue space with variable exponent is a Banach function space. For simplicity, we refer the reader to [2, Chapter 1] for the definition of Banach function space and its properties.

For any exponent function $p(\cdot) : \mathbb{R} \to (1,\infty)$, write

$$p_+ = \operatorname*{ess\,sup}_{x\in(0,\infty)} p(x) \quad \text{and} \quad p_- = \operatorname*{ess\,inf}_{x\in(0,\infty)} p(x).$$

For any Lebesgue measurable function $p(\cdot) : \mathbb{R} \to (1,\infty)$, define $p'(x) = \frac{p(x)}{p(x)-1}$, $x \in \mathbb{R}$. For the details of the Lebesgue spaces with variable exponents, the reader is referred to [3, 4].

DEFINITION 2. Let $p(\cdot):\mathbb{R}\to(1,\infty)$ be a Lebesgue measurable function. We write $p(\cdot)\in\mathscr{M}_+$ if $M^+:L^{p(\cdot)}\to L^{p(\cdot)}$ is bounded. We write $p(\cdot)\in\mathscr{M}_-$ if $M^-:L^{p(\cdot)}\to L^{p(\cdot)}$ is bounded.

We now give some concrete conditions that guarantee $p(\cdot)\in\mathscr{M}_+\cup\mathscr{M}_-$ from [29].

DEFINITION 3. Let $p(\cdot):\mathbb{R}\to(1,\infty)$ be a Lebesgue measurable function with $1<p_-\leqslant p_+<\infty$. We write $p(\cdot)\in\mathscr{L}_+$ if there is a constant $C>0$ such that

$$p(y)\geqslant p(x)+\frac{C}{\ln(y-x)}$$

for all $x,y\in\mathbb{R}$ with $0<y-x\leqslant\frac{1}{2}$.

We write $p(\cdot)\in\mathscr{L}_-$ if $\tilde{p}(\cdot)\in\mathscr{L}_+$ where $\tilde{p}(x)=p(-x)$, $x\in\mathbb{R}$.

DEFINITION 4. Let $p(\cdot):\mathbb{R}\to(1,\infty)$ be a Lebesgue measurable function. We write $p(\cdot)\in\mathscr{P}$ if there is a constant $c>0$ such that

$$\int_{\{x:p(x)\neq 0\}} c^{\frac{1}{p(x)}}\,dx<\infty.$$

We have the following result from [29, Theorem 1].

THEOREM 1. *Let $p(\cdot):\mathbb{R}\to(1,\infty)$ be a Lebesgue measurable function.*

1. *If $p(\cdot)\in\mathscr{L}_+$ and there exists a non-increasing function $q(\cdot)$ satisfying $1<q_-\leqslant q_+<\infty$, $|p(\cdot)-q(\cdot)|\in\mathscr{P}$, then $p(\cdot)\in\mathscr{M}_+$.*

2. *If $p(\cdot)\in\mathscr{L}_-$ and there exists a non-decreasing function $q(\cdot)$ satisfying $1<q_-\leqslant q_+<\infty$, $|p(\cdot)-q(\cdot)|\in\mathscr{P}$, then $p(\cdot)\in\mathscr{M}_-$.*

We have another condition that guarantees $p(\cdot)\in\mathscr{M}_+\cup\mathscr{M}_-$. We need to recall some notations from [29] to present this condition.

Let $e_0=1$ and $e_{k+1}=e^{e_k}$, $k\in\mathbb{N}\backslash\{0\}$. Write $\ln_0 x=x$. Let $k\in\mathbb{N}\backslash\{0\}$. For any $x\in(e_k,\infty)$, write $\ln_k x=\ln(\ln_{k-1}x)$. For any $\alpha>0$, write

$$b_{k,\alpha}(x)=-\frac{1}{\alpha}\frac{d}{dx}(\ln_k^{-\alpha}x),\quad x\in(e_k,\infty).$$

DEFINITION 5. Let $p(\cdot):\mathbb{R}\to(1,\infty)$ be a Lebesgue measurable function with $1<p_-\leqslant p_+<\infty$. We write $p(\cdot)\in\mathfrak{Q}$ if

1. $p(x)=p(-x)$ for all $x\in\mathbb{R}$,

2. there are $\alpha,C>0$ and $k\in\mathbb{N}\backslash\{0\}$ such that $p(\cdot)$ is monotone on (e_k,∞) and

$$\left|\frac{dp(x)}{dx}\right|\leqslant Cb_{k,\alpha}(x),\quad x\in(e_k,\infty).$$

The following result is from [29, Theorem 2].

