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SOME PROPERTIES OF GENERALIZATION CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let $\overline{\mathscr{A}}(n)$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{1+\frac{k}{n}} z^{1+\frac{k}{n}} \quad (n = 1, 2, 3, \cdots)$$

which are analytic in the open unit disc $\mathbb{U}$. If $a_{1+\frac{k}{n}} = 0$ for $k \neq n, 2n, 3n, \cdots$, then $f(z)$ is given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For such functions $f(z) \in \overline{\mathscr{A}}(n)$, some generalization classes $\overline{\mathscr{S}}^*(n,\alpha)$, $\overline{\mathscr{C}}(n,\alpha)$ and $\overline{\mathscr{R}}(n,\alpha)$ are defined. The object of present paper is to discuss some interesting properties of $f(z) \in \overline{\mathscr{A}}(n)$ concerning with subordinations and strongly functions.

1. Introduction

Let n be a natural number and $\overline{\mathscr{A}}(n)$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{1+\frac{k}{n}} z^{1+\frac{k}{n}} \quad (n = 1, 2, 3, \cdots) \tag{1}$$

that are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Here, we take the principal value for $\sqrt[n]{z}$. Let $\overline{\mathscr{S}}^*(n,\alpha)$ denote the subclass of $\overline{\mathscr{A}}(n)$ consisting of $f(z)$ that satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \ (z \in \mathbb{U}) \tag{2}$$

for some α $(0 \leqslant \alpha < 1)$. Also, $\overline{\mathscr{C}}(n,\alpha)$ is the subclass of $\overline{\mathscr{A}}(n)$ consisting of $f(z)$ that satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ (z \in \mathbb{U}) \tag{3}$$

for some α $(0 \leqslant \alpha < 1)$. By the definitions for $\overline{\mathscr{S}}^*(n,\alpha)$ and $\overline{\mathscr{C}}(n,\alpha)$, we see that $f(z) \in \overline{\mathscr{C}}(n,\alpha)$ if and only if $zf'(z) \in \overline{\mathscr{S}}^*(n,\alpha)$, and that $f(z) \in \overline{\mathscr{S}}^*(n,\alpha)$ if and only

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if $\int_0^z \frac{f(t)}{t}dt \in \overline{\mathscr{C}}(n,\alpha)$. If $a_{1+\frac{k}{n}} = 0$ for $k \neq n, 2n, 3n, \cdots$, then $f(z) \in \overline{\mathscr{A}}(n)$ can be written by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{4}$$

and write that $f(z) \in \mathscr{A}$.

Let us consider a function $f(z) \in \overline{\mathscr{A}}(n)$ given by

$$\begin{aligned} f(z) &= \frac{z}{(1-\sqrt[n]{z})^{2n(1-\alpha)}} \\ &= z + \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{k}(j+2n(1-\alpha)-1)}{k!} z^{1+\frac{k}{n}} \end{aligned} \tag{5}$$

with $0 \leqslant \alpha < 1$. If $n = 2$, then

$$\begin{aligned} f(z) &= \frac{z}{(1-\sqrt{z})^{4(1-\alpha)}} \\ &= z + \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{k}(j+3-4\alpha)}{k!} z^{1+\frac{k}{2}} \end{aligned} \tag{6}$$

and $f(z) \in \overline{\mathscr{S}}^*(2,\alpha)$ in Güney, Breaz and Owa [3]. Also, $n = 3$, then

$$\begin{aligned} f(z) &= \frac{z}{(1-\sqrt[3]{z})^{6(1-\alpha)}} \\ &= z + \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{k}(j+5-6\alpha)}{k!} z^{1+\frac{k}{3}} \end{aligned} \tag{7}$$

and $f(z) \in \overline{\mathscr{S}}^*(3,\alpha)$ in Güney and Owa [2]. Also, $f(z)$ given by (5) satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} = Re\left(\frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}\right) > \alpha\,, \ (z \in \mathbb{U}). \tag{8}$$

Thus we know that $f(z)$ given by (5) is in the class $\overline{\mathscr{S}}^*(n,\alpha)$. If we consider a function $f(z) \in \overline{\mathscr{A}}(n)$ given by

$$f'(z) = \frac{1}{(1-\sqrt[n]{z})^{2n(1-\alpha)}} \quad (0 \leqslant \alpha < 1), \tag{9}$$

then $f(z)$ satisfies

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} = Re\left(\frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}\right) > \alpha\,, \ (z \in \mathbb{U}) \tag{10}$$

and $f(z) \in \overline{\mathscr{C}}(n,\alpha)$. Further, if $f(z) \in \overline{\mathscr{A}}(n)$ given by

$$f(z) = (2\alpha-1)z + 2(1-\alpha)\int_0^z \frac{1}{1-\sqrt[n]{t}}dt, \tag{11}$$

then $f(z)$ satisfies

$$Ref'(z) = Re\left(\frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}\right) > \alpha\,,\ (z \in \mathbb{U}). \tag{12}$$

We denote by $f(z) \in \overline{\mathscr{R}}(n,\alpha)$ if $f(z) \in \overline{\mathscr{A}}(n)$ satisfies

$$Ref'(z) > \alpha\,,\ (z \in \mathbb{U}) \tag{13}$$

for $0 \leqslant \alpha < 1$.

For analytic functions $f(z)$ and $F(z)$ in $\mathbb{U}$, we introduce that $f(z)$ is subordinate to $F(z)$, written $f(z) \prec F(z)$ $(z \in \mathbb{U})$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ and such that $f(z) = F(w(z))$ (see [6]). With the definition for subordinations, if $f(z) \in \overline{\mathscr{A}}(n)$ satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}\,,\ (z \in \mathbb{U}), \tag{14}$$

with $0 \leqslant \alpha < 1$, then $f(z) \in \overline{\mathscr{S}}^*(n,\alpha)$, and if $f(z) \in \overline{\mathscr{A}}(n)$ satisfies

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}\,,\ (z \in \mathbb{U}), \tag{15}$$

with $0 \leqslant \alpha < 1$, then $f(z) \in \overline{\mathscr{C}}(n,\alpha)$. Further, if $f(z) \in \overline{\mathscr{A}}(n)$ satisfies

$$f'(z) \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}\,,\ (z \in \mathbb{U}), \tag{16}$$

with $0 \leqslant \alpha < 1$, then $f(z) \in \overline{\mathscr{R}}(n,\alpha)$.

