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NEW INEQUALITIES FOR THE HADAMARD PRODUCT OF HILBERT SPACE OPERATORS

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Abstract. The main goal of this paper is to present further investigations of the Hadamard product of Hilbert space operators and matrices. In particular, we prove a Cauchy-Schwarz-type inequality involving the Hadamard product.

Then, singular value and norm bounds will be obtained as an application of the aforementioned Cauchy-Schwarz inequality. For example, if A and B are compact operators on a separable Hilbert space, it is shown that

$$s_j(A\circ B)\leqslant \||A^*|\circ|B^*|\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(|A|\circ|B|)$$

where $\circ$, $\|\cdot\|$ and $|\cdot|$ denote the Hadamard product, the usual operator norm, and the absolute value, respectively.

After that, numerical radius and spectral radius bounds for operator forms involving the Hadamard product are presented.

1. Introduction

For a separable Hilbert space $\mathbb{H}$, the notation $\mathbb{B}(\mathbb{H})$ will stand for the C^*-algebra of bounded linear operators from $\mathbb{H}$ to itself. When $\mathbb{H}$ is finite-dimensional, say of dimension n, $\mathbb{B}(\mathbb{H})$ is identified with the algebra $\mathbb{M}_n$ of all $n\times n$ complex matrices.

A product operation on $\mathbb{B}(\mathbb{H})$ has been defined by many forms, with certain desirable properties for each form. The usual operation is defined for $A,B\in\mathbb{B}(\mathbb{H})$ by $(AB)x=A(Bx)$, for $x\in\mathbb{H}$.

Another important product is the so-called Hadamard product, which is defined via a fixed orthonormal basis of $\{e_j\}$ of $\mathbb{H}$. More precisely, if $A,B\in\mathbb{B}(\mathbb{H})$, then the Hadamard product of A,B is denoted by $A\circ B$, and is defined by $\left\langle (A\circ B)e_i,e_j\right\rangle = \left\langle Ae_i,e_j\right\rangle\left\langle Be_i,e_j\right\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the defined inner product on $\mathbb{H}$.

When $\mathbb{H}$ is finite-dimensional, the Hadamard product is simply the entry-wise multiplication of matrices.

The Hadamard product has received considerable attention in the literature, as the reader can see in [5, 20]. We also refer to [28] for an excellent overview and history.

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One can obtain the Hadamard product by filtering the Kronecker product through a positive linear map. More precisely, if $A,B \in \mathbb{B}(\mathbb{H})$, and if $A \otimes B$ denotes the Kronecker (or tensor) product of A and B, then

$$A \circ B = U^{*}(A \otimes B)U, \tag{1}$$

where $U : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the isometry determined by $Ue_i = e_i \otimes e_i$ for all $i \in \mathbb{N}$; see [11]. In what follows, the operator U will denote this particular isometry.

Utilizing (1), together with the properties of the Kronecker product, one can deduce some interesting properties of the Hadamard product.

Our concern in this paper is to show some new properties and inequalities that involve the Hadamard product. For example, we discuss sufficient conditions that ensure accretivity of $A \circ B$. We recall here that an operator $T \in \mathbb{B}(\mathbb{H})$ is said to be accretive provided that $\Re T \geqslant O$, where $\Re T$ is the real part of T, defined by $\Re T = \frac{T+T^*}{2}$, and O is the zero operator in $\mathbb{B}(\mathbb{H})$. By writing $T \geqslant O$ we mean that $\langle Tx, x\rangle \geqslant 0$ for all $x \in \mathbb{H}$. Such an operator is referred to as being a positive operator. Similarly, the imaginary part of T is defined by $\Im T = \frac{T-T^*}{2\mathrm{i}}$. When both $\Re T, \Im T \geqslant O$, we say that T is accretive-dissipative. An operator $T \in \mathbb{B}(\mathbb{H})$ is said to be sectorial if, for some $0 \leqslant \theta < \frac{\pi}{2}$, we have

$$W(T) \subset \mathbb{S}_\theta := \{z \in \mathbb{C} : \ |\Im z| \leqslant \tan\theta \, \Re z\},$$

where $W(T)$ is the numerical range of T, defined by $W(T) = \{\langle Tx, x\rangle : x \in \mathbb{H}, \|x\| = 1\}$. When $W(T) \subset \mathbb{S}_\theta$, we will write $T \in \mathbb{S}_\theta$.

This class of operators has received renowned attention in the recent advances in the field, and we refer the reader to [22] for some work dedicated to this concern.

Related to the real part, we state the following simple observations, whose proof follows immediately noting the distributive property of the Hadamard product.