THEOREM 2. *Let* $p(\cdot):\mathbb{R}\to(1,\infty)$ *be a Lebesgue measurable function.*

1. *If* $p(\cdot)\in\mathscr{L}_+$ *and there exists a* $q(\cdot)\in\mathfrak{Q}$ *such that* $|p(\cdot)-q(\cdot)|\in\mathscr{P}$, *then* $p(\cdot)\in\mathscr{M}_+$.

2. *If* $p(\cdot)\in\mathscr{L}_-$ *and there exists a* $q(\cdot)\in\mathfrak{Q}$ *such that* $|p(\cdot)-q(\cdot)|\in\mathscr{P}$, *then* $p(\cdot)\in\mathscr{M}_-$.

We now present the definition of the one-sided Herz spaces with variable exponents studied in this paper. For any $m\in\mathbb{N}\backslash\{0\}$, define

$$D_0=(-1,1),\quad D_m=[2^{m-1},2^m),\quad D_{-m}=(-2^m,-2^{m-1}].$$

DEFINITION 6. Let $\theta\in(0,\infty)$, $\alpha\in(-\infty,\infty)$ and $p(\cdot):\mathbb{R}\to(1,\infty)$ be a Lebesgue measurable function. The one-sided Herz space with variable exponent $\mathring{K}^{\alpha}_{p(\cdot),\theta}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{\mathring{K}^{\alpha}_{p(\cdot),\theta}}=\left(\sum_{k\in\mathbb{Z}}\|2^{k\alpha}f\chi_{D_k}\|^{\theta}_{L^{p(\cdot)}}\right)^{\frac{1}{\theta}}<\infty.$$

When $\alpha>0$, the Herz space with variable exponent studied in [1, 10, 11] is a subspace of $\mathring{K}^{\alpha}_{p(\cdot),\theta}$. Whenever $p(\cdot)=p$, $p\in(1,\infty)$, is a constant function, we denote $\mathring{K}^{\alpha}_{p(\cdot),\theta}$ by $\mathring{K}^{\alpha}_{p,\theta}$. It is an extension of the classical Herz spaces. For the studies of the classical Herz space and its generalizations, see [6, 9, 14, 16, 20, 21, 22, 30, 31, 35].

A number of important results in harmonic analysis, such as the boundedness of the Hardy-Littlewood maximal function, the singular integral operators and bilinear operators had been extended to the classical Herz spaces and Herz spaces with variable exponents studied in [1, 10, 11], see [12, 13, 17, 18, 19].

The following result assures that for any $f\in\mathring{K}^{\alpha}_{p(\cdot),\theta}$, M^+f and M^-f are Lebesgue measurable.

PROPOSITION 1. *Let* $\theta\in(0,\infty)$, $\alpha\in(-\infty,\infty)$ *and* $p(\cdot):\mathbb{R}\to(1,\infty)$ *be a Lebesgue measurable function. For any* $f\in\mathring{K}^{\alpha}_{p(\cdot),\theta}$, M^+f, M^-f *are Lebesgue measurable functions.*

Proof. We show that for any $m\in\mathbb{N}$ and $f\in\mathring{K}^{\alpha}_{p(\cdot),\theta}$, $\int_{-2^m}^{2^m}|f(y)|dy$ is finite.