EXAMPLE 1. We consider a function $f(z) \in \overline{\mathscr{A}}(n)$ given by

$$f(z) = \frac{z}{1-\sqrt[n]{z}} = z + \sum_{k=1}^{\infty} z^{1+\frac{k}{n}}. \tag{17}$$

Letting $z = e^{i\theta}$ $(0 \leqslant \theta < 2\pi)$, we have

$$\begin{aligned} Ref(z) &= Re\left(\frac{e^{i\theta}}{1-e^{i\frac{\theta}{n}}}\right) \\ &= Re\left(\frac{e^{i\left(1-\frac{1}{2n}\right)\theta}}{e^{-i\frac{\theta}{2n}}-e^{i\frac{\theta}{2n}}}\right) \\ &= -\frac{1}{2}\left(\frac{\sin\left(\frac{2n-1}{2n}\theta\right)}{\sin\left(\frac{1}{2n}\theta\right)}\right). \end{aligned} \tag{18}$$

If $n = 1$, then

$$Ref(z) = -\frac{1}{2}\left(\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta}{n}}\right) = -\frac{1}{2}. \tag{19}$$

If $n=2$, then

$$\begin{aligned} Ref(z) &= -\frac{1}{2}\left(\frac{\sin\frac{3}{4}\theta}{\sin\frac{\theta}{4}}\right) = -\frac{1}{2}\left(\frac{\sin\left(\frac{\theta}{4}+\frac{\theta}{2}\right)}{\sin\frac{\theta}{4}}\right) \\ &= -\frac{1}{2}\left(4\cos^2\frac{\theta}{4}-1\right) \\ &\geqslant -\frac{3}{2}. \end{aligned} \tag{20}$$

If $n=3$, then

$$\begin{aligned} Ref(z) &= -\frac{1}{2}\left(\frac{\sin\frac{5}{6}\theta}{\sin\frac{\theta}{6}}\right) = -\frac{1}{2}\left(\frac{\sin\left(\frac{\theta}{6}+\frac{2}{3}\theta\right)}{\sin\frac{\theta}{6}}\right) \\ &= -\frac{1}{2}\left(4\cos^2\frac{\theta}{3}+2\cos\frac{\theta}{3}-1\right) \\ &\geqslant -\frac{5}{2}. \end{aligned} \tag{21}$$

If $n=4$, then

$$\begin{aligned} Ref(z) &= -\frac{1}{2}\left(\frac{\sin\frac{7}{8}\theta}{\sin\frac{\theta}{8}}\right) = -\frac{1}{2}\left(\frac{\sin\left(\frac{\theta}{8}+\frac{3}{4}\theta\right)}{\sin\frac{\theta}{8}}\right) \\ &= -\frac{1}{2}\left(4\cos\frac{\theta}{4}\left(2\cos^2\frac{\theta}{4}-1\right)+\left(4\cos^2\theta-1\right)\right) \\ &\geqslant -\frac{7}{2}. \end{aligned} \tag{22}$$

If $n=5$, then

$$\begin{aligned} Ref(z) &= -\frac{1}{2}\left(\frac{\sin\frac{9}{10}\theta}{\sin\frac{\theta}{10}}\right) = -\frac{1}{2}\left(\frac{\sin\left(\frac{\theta}{10}+\frac{4}{5}\theta\right)}{\sin\frac{\theta}{10}}\right) \\ &= -\frac{1}{2}\left(4\cos^2\frac{\theta}{5}\left(4\cos^2\frac{\theta}{5}-3\right)+4\cos\frac{\theta}{5}\left(2\cos^2\frac{\theta}{5}-1\right)+1\right) \\ &\geqslant -\frac{9}{2}. \end{aligned} \tag{23}$$

Therefore, we can be expected that (18) gives us

$$\begin{aligned} Ref(z) &= -\frac{1}{2}\left(\frac{\sin\left(\frac{\theta}{2n}+\frac{n-1}{n}\theta\right)}{\sin\frac{\theta}{2n}}\right) \\ &\geqslant -\frac{2n-1}{2}. \end{aligned} \tag{24}$$

2. Some applications of subordinations

To consider some applications of subordinations, we need the following lemma due to Suffridge [10].

LEMMA 1. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0)=1$ and $h(z)$ be analytic and starlike in $\mathbb{U}$. If*

$$zp'(z) \prec h(z)\,,\,(z\in\mathbb{U}), \tag{25}$$

then

$$p(z) \prec \int_0^z \frac{h(t)}{t}dt\,,\,(z\in\mathbb{U}). \tag{26}$$

Applying the above lemma, we have the following theorem.