PROPOSITION 1. *Let $A,B \in \mathbb{B}(\mathbb{H})$. Then*

$$A \circ B = \Re A \circ \Re B - \Im A \circ \Im B + \mathrm{i}(\Im A \circ \Re B + \Re A \circ \Im B).$$

More precisely,

$$\Re(A \circ B) = \Re A \circ \Re B - \Im A \circ \Im B \tag{2}$$

and

$$\Im(A \circ B) = \Im A \circ \Re B + \Re A \circ \Im B. \tag{3}$$

When $A,B \in \mathbb{B}(\mathbb{H})$ are self-adjoint, we say that $A \geqslant B$ when $A - B \geqslant O$. Proposition 1 says that $A \circ B$ is accretive if $\Re A \circ \Re B \geqslant \Im A \circ \Im B$. On the other hand, the Hadamard product enjoys the property that $T_1 \circ T_2 \geqslant T_3 \circ T_4$, whenever $T_1 \geqslant T_3 \geqslant O$ and $T_2 \geqslant T_4 \geqslant O$. This can be proved easily noting distributivity of the Hadamard product, and the fact that the Hadamard product of two positive operators in necessarily positive; see [26] and [28, Theorem 3.1]. Indeed, if $T_1 \geqslant T_3 \geqslant O$ and $T_2 \geqslant T_4 \geqslant O$, it follows that

$$T_1 \circ T_2 - T_3 \circ T_2 = (T_1 - T_3) \circ T_2 \geqslant O \tag{4}$$

and

$$T_3 \circ T_2 - T_3 \circ T_4 = T_3 \circ (T_2 - T_4) \geqslant O. \tag{5}$$

Adding (4) and (5) implies that $T_1 \circ T_2 \geqslant T_3 \circ T_4$, provided that $T_1 \geqslant T_3 \geqslant O$ and $T_2 \geqslant T_4 \geqslant O$.

In conclusion, we can state the following result.

COROLLARY 1. *Let $A, B \in \mathbb{B}(\mathbb{H})$ be accretive-dissipative such that $\Re A \geqslant \Im A$ and $\Re B \geqslant \Im B$. Then $A \circ B$ is accretive.*

Before proceeding, we list some useful properties of the Kronecker product that we will use in this work.

LEMMA 1. *Let $T_1, T_2, T_3, T_4 \in \mathbb{B}(\mathbb{H})$. Then*

(i) $T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$ *and* $(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$.

(ii) $(T_1 \otimes T_2)^* = T_1^* \otimes T_2^*$.

(iii) $(T_1 T_2) \otimes (T_3 T_4) = (T_1 \otimes T_3)(T_2 \otimes T_4)$.

(iv) If r is a positive integer, then $(T_1 \otimes T_2)^r = T_1^r \otimes T_2^r$.

(v) $T_1 \otimes T_2 = (T_1 \otimes \mathbf{1})(\mathbf{1} \otimes T_2) = (\mathbf{1} \otimes T_2)(T_1 \otimes \mathbf{1})$, *where $\mathbf{1}$ is the identity operator in $\mathbb{B}(\mathbb{H})$.*

(vi) $\|T_1 \otimes T_2\| = \|T_1\| \, \|T_2\|$, *where $\|\cdot\|$ stands for the usual operator (spectral) norm.*

(vii) $|T_1 \otimes T_2| = |T_1| \otimes |T_2|$, *where $|\cdot|$ stands for the absolute value.*

(viii) If $T_1, T_2 \geqslant O$ and $r \geqslant 0$, then $(T_1 \otimes T_2)^r = T_1^r \otimes T_2^r$.

Among those useful tools in obtaining our results is the so-called Aluthge transform, which was defined in [4] for the operator T with polar decomposition $T = V|T|$ by $\widetilde{T} = |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}}$. This has been used in the literature by many authors to deduce new and sharp bounds for celebrated quantities, such as the numerical radius. We recall that if $T \in \mathbb{B}(\mathbb{H})$, the numerical radius of T is defined by $\omega(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle|$. We refer the reader to [18, 24, 25] as a list of references that utilized the Aluthge transform to study the numerical radius. In this paper, applications will involve some new bounds for the numerical radius of the Hadamard product.

Another application of the obtained results will be singular value bounds. Given a compact operator $T \in \mathbb{B}(\mathbb{H})$, its singular values are arranged in a non-increasing order according to their multiplicities. Thus, $s_1(T) \geqslant s_2(T) \geqslant \ldots$.

In the sequel, we will use the notation $\mathbb{K}(\mathbb{H})$ to denote the class of compact operators in $\mathbb{B}(\mathbb{H})$, where $\mathbb{H}$, as indicated earlier, is a separable Hilbert space.

We list some lemmas that we will need to prove our main results.

LEMMA 2. (See [16, Lemma 1] and [7, Proposition 1.3.2]) *Let $T_1, T_2, T_3 \in \mathbb{B}(\mathbb{H})$, where T_1 and T_2 are positive. Then the following statements are mutually equivalent:*

(i) $\begin{bmatrix} T_2 & T_3 \\ T_3^* & T_1 \end{bmatrix}$ *is a positive operator in $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$.*

(ii) $\begin{bmatrix} T_1 & T_3^* \\ T_3 & T_2 \end{bmatrix}$ *is a positive operator in $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$.*

(iii) $|\langle T_3 x, y\rangle|^2 \leqslant \langle T_1 x, x\rangle \langle T_2 y, y\rangle$ *for all $x, y \in \mathbb{H}$.*

(iv) There exists a contraction T_4 (i.e., $\|T_4\| \leqslant 1$) such that $T_3 = T_2^{\frac{1}{2}} T_4 T_1^{\frac{1}{2}}$.