The Hölder inequality for $L^{p(\cdot)}$ [3, Theorem 2.36] gives

$$\int_{-2^m}^{2^m}|f(y)|dy=\sum_{k=-m}^{m}\int_{\mathbb{R}}\chi_{D_k}(y)|f(y)|dy\leqslant C\sum_{k=-m}^{m}\|\chi_{D_k}\|_{L^{p'(\cdot)}}\|\chi_{D_k}f\|_{L^{p(\cdot)}}.$$

When $\theta \in (1,\infty)$, the Hölder inequality and the inequality $\|\chi_{D_k}\|_{L^{p'(\cdot)}} \leqslant \|\chi_{(-2^m,2^m)}\|_{L^{p'(\cdot)}}$ yield

$$\begin{aligned}
&\int_{-2^m}^{2^m} |f(y)|dy \\
&\leqslant C\left(\sum_{k=-m}^{m} (2^{-k\alpha}\|\chi_{D_k}\|_{L^{p'(\cdot)}})^{\theta'}\right)^{\frac{1}{\theta'}} \left(\sum_{k=-m}^{m} (2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^{\theta}\right)^{\frac{1}{\theta}} \\
&\leqslant C\left(\sum_{k=-m}^{m} (2^{-k\alpha}\|\chi_{(-2^m,2^m)}\|_{L^{p'(\cdot)}})^{\theta'}\right)^{\frac{1}{\theta'}} \|f\|_{\mathring{K}^{\alpha}_{p(\cdot),\theta}} \\
&\leqslant C(2m)^{\frac{1}{\theta'}} 2^{m|\alpha|}\|\chi_{(-2^m,2^m)}\|_{L^{p'(\cdot)}}\|f\|_{\mathring{K}^{\alpha}_{p(\cdot),\theta}} < \infty.
\end{aligned}$$

When $\theta \in (0,1]$, the θ-inequality gives

$$\begin{aligned}
\int_{-2^m}^{2^m} |f(y)|dy &\leqslant C\sum_{k=-m}^{m} \|\chi_{D_k}\|_{L^{p'(\cdot)}}\|\chi_{D_k}f\|_{L^{p(\cdot)}} \\
&\leqslant C2^{m|\alpha|}\|\chi_{(-2^m,2^m)}\|_{L^{p'(\cdot)}} \left(\sum_{k=-m}^{m} (2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^{\theta}\right)^{\frac{1}{\theta}} \\
&\leqslant C2^{m|\alpha|}\|\chi_{(-2^m,2^m)}\|_{L^{p'(\cdot)}}\|f\|_{\mathring{K}^{\alpha}_{p(\cdot),\theta}} < \infty.
\end{aligned}$$

Thus, $\mathring{K}^{\alpha}_{p(\cdot),\theta} \subset L^1_{\mathrm{loc}}$. As the one-sided maximal operators M^+ and M^- are defined for locally integrable functions, M^+ and M^- are defined on $\mathring{K}^{\alpha}_{p(\cdot),\theta}$.

We now use the idea from [7, p.91] to show that for any $f \in \mathring{K}^{\alpha}_{p(\cdot),\theta}$, M^+f and M^-f are Lebesgue measurable.

For any $x \in \mathbb{R}$ and $t > 0$, $F_+(x,t) = \frac{1}{t}\int_x^{x+t}|f(y)|dy$ and $F_-(x,t) = \frac{1}{t}\int_{x-t}^{x}|f(y)|dy$ are continuous in t for each x. In view of the Fubini's theorem, $F_+(x,t)$ and $F_-(x,t)$ are measurable in x for each t. As $F_+(x,t)$ and $F_-(x,t)$ are continuous in r, we see that

$$\begin{aligned}
\sup_{t\in\mathbb{Q}\cap(0,\infty)} F_+(x,t) &= \sup_{t>0} F_+(x,t) = M^+f(x) \\
\sup_{t\in\mathbb{Q}\cap(0,\infty)} F_-(x,t) &= \sup_{t>0} F_-(x,t) = M^-f(x).
\end{aligned}$$

Therefore, for any $f \in \mathring{K}^{\alpha}_{p(\cdot),\theta}$, M^+f and M^-f are Lebesgue measurable. $\square$

The above proposition assures that it does make sense to study the boundedness of the one-sided maximal operators M^+ and M^- on the one-sided Herz space with variable exponent $\mathring{K}^{\alpha}_{p(\cdot),\theta}$.

3. Main results

The main result of this paper, the boundedness of the one-sided maximal operators on the one-sided Herz spaces with variable exponent $\mathring{K}^{\alpha}_{p(\cdot),\theta}$, is established in this section.