THEOREM 1. *Let $f(z)\in\overline{\mathscr{A}}(n)$ and $h(z)$ be given by*

$$h(z) = \frac{z}{(1-\sqrt[n]{z})^{2n(1-\alpha)}}\,,\,(z\in\mathbb{U}) \tag{27}$$

with $0\leqslant\alpha<1$. If $f(z)$ satisfies

$$zf''(z) \prec h(z) \tag{28}$$

then

$$f'(z) \prec \int_0^z \frac{h(t)}{t}dt = \int_0^z \frac{1}{\left(1-\sqrt[n]{t}\right)^{2n(1-\alpha)}}dt\,,\,(z\in\mathbb{U}). \tag{29}$$

Proof. We define a function $p(z)=f'(z)$ for $f(z)\in\overline{\mathscr{A}}(n)$. Then $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=1$. Also, $h(z)$ given by (27) satisfies

$$Re\left(\frac{zh'(z)}{h(z)}\right) = Re\left(\frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}\right) > \alpha\,,\,(z\in\mathbb{U}). \tag{30}$$

Using Lemma 1, we see if

$$zp'(z) = zf''(z) \prec h(z)\,,\,(z\in\mathbb{U}), \tag{31}$$

then

$$p(z) = f'(z) \prec \int_0^z \frac{h(t)}{t}dt = \int_0^z \frac{1}{\left(1-\sqrt[n]{t}\right)^{2n(1-\alpha)}}dt\,,\,(z\in\mathbb{U}). \quad \square \tag{32}$$

Letting $n=2$ in Theorem 1, we see the following corollary.

COROLLARY 1. *If* $f(z) \in \overline{\mathscr{A}}(2)$ *satisfies*

$$zf''(z) \prec \frac{z}{(1-\sqrt{z})^{4(1-\alpha)}}, \ (z \in \mathbb{U}) \tag{33}$$

with $0 \leqslant \alpha < 1$, *then*

$$\begin{aligned} f'(z) &\prec \int_0^z \frac{1}{(1-\sqrt{t})^{4(1-\alpha)}} dt \\ &= \frac{1}{(2\alpha-1)(4\alpha-3)} \left(1-(1-\sqrt{z})^{4\alpha-3}(1-(3-4\alpha)\sqrt{z})\right), \ (z \in \mathbb{U}). \end{aligned} \tag{34}$$

Next, we have the following theorem.

THEOREM 2. *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies*

$$\frac{zf'(z)-f(z)}{z} \prec \frac{z}{(1-\sqrt[n]{z})^{2n(1-\alpha)}}, \ (z \in \mathbb{U}) \tag{35}$$

with $0 \leqslant \alpha < 1$, *then*

$$\frac{f(z)}{z} \prec \int_0^z \frac{1}{(1-\sqrt[n]{t})^{2n(1-\alpha)}} dt, \ (z \in \mathbb{U}). \tag{36}$$

Proof. Let us consider a function $p(z)$ by $p(z) = \frac{f(z)}{z}$ for $f(z) \in \overline{\mathscr{A}}(n)$. Then $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Since

$$zp'(z) = \frac{zf'(z)-f(z)}{z} \prec \frac{z}{(1-\sqrt[n]{z})^{2n(1-\alpha)}}, \ (z \in \mathbb{U}), \tag{37}$$

Lemma 1 implies that

$$p(z) = \frac{f(z)}{z} \prec \int_0^z \frac{1}{(1-\sqrt[n]{t})^{2n(1-\alpha)}} dt, \ (z \in \mathbb{U}). \quad \square \tag{38}$$

Next, we have to introduce the following lemma by Hallenbeck and Ruscheweyh [5].

LEMMA 2. *Let a function* $p(z)$ *be analytic in* $\mathbb{U}$ *with* $p(0) = 1$ *and* $h(z)$ *be analytic and convex in* $\mathbb{U}$. *If* $p(z)$ *satisfies*

$$p(z) + zp'(z) \prec h(z), \ (z \in \mathbb{U}), \tag{39}$$

then

$$p(z) \prec \frac{1}{z} \int_0^z h(t) dt, \ (z \in \mathbb{U}). \tag{40}$$

Applying Lemma 2, we derive the following theorem.

THEOREM 3. *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies*

$$f'(z) + zf''(z) \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}, \ (z \in \mathbb{U}) \tag{41}$$

for some real α $(0 \leqslant \alpha < 1)$, *then*

$$f'(z) \prec \frac{1}{z}\int_0^z \frac{1+(1-2\alpha)\sqrt[n]{t}}{1-\sqrt[n]{t}} dt, \ (z \in \mathbb{U}). \tag{42}$$

Proof. We consider a function $p(z) = f'(z)$ for $f(z) \in \overline{\mathscr{A}}(n)$. Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Note that the function $h(z)$ given by

$$h(z) = \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}, \ (z \in \mathbb{U}) \tag{43}$$

is convex in $\mathbb{U}$. By the condition (41), we say that

$$p(z) + zp'(z) = f'(z) + zf''(z) \prec h(z) = \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}. \tag{44}$$

Using Lemma 2, we see that

$$p(z) = f'(z) \prec \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{1+(1-2\alpha)\sqrt[n]{t}}{1-\sqrt[n]{t}} dt, \ (z \in \mathbb{U}). \quad \square \tag{45}$$

Taking $n = 2$ in Theorem 3, we obtain the following corollary.

COROLLARY 2. *If* $f(z) \in \overline{\mathscr{A}}(2)$ *satisfies*

$$f'(z) + zf''(z) \prec \frac{1+(1-2\alpha)\sqrt{z}}{1-\sqrt{z}}, \ (z \in \mathbb{U}) \tag{46}$$

with $0 \leqslant \alpha < 1$, *then*

$$f'(z) \prec (2\alpha - 1) - 4(1-\alpha)\frac{\log(1-\sqrt{z})}{z} + \frac{4(1-\alpha)}{\sqrt{z}}. \tag{47}$$

Considering $p(z) = \frac{f(z)}{z}$ for $f(z) \in \overline{\mathscr{A}}(n)$ and

$$h(z) = \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}} \tag{48}$$

with $0 \leqslant \alpha < 1$ in Lemma 2, we have the following theorem.