The max-min principle is a useful tool in obtaining singular value bounds, which can be found in [27, Theorem 1.5] or [12, Theorem 9.1].

LEMMA 3. *Let $T \in \mathbb{K}(\mathbb{H})$. Then for $j = 1, 2, \ldots,$*

$$s_j(T) = \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\|=1}} \|Tx\|.$$

LEMMA 4. [14, (4.6)] *Let $T_1, T_2 \in \mathbb{B}(\mathbb{H})$. Then*

$$\frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\| T_1 + e^{\mathrm{i}\theta} T_2^* \right\| = \omega\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix} \right).$$

The organization of the subsequent sections will be as follows. In the first part, we show a Cauchy-Schwarz-type inequality for the Hadamard product, where we show that

$$|\langle (T_1 T_2 \circ T_3 T_4) x, y\rangle| \leqslant \sqrt{\left\langle \left(|T_2|^2 \circ |T_4|^2 \right) x, x \right\rangle \left\langle \left(|T_1^*|^2 \circ |T_3^*|^2 \right) y, y \right\rangle},$$

where $T_1, T_2, T_3, T_4 \in \mathbb{B}(\mathbb{H})$ and $x, y \in \mathbb{H}$. This will be employed to demonstrate some simpler forms with some further applications. For example, it is shown that the operator matrix $\begin{bmatrix} |T_2|^2 \circ |T_4|^2 & T_1 T_2 \circ T_3 T_4 \\ T_2^* T_1^* \circ T_4^* T_3^* & |T_1^*|^2 \circ |T_3^*|^2 \end{bmatrix}$ is positive in $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$. After that, the above Cauchy-Schwarz inequality discusses possible bounds for the singular values of certain matrix forms involving the Hadamard product. This discussion will lead to the relation

$$\|\Re (A \circ B)\| \leqslant \sec^2\theta \ \|\Re A \circ \Re B\|,$$

where $A, B \in \mathbb{S}_\theta$. Then, numerical radius bounds are studied, and interestingly, as an application, we strengthen the above inequality by removing the factor $\sec^2\theta$ for the class of accretive-dissipative operators.

2. Main results

2.1. Cauchy-Schwarz type inequalities

In this subsection, we present several inequalities that can be considered of the Cauchy-Schwarz type for the Hadamard product. Applications of the obtained inequalities on singular values and numerical radii will follow. We remind the reader of the celebrated inequalities [15], known as the mixed Cauchy-Schwarz inequality, which states

$$|\langle Ax,y\rangle|^2 \leqslant \left\langle |A|^{2(1-t)}x,x\right\rangle \left\langle |A^*|^{2t}y,y\right\rangle, \tag{6}$$

where $A \in \mathbb{B}(\mathbb{H})$ and $0 \leqslant t \leqslant 1$. In what follows, we prove a mixed Cauchy-Schwarz inequality for the Hadamard product.

THEOREM 1. *Let $T_1,T_2,T_3,T_4 \in \mathbb{B}(\mathbb{H})$. Then for any $x,y \in \mathbb{H}$,*

$$|\langle (T_1T_2 \circ T_3T_4)x,y\rangle| \leqslant \sqrt{\left\langle \left(|T_2|^2 \circ |T_4|^2\right)x,x\right\rangle \left\langle \left(|T_1^*|^2 \circ |T_3^*|^2\right)y,y\right\rangle}.$$

Proof. Let $x,y \in \mathbb{H}$. By the Cauchy-Schwarz inequality, the relation (1) and Lemma 1, we have

$$\begin{aligned}
|\langle (T_1T_2 \circ T_3T_4)x,y\rangle| &= |\langle U^*(T_1T_2 \otimes T_3T_4)Ux,y\rangle| \\
&= |\langle U^*(T_1 \otimes T_3)(T_2 \otimes T_4)Ux,y\rangle| \\
&= |\langle (T_2 \otimes T_4)Ux,(T_1^* \otimes T_3^*)Uy\rangle| \\
&\leqslant \|(T_2 \otimes T_4)Ux\| \, \|(T_1^* \otimes T_3^*)Uy\| \\
&= \sqrt{\left\langle U^*|T_2 \otimes T_4|^2Ux,x\right\rangle \left\langle U^*|T_1^* \otimes T_3^*|^2Uy,y\right\rangle} \\
&= \sqrt{\left\langle U^*\left(|T_2|^2 \otimes |T_4|^2\right)Ux,x\right\rangle \left\langle U^*\left(|T_1^*|^2 \otimes |T_3^*|^2\right)Uy,y\right\rangle} \\
&= \sqrt{\left\langle \left(|T_2|^2 \circ |T_4|^2\right)x,x\right\rangle \left\langle \left(|T_1^*|^2 \circ |T_3^*|^2\right)y,y\right\rangle},
\end{aligned}$$