THEOREM 3. *Let* $\theta \in (0,\infty)$, $\alpha \in (0,\infty)$, $p(\cdot) : \mathbb{R} \to (1,\infty)$ *be a Lebesgue measurable function. If* $1 < p_- \leqslant p_+ < \infty$ *and* $p(\cdot) \in \mathscr{M}_+$, *then* $M^+ : \mathring{K}^{\alpha}_{p(\cdot),\theta} \to \mathring{K}^{\alpha}_{p(\cdot),\theta}$ *is bounded.*

Proof. We first establish that for any $f \in \mathring{K}^{\alpha}_{p(\cdot),\theta}$ and $x \in D_m$, $m \in \mathbb{Z}$, we have

$$\chi_{D_m}(x) M^+ f(x) \leqslant M^+(\chi_{\cup_{k=m}^{\infty} D_k} f)(x), \quad x \in \mathbb{R}. \tag{1}$$

When $x \notin D_m$, we have

$$\chi_{D_m}(x) M^+ f(x) = 0 \leqslant M^+(\chi_{\cup_{k=m}^{\infty} D_k} f)(x). \tag{2}$$

Whenever $x \in D_m$, we find that for any $h > 0$

$$\begin{aligned} \frac{1}{h}\int_x^{x+h} |f(y)|dy &= \frac{1}{h}\int_x^{x+h} \chi_{\cup_{k=m}^{\infty} D_k}(y)|f(y)|dy \\ &\leqslant M^+(\chi_{\cup_{k=m}^{\infty} D_k} f)(x) \end{aligned} \tag{3}$$

because $x \in D_m$ asserts that $(x, x+h) \subset (x,\infty) \subset \cup_{k=m}^{\infty} D_k$. By taking the supremum over $h > 0$ on (3), we find that for any $x \in D_m$

$$\chi_{D_m}(x) M^+ f(x) \leqslant M^+(\chi_{\cup_{k=m}^{\infty} D_k} f)(x). \tag{4}$$

Consequently, (2) and (4) yield (1).

Next, we apply the norm $\|\cdot\|_{L^{p(\cdot)}}$ on both sides of (1), we obtain

$$\|\chi_{D_m} M^+ f\|_{L^{p(\cdot)}} \leqslant \|M^+(\chi_{\cup_{k=m}^{\infty} D_k} f)\|_{L^{p(\cdot)}}.$$

As $\alpha > 0$, the Hölder inequality on ℓ^{θ} for $\theta \geqslant 1$ and the θ-inequality for $0 < \theta < 1$ yield $\chi_{\cup_{k=m}^{\infty} D_k} f \in L^{p(\cdot)}$. Consequently, $p(\cdot) \in \mathscr{M}_+$ gives

$$\|\chi_{D_m} M^+ f\|_{L^{p(\cdot)}} \leqslant C\|\chi_{\cup_{k=m}^{\infty} D_k} f\|_{L^{p(\cdot)}} \leqslant C \sum_{k=m}^{\infty} \|\chi_{D_k} f\|_{L^{p(\cdot)}} \tag{5}$$

for some $C > 0$.

We now consider the case when $\theta \in (1,\infty)$. (5) gives

$$\begin{aligned} \sum_{m\in\mathbb{Z}} (2^{m\alpha}\|\chi_{D_m} M^+ f\|_{L^{p(\cdot)}})^{\theta} &\leqslant C \sum_{m\in\mathbb{Z}} \left(\sum_{k=m}^{\infty} 2^{m\alpha} \|\chi_{D_k} f\|_{L^{p(\cdot)}} \right)^{\theta} \\ &= C \sum_{m\in\mathbb{Z}} \left(\sum_{k=m}^{\infty} 2^{(m-k)\alpha} 2^{k\alpha} \|\chi_{D_k} f\|_{L^{p(\cdot)}} \right)^{\theta}. \end{aligned}$$

We now use the idea in [32, Proposition 1.2]. The Hölder inequality guarantees that

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}}(2^{m\alpha}\|\chi_{D_m}M^+f\|_{L^{p(\cdot)}})^\theta \\
&\leqslant C\sum_{m\in\mathbb{Z}}\left(\sum_{k=m}^{\infty}2^{\frac{\theta}{2}(m-k)\alpha}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta\right)\left(\sum_{k=m}^{\infty}2^{\frac{\theta'}{2}(m-k)\alpha}\right)^{\frac{\theta}{\theta'}}.
\end{aligned}$$