THEOREM 4. *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies*

$$f'(z) \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}, \ (z \in \mathbb{U}) \tag{49}$$

with $0 \leqslant \alpha < 1$, *then*

$$f(z) \prec \int_0^z \frac{1+(1-2\alpha)\sqrt[n]{t}}{1-\sqrt[n]{t}} dt, \ (z \in \mathbb{U}). \tag{50}$$

REMARK 1. If we consider $f(z) \in \overline{\mathscr{A}}(2)$ given by

$$f(z) = (2\alpha-1)z - 4(1-\alpha)\log(1-\sqrt{z}) + 4(1-\alpha)\sqrt{z}. \tag{51}$$

with $0 \leqslant \alpha < 1$, then

$$f'(z) = \frac{1+(1-2\alpha)\sqrt{z}}{1-\sqrt{z}}, \ (z \in \mathbb{U}), \tag{52}$$

that is $f(z) \in \overline{\mathscr{R}}(2,\alpha)$.

Next, we introduce the lemma due to Miller and Mocanu [7].

LEMMA 3. *Let* $F(z)$ *be analytic in* $\mathbb{U}$ *and* $G(z)$ *be analytic in* $\mathbb{U}$ *and the boundary of* $\mathbb{U}$ *with* $F(0) = G(0)$. *If* $F(z)$ *is not subordinate to* $G(z)$, *then there exist points* $z_0 \in \mathbb{U}$ *and* $\xi_0 \in \partial\mathbb{U}$, *and a real* $m \geqslant 1$ *for which* $F(|z| < |z_0|) \subset G(\mathbb{U})$,

$$(i) \quad F(z_0) = G(\xi_0)$$
$$(ii) \quad z_0 F'(z_0) = m\xi_0 G'(\xi_0).$$

Using the above lemma, we prove the following theorem.

THEOREM 5. *Let* β_0 *be the solution of*

$$\beta\pi = \frac{3}{2}\pi - \mathrm{Tan}^{-1}\left(\frac{\beta}{n}\right) \tag{53}$$

and let

$$\alpha = \beta + \frac{2}{\pi}\mathrm{Tan}^{-1}\left(\frac{\beta}{n}\right), \ (0 < \beta \leqslant \beta_0) \tag{54}$$

with $n = 1,2,3,\cdots$. *If* $p(z)$ *is analytic in* $\mathbb{U}$ *with* $p(0) = 1$ *and*

$$p(z) + zp'(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\alpha} \ (z \in \mathbb{U}) \tag{55}$$

then

$$p(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} \ (z \in \mathbb{U}). \tag{56}$$

Proof. We use the same method of the proof by Miller and Mocanu (Theorem 5 in [8]). Note that (53) implies

$$\frac{2}{\pi}\mathrm{Tan}^{-1}\left(\frac{\beta}{n}\right)=3-2\beta \tag{57}$$

and

$$\beta\leqslant 3-\alpha \ (0<\beta\leqslant\beta_0) \tag{58}$$

by (54). We consider the functions

$$h(z)=\left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\alpha} \tag{59}$$

and

$$g(z)=\left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta}. \tag{60}$$

Then we see that $|\arg h(z)|<\frac{\pi}{2}\alpha$ and $|\arg g(z)|<\frac{\pi}{2}\beta$ in $\mathbb{U}$. For such $h(z)$ and $g(z)$, we need to show that $p(z)\prec g(z)$ $(z\in\mathbb{U})$. Since $p(0)=g(0)=1$, by Lemma 3, if $p(z)$ is not subordinate to $g(z)$, then there exist points $z_0\in\mathbb{U}$ and $\xi_0\in\partial\mathbb{U}$, and a real $m\geqslant 1$ for which $p(|z|<|z_0|)\subset g(\mathbb{U})$, $p(z_0)=g(\xi_0)$ and $z_0p'(z_0)=m\xi_0g'(\xi_0)$. Let $p(z_0)\neq 0$, then $g(\xi_0)\neq 0$ and $\sqrt[n]{\xi_0}\neq\pm 1$. Noting that

$$|\arg g(\xi_0)|=\beta\left|\arg\left(\frac{1+\sqrt[n]{\xi_0}}{1-\sqrt[n]{\xi_0}}\right)\right|\leqslant\frac{\pi}{2}\beta \ \ (\xi_0\in\partial\mathbb{U}), \tag{61}$$

we consider

$$ir=\frac{1+\sqrt[n]{\xi_0}}{1-\sqrt[n]{\xi_0}} \ \ (\xi_0\in\partial\mathbb{U}). \tag{62}$$

Then, we obtain that

$$\begin{aligned}p(z_0)+z_0p'(z_0)&=g(\xi_0)+m\xi_0g'(\xi_0)\\&=g(\xi_0)\left(1+m\frac{\xi_0g'(\xi_0)}{g(\xi_0)}\right)\\&=(ir)^{\beta}\left(1+\frac{2m\beta}{n}\frac{\sqrt[n]{\xi_0}}{1-\left(\sqrt[n]{\xi_0}\right)^2}\right)\\&=(ir)^{\beta}\left(1+i\frac{m\beta}{2n}\left(r+\frac{1}{r}\right)\right).\end{aligned} \tag{63}$$

It follows from (63), that

$$\arg(p(z_0)+z_0p'(z_0))=\frac{\pi}{2}\beta+\mathrm{Tan}^{-1}\left(\frac{m\beta\left(r+\frac{1}{r}\right)}{2n}\right) \tag{64}$$

and

$$\frac{\pi}{2}\beta + \mathrm{Tan}^{-1}\left(\frac{\beta}{n}\right) \leqslant \left|\arg(p(z_0)+z_0p'(z_0))\right| \leqslant \frac{\pi}{2}\beta + \frac{\pi}{2}. \tag{65}$$

Thus, using (57) and (58), we have

$$\frac{\pi}{2}\alpha \leqslant \left|\arg(p(z_0)+z_0p'(z_0))\right| \leqslant 2\pi - \frac{\pi}{2}\alpha. \tag{66}$$

Since $|\arg h(z)| < \frac{\pi}{2}\alpha \ (z \in \mathbb{U})$, (66) contradicts (55). Therefore, the subordination (56) is true for $p(z_0) \neq 0 \ (z \in \mathbb{U})$. If $p(z_0) = 0 \ (z \in \mathbb{U})$, with the same reason of the proof by Miller and Mocanu (Theorem 5 in [8]), we have

$$\frac{\pi}{2}\alpha \leqslant \left|\arg(p(z_0)+z_0p'(z_0))\right| \leqslant 2\pi - \frac{\pi}{2}\alpha. \quad \square \tag{67}$$

Taking $\beta = n$ in Theorem 5, then we have the following corollary.