as required. □

COROLLARY 2. *Let $A,B \in \mathbb{B}(\mathbb{H})$ and let $0 \leqslant t \leqslant 1$. Then for any $x,y \in \mathbb{H}$,*

$$|\langle (A \circ B)x,y\rangle| \leqslant \sqrt{\left\langle \left(|A|^{2t} \circ |B|^{2(1-t)}\right)x,x\right\rangle \left\langle \left(|A^*|^{2(1-t)} \circ |B^*|^{2t}\right)y,y\right\rangle}. \tag{7}$$

In particular,

$$|\langle (A \circ B)x,y\rangle| \leqslant \sqrt{\langle (|A| \circ |B|)x,x\rangle \langle (|A^*| \circ |B^*|)y,y\rangle}$$

and

$$|\langle (A \circ B)x,y\rangle| \leqslant \sqrt{\left\langle \left(\mathbf{1} \circ |B|^2\right)x,x\right\rangle \left\langle \left(|A^*|^2 \circ \mathbf{1}\right)y,y\right\rangle}.$$

Proof. Let $A = V_1|A|$ and $B = V_2|B|$ be the polar decompositions of A and B, respectively. Letting $T_1 = V_1|A|^{1-t}$, $T_2 = |A|^t$, $T_3 = V_2|B|^t$, and $T_4 = |B|^{1-t}$, in Theorem 1, we deduce the desired result. □

REMARK 1. Letting $x = y = e_i$, in Theorem 1, we infer that

$$|\langle (T_1T_2 \circ T_3T_4)e_i, e_i\rangle| \leqslant \sqrt{\left\langle \left(|T_2|^2 \circ |T_4|^2\right)e_i, e_i\right\rangle \left\langle \left(|T_1^*|^2 \circ |T_3^*|^2\right)e_i, e_i\right\rangle}$$
$$= \sqrt{\left\langle \left(\left(|T_2|^2 \circ |T_4|^2\right) \circ \left(|T_1^*|^2 \circ |T_3^*|^2\right)\right)e_i, e_i\right\rangle}.$$

Now Theorem 1, together with Lemma 2, implies the following result, which presents a new positive operator matrix involving the Hadamard product.

COROLLARY 3. *Let $T_1, T_2, T_3, T_4 \in \mathbb{B}(\mathbb{H})$. Then* $\begin{bmatrix} |T_2|^2 \circ |T_4|^2 & T_1T_2 \circ T_3T_4 \\ T_2^*T_1^* \circ T_4^*T_3^* & |T_1^*|^2 \circ |T_3^*|^2 \end{bmatrix}$ *is a positive operator in $\mathbb{B}(\mathbb{H} \oplus \mathbb{H})$.*

We recall that when $A, B \in \mathbb{B}(\mathbb{H})$ are positive, and $0 \leqslant t \leqslant 1$, the quantity $\frac{1}{2}\left(A^tB^{1-t} + A^{1-t}B^t\right)$, which represents the Heinz means for A and B, has received significant attention in the literature, due to its relation with other operator means. The reader is referred to [7, 10, 19, 17, 23] for some work treating this quantity and to [6] for treating this quantity when the Hadamard product replaces the usual product. In the following result, we present an upper bound for $\Re(A \circ B)$ in terms of a quantity that simulates the above Heinz quantity.

THEOREM 2. *Let $A, B \in \mathbb{B}(\mathbb{H})$ and let $0 \leqslant t \leqslant 1$. Then*

$$\Re(A \circ B) \leqslant \frac{|A|^{2t} \circ |B|^{2(1-t)} + |A^*|^{2(1-t)} \circ |B^*|^{2t}}{2}.$$

Proof. Implementing the arithmetic-geometric mean inequality, it follows from (7) that

$$\begin{aligned}
\langle \Re(A \circ B)x, x\rangle &= \Re\langle (A \circ B)x, x\rangle \\
&\leqslant |\langle (A \circ B)x, x\rangle| \\
&\leqslant \sqrt{\left\langle \left(|A|^{2t} \circ |B|^{2(1-t)}\right)x, x\right\rangle \left\langle \left(|A^*|^{2(1-t)} \circ |B^*|^{2t}\right)x, x\right\rangle} \\
&\leqslant \frac{1}{2}\left\langle \left(|A|^{2t} \circ |B|^{2(1-t)} + |A^*|^{2(1-t)} \circ |B^*|^{2t}\right)x, x\right\rangle.
\end{aligned}$$

This implies the desired result. □

We have the following result as a direct consequence of Theorem 2.