As $\alpha>0$, we find that $\sum_{k=m}^{\infty}2^{\frac{\theta'}{2}(m-k)\alpha}<C$ for some $C>0$ independent of m. Therefore, by interchanging the summations, we obtain

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}(2^{m\alpha}\|\chi_{D_m}M^+f\|_{L^{p(\cdot)}})^\theta &\leqslant C\sum_{m\in\mathbb{Z}}\sum_{k=m}^{\infty}2^{\frac{\theta}{2}(m-k)\alpha}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta \\
&\leqslant C\sum_{k\in\mathbb{Z}}\sum_{m=-\infty}^{k}2^{\frac{\theta}{2}(m-k)\alpha}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta \\
&= C\sum_{k\in\mathbb{Z}}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta\sum_{m=-\infty}^{k}2^{\frac{\theta}{2}(m-k)\alpha} \\
&\leqslant C\sum_{k\in\mathbb{Z}}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta
\end{aligned}$$

where we have the last inequality because $\alpha>0$. Hence,

$$\begin{aligned}
\|M^+f\|_{\mathring{K}^{\alpha}_{p(\cdot),\theta}} &= \left(\sum_{m\in\mathbb{Z}}(2^{m\alpha}\|\chi_{D_m}M^+f\|_{L^{p(\cdot)}})^\theta\right)^{\frac{1}{\theta}} \\
&\leqslant C\left(\sum_{k\in\mathbb{Z}}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta\right)^{\frac{1}{\theta}} = C\|f\|_{\mathring{K}^{\alpha}_{p(\cdot),\theta}}.
\end{aligned}$$

Therefore, we establish the boundedness of $M^+:\mathring{K}^{\alpha}_{p(\cdot),\theta}\to\mathring{K}^{\alpha}_{p(\cdot),\theta}$ when $\theta\in(1,\infty)$.

We consider the case $\theta\in(0,1]$. The θ-inequality and (5) yield

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}(2^{m\alpha}\|\chi_{D_m}M^+f\|_{L^{p(\cdot)}})^\theta &\leqslant C\sum_{m\in\mathbb{Z}}\left(\sum_{k=m}^{\infty}2^{m\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}}\right)^\theta \\
&= C\sum_{m\in\mathbb{Z}}\left(\sum_{k=m}^{\infty}2^{(m-k)\alpha}2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}}\right)^\theta \\
&\leqslant C\sum_{m\in\mathbb{Z}}\sum_{k=m}^{\infty}2^{\theta(m-k)\alpha}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta.
\end{aligned}$$

By interchanging the summations, we get

$$\sum_{m\in\mathbb{Z}}(2^{m\alpha}\|\chi_{D_m}M^+f\|_{L^{p(\cdot)}})^\theta \leqslant C\sum_{k\in\mathbb{Z}}\sum_{m=-\infty}^{k}2^{\theta(m-k)\alpha}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta$$
$$= C\sum_{k\in\mathbb{Z}}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta\sum_{m=-\infty}^{k}2^{\theta(m-k)\alpha}.$$

Since $\alpha>0$, we find that $\sum_{m=-\infty}^{k}2^{\theta(m-k)\alpha}<C$ for some $C>0$ independent of k. Consequently,

$$\|M^+f\|_{\mathring{K}^\alpha_{p(\cdot),\theta}} = \left(\sum_{m\in\mathbb{Z}}(2^{m\alpha}\|\chi_{D_m}M^+f\|_{L^{p(\cdot)}})^\theta\right)^{\frac{1}{\theta}}$$
$$\leqslant C\left(\sum_{k\in\mathbb{Z}}(2^{k\alpha}\|\chi_{D_k}f\|_{L^{p(\cdot)}})^\theta\right)^{\frac{1}{\theta}} = C\|f\|_{\mathring{K}^\alpha_{p(\cdot),\theta}}.$$

Thus, we establish the boundedness of $M^+:\mathring{K}^\alpha_{p(\cdot),\theta}\to\mathring{K}^\alpha_{p(\cdot),\theta}$ when $\theta\in(0,1]$. □

We also have the boundedness of M^- on the one-sided Herz spaces with variable exponents $\mathring{K}^\alpha_{p(\cdot),\theta}$.