COROLLARY 3. *If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$p(z) + zp'(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\frac{2n+1}{2}} \quad (z \in \mathbb{U}) \tag{68}$$

for $n = 1, 2, 3, \cdots$, then

$$p(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{n} \quad (z \in \mathbb{U}). \tag{69}$$

Taking $\beta = \sqrt{3}n$ in Theorem 5, then we obtain the following corollary.

COROLLARY 4. *If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$p(z) + zp'(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\frac{2}{3}+\sqrt{3}n} \quad (z \in \mathbb{U}) \tag{70}$$

for $n = 1, 2, 3, \cdots$, then

$$p(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\sqrt{3}n} \quad (z \in \mathbb{U}). \tag{71}$$

Taking $p(z) = f'(z)$ in Theorem 5, then we have the following corollary.

COROLLARY 5. *Let α and β be define (53) and (54). If $f(z) \in \overline{\mathscr{A}}(n)$ satisfies*

$$f'(z) + zf''(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\alpha} \quad (z \in \mathbb{U}) \tag{72}$$

for $n = 1, 2, 3, \cdots$, then

$$f'(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} \quad (z \in \mathbb{U}), \tag{73}$$

that is, $f(z) \in \overline{\mathscr{R}}(n,0)$.

If we consider $p(z) = \frac{f(z)}{z}$ for $f(z) \in \overline{\mathscr{A}}(n)$ in Theorem 5, then we have the following corollary.

COROLLARY 6. *Let* α *and* β *be define* (53) *and* (54). *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies*

$$f'(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\alpha} \quad (z \in \mathbb{U}) \tag{74}$$

for $n = 1,2,3,\cdots$, *then*

$$\frac{f(z)}{z} \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} \quad (z \in \mathbb{U}). \tag{75}$$

Next, we show the following theorem.

THEOREM 6. *Let* $p(z)$ *be analytic in* $\mathbb{U}$ *with* $p(0) = 1$. *If* $p(z)$ *satisfies*

$$p(z) + zp'(z) \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}} \quad (z \in \mathbb{U}) \tag{76}$$

for some real α $(0 \leqslant \alpha < 1)$, *then*

$$p(z) \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}} \quad (z \in \mathbb{U}) \tag{77}$$

where $n = 1,2,3,\cdots$.

Proof. If we suppose that $p(z)$ is not subordinate to a function $g(z)$ given by

$$g(z) = \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}} \quad (z \in \mathbb{U}), \tag{78}$$

Lemma 3 gives us that there exist points $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U}$, and a real $m \geqslant 1$ for which $p(|z| < |z_0|) \subset g(\mathbb{U})$, $p(z_0) = g(\xi_0)$ and $z_0 p'(z_0) = m\xi_0 g'(\xi_0)$. It follows that

$$\begin{aligned} p(z_0) + z_0 p'(z_0) &= g(\xi_0) + m\xi_0 g'(\xi_0) \\ &= \frac{1+(1-2\alpha)\sqrt[n]{\xi_0}}{1-\sqrt[n]{\xi_0}} \left\{ 1 + \frac{m}{n} \left(\frac{(1-2\alpha)\sqrt[n]{\xi_0}}{1+(1-2\alpha)\sqrt[n]{\xi_0}} + \frac{\sqrt[n]{\xi_0}}{1-\sqrt[n]{\xi_0}} \right) \right\}. \end{aligned} \tag{79}$$

Letting $\xi_0 = e^{i\theta}$ $(0 \leqslant \theta < 2\pi)$, we have

$$\begin{aligned} p(z_0) + z_0 p'(z_0) =& \frac{1+(1-2\alpha)e^{i\frac{\theta}{n}}}{1-e^{i\frac{\theta}{n}}} \\ &+ \frac{m}{n} \left(\frac{(1-2\alpha)e^{i\frac{\theta}{n}}}{1-e^{i\frac{\theta}{n}}} + \frac{e^{i\frac{\theta}{n}}\left(1+(1-2\alpha)e^{i\frac{\theta}{n}}\right)}{\left(1-e^{i\frac{\theta}{n}}\right)^2} \right). \end{aligned} \tag{80}$$

This implies that

$$Re(p(z_0)+z_0p'(z_0)) = \alpha - \frac{m}{2n}\left(1-2\alpha+\frac{1+(1-2\alpha)\cos\frac{\theta}{n}}{1-\cos\frac{\theta}{n}}\right). \tag{81}$$

If we define a function $k(t)$ given by

$$k(t) = \frac{1+(1-2\alpha)t}{1-t} \quad \left(t=\cos\frac{\theta}{n}\right), \tag{82}$$

then $k(t) > \alpha$ and

$$k'(t) = \frac{2(1-\alpha)}{(1-t)^2} > 0. \tag{83}$$

Thus, we have

$$Re(p(z_0)+z_0p'(z_0)) < \alpha - \frac{m(1-\alpha)}{2n} < \alpha. \tag{84}$$

Noting that $Reg(z) > \alpha \ (z \in \mathbb{U})$, (84) contradicts the condition (76). Therefore, $p(z)$ satisfies the subordination (77). □

Taking $p(z) = f'(z)$ in Theorem 6, then we have the following corollary.