COROLLARY 4. *Let* $A, B \in \mathbb{B}(\mathbb{H})$. *Then*

$$\Re(A \circ B) \leqslant \frac{|A| \circ |B| + |A^*| \circ |B^*|}{2}.$$

In particular, if A, B *are two normal operators, then*

$$\Re(A \circ B) \leqslant |A| \circ |B|.$$

Notice that using the results in [9], we can also get several bounds for $|\Re(A \circ B)|$ in terms of $|A| \circ |B|$ and $|A^*| \circ |B^*|$.

2.2. Singular value bounds

In this subsection, we present some bounds for the singular values of operator forms that involve the Hadamard product. We recall, here, that if $T_1, T_2 \in \mathbb{K}(\mathbb{H})$, then [27, Theorem 1.6]

$$s_j(T_1 T_2) \leqslant \|T_1\| s_j(T_2) \text{ and } s_j(T_1 T_2) \leqslant \|T_2\| s_j(T_1),\ j = 1, 2, \ldots. \tag{8}$$

This clearly extends the sub-multiplicative property of the operator norm. The above inequalities together imply that if $T_1, T_2, T_3 \in \mathbb{K}(\mathbb{H})$, then [13, p. 27].

$$s_j(T_1 T_2 T_3) \leqslant \|T_1\|\ \|T_3\| s_j(T_2),\ j = 1, 2, \ldots.$$

For up-to-date discussion regarding singular value bounds, the reader may refer to [2, 3, 21, 30].

THEOREM 3. *Let* $T_1, T_2, T_3, T_4 \in \mathbb{K}(\mathbb{H})$. *Then for* $j = 1, 2, \ldots$,

$$s_j(T_1 T_2 \circ T_3 T_4) \leqslant \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\left(|T_2|^2 \circ |T_4|^2 \right).$$

Proof. If we take supremum over unit vectors $y \in \mathbb{H}$ in Theorem 2, we obtain, for all unit vectors $x \in \mathbb{H}$,

$$\begin{aligned} \|(T_1 T_2 \circ T_3 T_4) x\| &\leqslant \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|^{\frac{1}{2}} \left\langle \left(|T_2|^2 \circ |T_4|^2 \right) x, x \right\rangle^{\frac{1}{2}} \\ &\leqslant \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|^{\frac{1}{2}} \left\| \left(|T_2|^2 \circ |T_4|^2 \right) x \right\|^{\frac{1}{2}} \end{aligned}$$

where the second inequality is obtained via the Cauchy-Schwarz inequality. That is,

$$\|(T_1 T_2 \circ T_3 T_4) x\| \leqslant \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|^{\frac{1}{2}} \left\| \left(|T_2|^2 \circ |T_4|^2 \right) x \right\|^{\frac{1}{2}}$$

for all unit vectors $x \in \mathbb{H}$. Thus, Lemma 3 implies

$$\begin{aligned} s_j\left(T_1T_2 \circ T_3T_4\right) &= \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\|=1}} \left\|\left(T_1T_2 \circ T_3T_4\right)x\right\| \\ &\leqslant \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|^{\frac{1}{2}} \max_{\dim M = j} \min_{\substack{x \in M \\ \|x\|=1}} \left\| \left(|T_2|^2 \circ |T_4|^2 \right) x \right\|^{\frac{1}{2}} \\ &= \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|^{\frac{1}{2}} \left(\max_{\dim M = j} \min_{\substack{x \in M \\ \|x\|=1}} \left\| \left(|T_2|^2 \circ |T_4|^2 \right) x \right\| \right)^{\frac{1}{2}} \\ &= \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|^{\frac{1}{2}} s_j^{\frac{1}{2}} \left(|T_2|^2 \circ |T_4|^2 \right), \end{aligned}$$

as required. □

COROLLARY 5. *Let* $A, B \in \mathbb{K}(\mathbb{H})$ *and let* $0 \leqslant t \leqslant 1$. *Then for* $j = 1, 2, \ldots,$

$$s_j\left(A \circ B\right) \leqslant \left\| |A^*|^{2(1-t)} \circ |B^*|^{2t} \right\|^{\frac{1}{2}} s_j^{\frac{1}{2}} \left(|A|^{2t} \circ |B|^{2(1-t)} \right).$$

Proof. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B, respectively. Letting $T_1 = U|A|^{1-t}$, $T_2 = |A|^t$, $T_3 = V|B|^t$, and $T_4 = |B|^{1-t}$, in Theorem 3, we deduce the desired result. □

REMARK 2. We notice that Corollary 5 gives a possible extension of (8) to the Hadamard product. For example, when $t = \frac{1}{2}$, we obtain

$$s_j(A \circ B) \leqslant \| |A^*| \circ |B^*| \|^{\frac{1}{2}} s_j^{\frac{1}{2}} \left(|A| \circ |B| \right).$$

In particular,

$$\|A \circ B\| \leqslant \sqrt{\| \, |A^*| \circ |B^*| \, \| \, \| \, |A| \circ |B| \, \|}. \tag{9}$$