THEOREM 4. *Let $\theta\in(0,\infty)$, $\alpha\in(-\infty,0)$, $p(\cdot):\mathbb{R}\to(1,\infty)$ be a Lebesgue measurable function. If $1<p_-\leqslant p_+<\infty$ and $p(\cdot)\in\mathscr{M}_-$, then $M^-:\mathring{K}^\alpha_{p(\cdot),\theta}\to\mathring{K}^\alpha_{p(\cdot),\theta}$ is bounded.*

As the proof of the above result is similar to the proof of Theorem 3, for simplicity, we omit the proof.

In view of Theorems 1, 2, 3 and 4, we have the following boundedness results for the one-sided maximal operators M^+ and M^- on the one-sided Herz space with variable exponent $\mathring{K}^\alpha_{p(\cdot),\theta}$.

COROLLARY 1. *Let $\alpha\in\mathbb{R}$, $\theta\in(0,\infty)$ and $p(\cdot):\mathbb{R}\to(1,\infty)$ be a Lebesgue measurable function.*

1. *If $\alpha\in(0,\infty)$, $p(\cdot)\in\mathscr{L}_+$ and there exists a non-increasing function $q(\cdot)$ satisfying $1<q_-\leqslant q_+<\infty$, $|p(\cdot)-q(\cdot)|\in\mathscr{P}$, then $M^+:\mathring{K}^\alpha_{p(\cdot),\theta}\to\mathring{K}^\alpha_{p(\cdot),\theta}$ is bounded.*
2. *If $\alpha\in(-\infty,0)$, $p(\cdot)\in\mathscr{L}_-$ and there exists a non-decreasing function $q(\cdot)$ satisfying $1<q_-\leqslant q_+<\infty$, $|p(\cdot)-q(\cdot)|\in\mathscr{P}$, then $M^-:\mathring{K}^\alpha_{p(\cdot),\theta}\to\mathring{K}^\alpha_{p(\cdot),\theta}$ is bounded.*

COROLLARY 2. *Let $\alpha\in\mathbb{R}$, $\theta\in(0,\infty)$ and $p(\cdot):\mathbb{R}\to(1,\infty)$ be a Lebesgue measurable function.*

1. *If* $\alpha \in (0,\infty)$, $p(\cdot) \in \mathscr{L}_+$ *and there exists a* $q(\cdot) \in \mathfrak{Q}$ *such that* $|p(\cdot)-q(\cdot)| \in \mathscr{P}$, *then* $M^+ : \mathring{K}^{\alpha}_{p(\cdot),\theta} \to \mathring{K}^{\alpha}_{p(\cdot),\theta}$ *is bounded.*

2. *If* $\alpha \in (-\infty,0)$, $p(\cdot) \in \mathscr{L}_-$ *and there exists a* $q(\cdot) \in \mathfrak{Q}$ *such that* $|p(\cdot)-q(\cdot)| \in \mathscr{P}$, *then* $M^- : \mathring{K}^{\alpha}_{p(\cdot),\theta} \to \mathring{K}^{\alpha}_{p(\cdot),\theta}$ *is bounded.*

Moreover, Theorems 3 and 4 also give the boundedness result for M^+ and M^- on the one-sided Herz spaces $\mathring{K}^{\alpha}_{p,\theta}$.

COROLLARY 3. *Let* $\theta \in (0,\infty)$, $\alpha \in \mathbb{R}$ *and* $p \in (1,\infty)$.

1. *If* $\alpha \in (0,\infty)$, *then* $M^+ : \mathring{K}^{\alpha}_{p,\theta} \to \mathring{K}^{\alpha}_{p,\theta}$ *is bounded.*

2. *If* $\alpha \in (-\infty,0)$, *then* $M^- : \mathring{K}^{\alpha}_{p,\theta} \to \mathring{K}^{\alpha}_{p,\theta}$ *is bounded.*

The mapping properties of M^- on the weighted Herz spaces were given in [14].

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Kwok-Pun Ho
Department of Mathematics and Information Technology
The Education University of Hong Kong
10 Lo Ping Road, Tai Po, Hong Kong, China
e-mail: vkpho@eduhk.hk