COROLLARY 7. *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies*

$$f'(z)+zf''(z) \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}} \ (z \in \mathbb{U}) \tag{85}$$

for some real α $(0 \leqslant \alpha < 1)$, *then*

$$f'(z) \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}} \ (z \in \mathbb{U}), \tag{86}$$

that is, $f(z) \in \overline{\mathscr{R}}(n,\alpha)$.

Further, we have the following corollary.

COROLLARY 8. *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies*

$$f'(z) \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}} \ (z \in \mathbb{U}) \tag{87}$$

for some real α $(0 \leqslant \alpha < 1)$, *then*

$$\frac{f(z)}{z} \prec \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}} \ (z \in \mathbb{U}). \tag{88}$$

To discuss the next our results for subordinations, we have to introduce the lemma due to Nunokawa, Owa and Sokol [9].

LEMMA 4. *Let* $p(z)$ *be analytic in* $\mathbb{U}$ *with* $p(0) = 1$ *and* $p(z) \neq 0$ *in* $\mathbb{U}$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$|\arg p(z)| < \frac{\pi}{2}\beta \ (|z| < |z_0| < 1) \tag{89}$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\beta \tag{90}$$

for some $\beta > 0$, *then*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta, \tag{91}$$

where

$$k \geqslant \frac{a^2+1}{2a} \geqslant 1 \ (\arg p(z_0) = \frac{\pi}{2}\beta) \tag{92}$$

and

$$k \leqslant -\frac{a^2+1}{2a} \leqslant -1 \ (\arg p(z_0) = -\frac{\pi}{2}\beta), \tag{93}$$

where

$$(p(z_0))^{\frac{1}{\beta}} = \pm ia \ (a > 0). \tag{94}$$

Now, we derive the following theorem.

THEOREM 7. *Let* $p(z)$ *be analytic with* $p(0) = 1$ *and* $p(z) \neq 0$ *in* $\mathbb{U}$. *If* $p(z)$ *satisfies*

$$p(z) + \frac{zp'(z)}{p(z)} \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} + \frac{2\beta\sqrt[n]{z}}{1-(\sqrt[n]{z})^2} \ (z \in \mathbb{U}) \tag{95}$$

with $0 < \beta \leqslant 1$ *and* $n = 1, 2, 3\cdots$, *then*

$$p(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} \ (z \in \mathbb{U}). \tag{96}$$

Proof. We consider a function $p(z)$ which is not satisfy the subordination (96). Then there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\beta \ (|z| < |z_0| < 1) \tag{97}$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\beta. \tag{98}$$

Applying Lemma 4, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta, \tag{99}$$

where k satisfies (92) and (93) in Lemma 4. If $p(z_0) = \frac{\pi}{2}\beta$, then

$$p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} = (ia)^{\beta} + ik\beta \;\; (a > 0). \tag{100}$$

Considering a boundary point $z = e^{i\theta}$ in $\mathbb{U}$, we see that

$$\left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} + \frac{2\beta\sqrt[n]{z}}{1-(\sqrt[n]{z})^2} = \left(i\frac{\sin\frac{\theta}{n}}{1-\cos\frac{\theta}{n}}\right)^{\beta} + i\frac{\beta}{\sin\frac{\theta}{n}}. \tag{101}$$

Thus, if the subordination (95) is satisfied, then

$$a = \frac{\sin\frac{\theta}{n}}{1-\cos\frac{\theta}{n}} > 0 \tag{102}$$

and

$$k \geqslant \frac{a^2+1}{2a} = \frac{1}{\sin\frac{\theta}{n}} \geqslant 1. \tag{103}$$

This implies that $p(z)$ is not satisfy the condition (95). For the case $\arg p(z_0) = -\frac{\pi}{2}\beta$, using the same way for the case $\arg p(z_0) = \frac{\pi}{2}\beta$, we say that $p(z)$ is not satisfy the condition (95). Therefore, we complete the proof of the theorem. □

Letting $p(z) = f'(z)$ in Theorem 7, we have the following corollary.

COROLLARY 9. *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies* $f'(z) \neq 0$ $(z \in \mathbb{U})$ *and*

$$f'(z) + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} + \frac{2\beta\sqrt[n]{z}}{1-(\sqrt[n]{z})^2} \;(z \in \mathbb{U}) \tag{104}$$

with $0 < \beta \leqslant 1$ *then*

$$f'(z) \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} \;(z \in \mathbb{U}), \tag{105}$$

that is, $f(z) \in \overline{\mathscr{R}}(n,0)$.

COROLLARY 10. *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies* $\frac{zf'(z)}{f(z)} \neq 0$ $(z \in \mathbb{U})$ *and*

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} + \frac{2\beta\sqrt[n]{z}}{1-(\sqrt[n]{z})^2} \;(z \in \mathbb{U}) \tag{106}$$

with $0 < \beta \leqslant 1$ *then*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)^{\beta} \;(z \in \mathbb{U}), \tag{107}$$

that is, $f(z) \in \overline{\mathscr{S}}^{*}(n,0)$.

3. Consideration for strongly functions

Let us consider a function $f(z) \in \overline{\mathscr{A}}(n)$ given by

$$f(z) = \int_0^z \left(\frac{1+\sqrt[n]{t}}{1-\sqrt[n]{t}} \right)^{\alpha} dt \, , \, (z \in \mathbb{U}). \tag{108}$$

with $0 \leqslant \alpha < 1$. Then $f(z)$ satisfies

$$\left| arg f'(z) \right| = \alpha \left| arg \left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}} \right) \right| < \frac{\pi}{2}\alpha \, , \, (z \in \mathbb{U}). \tag{109}$$

We say that $f(z)$ is strongly of order α in $\mathbb{U}$ if $f(z) \in \overline{\mathscr{A}}(n)$ satisfies

$$\left| arg f'(z) \right| < \frac{\pi}{2}\alpha \, , \, (z \in \mathbb{U}) \tag{110}$$

for $0 \leqslant \alpha < 1$.