In [29, (19)], it has been shown that if $A, B \in \mathbb{M}_n$, then

$$\|A \circ B\|^2 \leqslant \frac{1}{2} \left(\|AA^* \circ BB^*\| + \|AB^* \circ BA^*\| \right). \tag{10}$$

Now we give two examples to show that (9) and (10) are not comparable, in general. For this, let $A = \begin{bmatrix} [\,-1 & -1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then direct numerical calculations show that

$$\| \, |A^*| \circ |B^*| \, \| \, \| \, |A| \circ |B| \, \| = 1 \text{ and } \frac{1}{2} \left(\|AA^* \circ BB^*\| + \|AB^* \circ BA^*\| \right) = 1.5,$$

showing that (9) is better than (10) in this example.

On the other hand, letting $A=\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ shows that

$$\| \, |A^*| \circ |B^*| \, \| \| \, |A| \circ |B| \, \| = 0.8 \text{ and } \frac{1}{2}\left(\|AA^* \circ BB^*\| + \|AB^* \circ BA^*\|\right) = 0.5,$$

which indicates that (10) is better than (9) for this choice. Thus, neither (9) nor (10) is uniformly better than the other.

The inequality in Remark 2, in the matrix case, is also a consequence of [8, Corollary 2.7], which states that

$$|\Phi(Z)| \leqslant \Phi\left(|Z^*|^{1+p}\right) \sharp \Phi\left(|Z^*|^{1-p}\right)$$

for all 2-positive linear map Φ, and all real numbers p. With $Z = A \otimes B$, $p = 0$, and Φ being the compression map extracting $A \circ B$, we derive easily, for $j,k = 0,1,\ldots$

$$s_{1+j+k}^2 (A \circ B) \leqslant s_{1+j} (|A^*| \circ |B^*|) \, s_{1+k} (|A| \circ |B|).$$

COROLLARY 6. *Let $A \in \mathbb{K}(\mathbb{H})$ and let $0 \leqslant t \leqslant 1$. Then for $j = 1,2,\ldots,$*

$$s_j (A \circ A^*) \leqslant \left\| |A^*|^{2(1-t)} \circ |A|^{2t} \right\|^{\frac{1}{2}} s_j^{\frac{1}{2}} \left(|A|^{2t} \circ |A^*|^{2(1-t)} \right).$$

In particular,

$$s_j (A \circ A^*) \leqslant \|A\| \; s_j^{\frac{1}{2}} \left(|A|^{2t} \circ |A^*|^{2(1-t)} \right).$$

Proof. The first inequality follows from Corollary 5 by putting $B = A^*$. The second inequality is obtained from the following observations, where we use (1) and Lemma 1,

$$\begin{aligned}
\left\| |A^*|^{2(1-t)} \circ |A|^{2t} \right\| &= \left\| U^* \left(|A^*|^{2(1-t)} \otimes |A|^{2t} \right) U \right\| \\
&\leqslant \left\| |A^*|^{2(1-t)} \otimes |A|^{2t} \right\| \\
&= \left\| |A^*|^{2(1-t)} \right\| \left\| |A|^{2t} \right\| \\
&= \| \, |A^*| \, \|^{2(1-t)} \| \, |A| \, \|^{2t} \\
&= \|A\|^2. \qquad \square
\end{aligned}$$

On the other hand, singular value bounds for sums of products have been well-studied in the literature. Below, we present such bound for the Hadamard product.

THEOREM 4. *Let $T_1, T_2, T_3, T_4 \in \mathbb{K}(\mathbb{H})$ and let $0 \leqslant t \leqslant 1$. Then for $j = 1,2,\ldots,$*

$$\begin{aligned}
& s_j ((T_1 \circ T_2) \pm (T_3 \circ T_4)) \\
& \leqslant \left\| \left(|T_1^*|^{2(1-t)} \circ |T_2^*|^{2t} \right) + \left(|T_3^*|^{2(1-t)} \circ |T_4^*|^{2t} \right) \right\|^{\frac{1}{2}} s_j^{\frac{1}{2}} \\
& \quad \times \left(\left(|T_1|^{2t} \circ |T_2|^{2(1-t)} \right) + \left(|T_3|^{2t} \circ |T_4|^{2(1-t)} \right) \right).
\end{aligned}$$