To discuss about strongly functions in $\mathbb{U}$ we need the following lemma due to Fejér and Riesz [1] (also by Tsuji [11].)

LEMMA 5. *Let a function $f(z)$ be analytic in $|z| \leqslant 1$. Then, $f(z)$ satisfies*

$$\int_{-1}^{1} |f(z)|^{\rho} |dz| \leqslant \frac{1}{2} \int_{|z|=1} |f(z)|^{\rho} |dz| \, , \, (\rho > 0), \tag{111}$$

where the above integral on the left hand side is considered along the real axis.

REMARK 2. If we make a change of variables in Lemma 5, then the inequality (111) becomes

$$\int_{-r}^{r} |f(\rho e^{i\theta})|^{\rho} d\rho \leqslant \frac{r}{2} \int_0^{2\pi} |f(re^{i\theta})|^{\rho} d\theta. \tag{112}$$

Also, we use the following lemma by Gwynme [4].

LEMMA 6. *Let $f(z)$ be a complex valued harmonic function defined on a neighborhood of a closed disk of radius 1 and center 0 in the complex plane. Then*

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\rho}) \frac{1-r^2}{1+r^2-2r\cos(\theta-\rho)} d\rho \tag{113}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos\rho} d\rho = 1. \tag{114}$$

Now, we derive the following theorem.

THEOREM 8. *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqslant \frac{\alpha}{2}, \ (z \in \mathbb{U}) \tag{115}$$

for some real α $(0 < \alpha \leqslant 1)$, *then*

$$\left|arg f'(z)\right| \leqslant \frac{\pi}{2}\alpha, \ (z \in \mathbb{U}). \tag{116}$$

Proof. We note that

$$\log f'(z) = \log|f'(z)| + i \arg f'(z) \tag{117}$$

and

$$\log f'(z) = \int_0^z \left(\log f'(t)\right)' dt = \int_0^z \frac{f''(t)}{f'(t)} dt. \tag{118}$$

Thus we have

$$\begin{aligned} \left|arg f'(z)\right| &= \left|\mathrm{Im}\left(\log f'(z)\right)\right| \\ &= \left|\mathrm{Im} \int_0^z \frac{f''(t)}{f'(t)} dt\right| \\ &\leqslant \left|\mathrm{Im} \int_0^r \frac{f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})} e^{i\theta} d\rho\right| \\ &= \int_0^r \left|\mathrm{Im}\left(\frac{e^{i\theta} f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right)\right| d\rho \\ &\leqslant \int_{-r}^r \left|\mathrm{Im}\left(\frac{e^{i\theta} f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right)\right| d\rho \\ &\leqslant \int_{-r}^r \left|\frac{f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right| d\rho, \end{aligned} \tag{119}$$

where $z = re^{i\theta}$ $(0 \leqslant \theta < 2\pi)$, $0 \leqslant r < 1$ and $0 \leqslant \rho \leqslant r$. Using the inequality (112) with $\rho = 1$, we obtain

$$\begin{aligned} \left|arg f'(z)\right| &\leqslant \frac{r}{2} \int_0^{2\pi} \left|\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left|\frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta \\ &\leqslant \frac{\alpha}{4} \int_0^{2\pi} d\theta = \frac{\pi}{2}\alpha. \end{aligned} \tag{120}$$

This completes the proof of theorem. □

Taking $\alpha = 1$ in Theorem 8, we obtain the following corollary.

COROLLARY 11. *Let* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqslant \frac{1}{2}, \ (z \in \mathbb{U}) \tag{121}$$

then

$$\left|\arg f'(z)\right| < \frac{\pi}{2}, \ (z \in \mathbb{U}). \tag{122}$$

REMARK 3. If we take $n = 3$ in Theorem 8, then we have the result by Güney and Owa [2].

Next, we derive the following theorem.

THEOREM 9. *If* $f(z) \in \overline{\mathscr{A}}(n)$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{\alpha}{2} Re\left(\frac{1+\beta\sqrt[n]{z}}{1-\sqrt[n]{z}}\right), \ (z \in \mathbb{U}) \tag{123}$$

for some real α $(0 < \alpha \leqslant 1)$ *and some real* β $(\beta \neq -1)$, *then*

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\alpha, \ (z \in \mathbb{U}). \tag{124}$$

Proof. If we use the same method of the proof in Theorem 8, we have that

$$\left|\arg f'(z)\right| < \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta. \tag{125}$$

Thus, using the inequality (123), we know that

$$\begin{aligned}
\left|\arg f'(z)\right| &< \frac{\alpha}{4}\int_0^{2\pi} Re\left(\frac{1+\beta\sqrt[n]{r}e^{i\frac{\theta}{n}}}{1-\sqrt[n]{r}e^{i\frac{\theta}{n}}}\right) d\theta \\
&= \frac{\alpha}{4}\int_0^{2\pi}\left(\frac{1+(\beta-1)\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)-\beta\left(\sqrt[n]{r}\right)^2}{1-2\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)+\left(\sqrt[n]{r}\right)^2}\right) d\theta \\
&= \frac{\alpha}{4}\int_0^{2\pi}\left\{\frac{1-\beta}{2}+\frac{1+\beta}{2}\,\frac{1-\left(\sqrt[n]{r}\right)^2}{1+\left(\sqrt[n]{r}\right)^2-2\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)}\right\} d\theta.
\end{aligned} \tag{126}$$

Further, we note that

$$\int_0^{2\pi}\frac{1-\left(\sqrt[n]{r}\right)^2}{1+\left(\sqrt[n]{r}\right)^2-2\sqrt[n]{r}\cos\frac{\theta}{n}}d\theta = n\int_0^{\frac{2\pi}{n}}\frac{1-\left(\sqrt[n]{r}\right)^2}{1+\left(\sqrt[n]{r}\right)^2-2\sqrt[n]{r}\cos\rho}d\rho \leqslant 2\pi \tag{127}$$

by Lemma 6. Therefore, we obtain that

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\alpha, \ (z \in \mathbb{U}). \quad \square \tag{128}$$

REMARK 4. If we consider $n = 3$ in Theorem 9, then we obtain the result for $f(z) \in \overline{\mathscr{A}}(3)$ in Güney and Owa [2].

EXAMPLE 2. We consider a function $f(z) \in \overline{\mathscr{A}}(n)$ given by

$$f'(z) = \left(\frac{2}{2-\sqrt[n]{z}}\right)^{3\alpha}, (z \in \mathbb{U}) \tag{129}$$

with α $(0 < \alpha \leqslant 1)$. It follows that

$$\left|argf'(z)\right| = 3\alpha \left|\arg\left(2-\sqrt[n]{z}\right)\right| < 3\alpha\frac{\pi}{6} = \frac{\pi}{2}\alpha, (z \in \mathbb{U}) \tag{130}$$

and

$$\left|\frac{zf''(z)}{f'(z)}\right| = \frac{3\alpha}{n}\left|\frac{\sqrt[n]{z}}{2-\sqrt[n]{z}}\right| < \frac{3}{n}\alpha, (z \in \mathbb{U}). \tag{131}$$

If we consider some real β such that $\beta \leqslant \frac{n-12}{n}$, then we have

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3}{n}\alpha \leqslant \frac{\alpha(1-\beta)}{4} < \frac{\alpha}{2}Re\left(\frac{1+\beta\sqrt[n]{z}}{1-\sqrt[n]{z}}\right), (z \in \mathbb{U}). \tag{132}$$

Next, we derive the following theorem.

THEOREM 10. *If $f(z) \in \overline{\mathscr{A}}(n)$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{\alpha}{2}Re\left(\frac{1+\beta\sqrt[n]{z}}{1-\sqrt[n]{z}}\right), (z \in \mathbb{U}) \tag{133}$$

for some real α $(0 < \alpha \leqslant 1)$ and some real β $(\beta \neq -1)$, then

$$\left|\arg\frac{f(z)}{z}\right| < \frac{\pi}{2}\alpha, (z \in \mathbb{U}). \tag{134}$$

Proof. We note that

$$\log\left(\frac{f(z)}{z}\right) = \log\left|\frac{f(z)}{z}\right| + i\arg\left(\frac{f(z)}{z}\right) \tag{135}$$

and

$$\begin{aligned} \log\left(\frac{f(z)}{z}\right) &= \int_0^z \left(\log\left(\frac{f(t)}{t}\right)\right)' dt \\ &= \int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right) dt. \end{aligned} \tag{136}$$

Thus, we have

$$\begin{aligned}
\left|\arg\left(\frac{f(z)}{z}\right)\right| &= \left|\operatorname{Im}\left(\log\left(\frac{f(z)}{z}\right)\right)\right| \\
&= \left|\operatorname{Im}\int_0^z\left(\frac{f'(t)}{f(t)}-\frac{1}{t}\right)dt\right| \\
&= \left|\operatorname{Im}\int_0^r\left(\frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}-\frac{1}{\rho e^{i\theta}}\right)e^{i\theta}d\rho\right| \\
&\leqslant \int_0^r\left|\operatorname{Im}\left(\frac{f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}-\frac{1}{\rho e^{i\theta}}\right)e^{i\theta}\right|d\rho \\
&< \int_{-r}^r\left|\operatorname{Im}\left(\frac{e^{i\theta}f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}-\frac{1}{\rho}\right)\right|d\rho \\
&\leqslant \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}-1\right|d\theta \\
&< \frac{\alpha}{4}\int_0^{2\pi}Re\left(\frac{1+\beta\sqrt[n]{r}e^{i\frac{\theta}{n}}}{1-\sqrt[n]{r}e^{i\frac{\theta}{n}}}\right)d\theta \\
&\leqslant \frac{\pi}{2}\alpha.
\end{aligned} \tag{137}$$

Thus, we complete the proof of the theorem. □

EXAMPLE 3. We consider a function $f(z)\in\overline{\mathscr{A}}(n)$ given by

$$f(z)=z\left(\frac{2}{2-\sqrt[n]{z}}\right)^{3\alpha},\ (z\in\mathbb{U}) \tag{138}$$

with $\alpha\ (0<\alpha\leqslant 1)$. Then

$$w(z)=\frac{2}{2-\sqrt[n]{z}} \tag{139}$$

satisfies

$$\left|w(z)-\frac{4}{3}\right|<\frac{2}{3},\ (z\in\mathbb{U}). \tag{140}$$

Thus $w(z)$ satisfies

$$|\arg w(z)|=\left|\arg\left(\frac{2}{2-\sqrt[n]{z}}\right)\right|<\frac{\pi}{6},\ (z\in\mathbb{U}). \tag{141}$$

This gives us that

$$\left|\arg\left(\frac{f(z)}{z}\right)\right|=3\alpha\left|\arg\left(\frac{2}{2-\sqrt[n]{z}}\right)\right|<\frac{\pi}{2}\alpha,\ (z\in\mathbb{U}). \tag{142}$$

Also, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \frac{3\alpha}{n}\left|\frac{\sqrt[n]{z}}{2-\sqrt[n]{z}}\right| < \frac{3}{n}\alpha, \ (z \in \mathbb{U}). \tag{143}$$

Therefore, considering β such that $\beta \leqslant \frac{n-12}{n}$, we see

$$\left|\frac{zf'(z)}{f(z)}-1\right| < \frac{3}{n}\alpha \leqslant \frac{\alpha(1-\beta)}{4} < \frac{\alpha}{2}Re\left(\frac{1+\beta\sqrt[n]{z}}{1-\sqrt[n]{z}}\right), \ (z \in \mathbb{U}). \tag{144}$$

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