Proof. Let $x, y \in \mathbb{H}$. We have

$$\begin{aligned}
&|\langle((T_1\circ T_2)+(T_3\circ T_4))x,y\rangle|\\
&=|\langle(T_1\circ T_2)x,y\rangle+\langle(T_3\circ T_4)x,y\rangle|\\
&\leqslant|\langle(T_1\circ T_2)x,y\rangle|+|\langle(T_3\circ T_4)x,y\rangle|\\
&\quad\text{(by the triangle inequality)}\\
&\leqslant\sqrt{\left\langle\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)x,x\right\rangle\left\langle\left(\left|T_1^*\right|^{2(1-t)}\circ\left|T_2^*\right|^{2t}\right)y,y\right\rangle}\\
&\quad+\sqrt{\left\langle\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)x,x\right\rangle\left\langle\left(\left|T_3^*\right|^{2(1-t)}\circ\left|T_4^*\right|^{2t}\right)y,y\right\rangle}\\
&\quad\text{(by (7))}\\
&\leqslant\left(\left\langle\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)x,x\right\rangle+\left\langle\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)x,x\right\rangle\right)^{\frac{1}{2}}\\
&\quad\times\left(\left\langle\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)y,y\right\rangle+\left\langle\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)y,y\right\rangle\right)^{\frac{1}{2}}\\
&\quad\text{(by the Cauchy-Schwarz inequality)}\\
&=\sqrt{\left\langle\left(\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right)x,x\right\rangle}\\
&\quad\times\sqrt{\left\langle\left(\left(\left|T_1^*\right|^{2(1-t)}\circ\left|T_2^*\right|^{2t}\right)+\left(\left|T_3^*\right|^{2(1-t)}\circ\left|T_4^*\right|^{2t}\right)\right)y,y\right\rangle}.
\end{aligned}$$

That is,

$$\begin{aligned}
&|\langle((T_1\circ T_2)+(T_3\circ T_4))x,y\rangle|\\
&\leqslant\sqrt{\left\langle\left(\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right)x,x\right\rangle}\\
&\quad\times\sqrt{\left\langle\left(\left(\left|T_1^*\right|^{2(1-t)}\circ\left|T_2^*\right|^{2t}\right)+\left(\left|T_3^*\right|^{2(1-t)}\circ\left|T_4^*\right|^{2t}\right)\right)y,y\right\rangle}.
\end{aligned}\tag{11}$$

If we take supremum over unit vectors $y \in \mathbb{H}$, in the above inequality, we obtain

$$\begin{aligned}
&\|((T_1\circ T_2)+(T_3\circ T_4))x\|\\
&\leqslant\left\|\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)+\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)\right\|^{\frac{1}{2}}\\
&\quad\times\left\|\left(\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right)x\right\|^{\frac{1}{2}}.
\end{aligned}$$

Consequently, using Lemma 3,

$$\begin{aligned}
& s_j\left((T_1\circ T_2)+(T_3\circ T_4)\right)\\
&= \max_{\dim M=j}\ \min_{\substack{x\in M\\ \|x\|=1}} \left\|\left((T_1\circ T_2)+(T_3\circ T_4)\right)x\right\|\\
&\leqslant \left\|\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)+\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)\right\|^{\frac{1}{2}}\\
&\quad\times \max_{\dim M=j}\ \min_{\substack{x\in M\\ \|x\|=1}} \left\|\left(\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right)x\right\|^{\frac{1}{2}}\\
&= \left\|\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)+\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)\right\|^{\frac{1}{2}}\\
&\quad\times \left(\max_{\dim M=j}\ \min_{\substack{x\in M\\ \|x\|=1}} \left\|\left(\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right)x\right\|\right)^{\frac{1}{2}}\\
&= \left\|\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)+\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\\
&\quad\times\left(\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right).
\end{aligned}$$

Namely,

$$\begin{aligned}
& s_j\left((T_1\circ T_2)+(T_3\circ T_4)\right)\\
&\leqslant \left\|\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)+\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\\
&\quad\times\left(\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right).
\end{aligned}$$

Now by replacing T_3 and T_4 by $\mathrm{i}T_3$ and $\mathrm{i}T_4$, respectively, we deduce the desired result. □

Theorem 4 may be used to obtain the following upper bound for the singular values of the real and imaginary parts of the Hadamard product.

COROLLARY 7. *Let $A,B\in\mathbb{K}(\mathbb{H})$ and let $0\leqslant t\leqslant 1$. Then for $j=1,2,\ldots$,*

$$\begin{aligned}
& s_j\left(\Re(A\circ B)\right)\\
&\leqslant \left\|\left(|\Re A|^{2(1-t)}\circ|\Re B|^{2t}\right)+\left(|\Im A|^{2(1-t)}\circ|\Im B|^{2t}\right)\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\\
&\quad\times\left(\left(|\Re A|^{2t}\circ|\Re B|^{2(1-t)}\right)+\left(|\Im A|^{2t}\circ|\Im B|^{2(1-t)}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
& s_j\left(\Im(A\circ B)\right)\\
&\leqslant \left\|\left(|\Im A|^{2(1-t)}\circ|\Re B|^{2t}\right)+\left(|\Re A|^{2(1-t)}\circ|\Im B|^{2t}\right)\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}}\\
&\quad\times\left(\left(|\Im A|^{2t}\circ|\Re B|^{2(1-t)}\right)+\left(|\Re A|^{2t}\circ|\Im B|^{2(1-t)}\right)\right).
\end{aligned}$$

Proof. Letting $T_1 = \Re A$, $T_2 = \Re B$, $T_3 = \Im A$, and $T_4 = \Im B$, in Theorem 4, we have by (2) that

$$\begin{aligned}
&s_j(\Re(A\circ B)) \\
&= s_j((\Re A\circ \Re B) - (\Im A\circ \Im B)) \\
&\leqslant \left\| \left(|\Re A|^{2(1-t)}\circ |\Re B|^{2t}\right) + \left(|\Im A|^{2(1-t)}\circ |\Im B|^{2t}\right)\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}} \\
&\quad \times \left(\left(|\Re A|^{2t}\circ |\Re B|^{2(1-t)}\right) + \left(|\Im A|^{2t}\circ |\Im B|^{2(1-t)}\right)\right).
\end{aligned}$$

Letting $T_1 = \Im A$, $T_2 = \Re B$, $T_3 = \Re A$, and $T_4 = \Im B$, in Theorem 4, we have by (3) that

$$\begin{aligned}
&s_j(\Im(A\circ B)) \\
&= s_j((\Im A\circ \Re B) + (\Re A\circ \Im B)) \\
&\leqslant \left\| \left(|\Im A|^{2(1-t)}\circ |\Re B|^{2t}\right) + \left(|\Re A|^{2(1-t)}\circ |\Im B|^{2t}\right)\right\|^{\frac{1}{2}} s_j^{\frac{1}{2}} \\
&\quad \times \left(\left(|\Im A|^{2t}\circ |\Re B|^{2(1-t)}\right) + \left(|\Re A|^{2t}\circ |\Im B|^{2(1-t)}\right)\right).
\end{aligned}$$

This completes the proof. □

COROLLARY 8. *Let $A,B\in \mathbb{K}(\mathbb{H})\cap \mathbb{S}_\theta$ for some $\theta\in \left[0,\frac{\pi}{2}\right)$. Then for $j=1,2,\ldots,$*

$$s_j(\Re(A\circ B)) \leqslant \sec^2\theta\ \|\Re A\circ \Re B\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(\Re A\circ \Re B).$$

Proof. We have by Corollary 7 and the definition of $\mathbb{S}_\theta$ that

$$\begin{aligned}
&s_j(\Re(A\circ B)) \\
&\leqslant \|(\Re A\circ \Re B) + (\Im A\circ \Im B)\|^{\frac{1}{2}} s_j^{\frac{1}{2}}((\Re A\circ \Re B) + (\Im A\circ \Im B)) \\
&\leqslant \left(1+\tan^2\theta\right)\|\Re A\circ \Re B\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(\Re A\circ \Re B) \\
&= \sec^2\theta\ \|\Re A\circ \Re B\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(\Re A\circ \Re B),
\end{aligned}$$

as required. □

This entails the following bound, which tells that the real part is sub-multiplicative on the class of sectorial matrices up to a certain constant that depends on the sector.

COROLLARY 9. *Let $A,B\in \mathbb{K}(\mathbb{H})\cap \mathbb{S}_\theta$. Then*

$$\|\Re(A\circ B)\| \leqslant \sec^2\theta\ \|\Re A\circ \Re B\|.$$

2.3. Numerical radius bounds

In this subsection, we present some bounds for the numerical radius of the Hadamard product.

THEOREM 5. *Let* $T_1, T_2, T_3, T_4 \in \mathbb{B}(\mathbb{H})$. *Then*

$$\omega(T_1T_2 \circ T_3T_4) \leqslant \sqrt{\left\| |T_2|^2 \circ |T_4|^2 \right\| \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|}.$$

and

$$\omega(T_1T_2 \circ T_3T_4) \leqslant \frac{1}{2}\left\| \left(|T_2|^2 \circ |T_4|^2\right) + \left(|T_1^*|^2 \circ |T_3^*|^2\right) \right\|.$$

Proof. For the first inequality, let $x \in \mathbb{H}$ be a unit vector. It follows from Theorem 1 that

$$\begin{aligned}
|\langle (T_1T_2 \circ T_3T_4)x, x\rangle| &\leqslant \sqrt{\left\langle \left(|T_2|^2 \circ |T_4|^2\right)x, x\right\rangle \left\langle \left(|T_1^*|^2 \circ |T_3^*|^2\right)x, x\right\rangle} \\
&\leqslant \sqrt{\left\| \left(|T_2|^2 \circ |T_4|^2\right)x \right\| \left\| \left(|T_1^*|^2 \circ |T_3^*|^2\right)x \right\|} \\
&\leqslant \sqrt{\left\| |T_2|^2 \circ |T_4|^2 \right\| \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|}.
\end{aligned}$$

That is,

$$|\langle (T_1T_2 \circ T_3T_4)x, x\rangle| \leqslant \sqrt{\left\| |T_2|^2 \circ |T_4|^2 \right\| \left\| |T_1^*|^2 \circ |T_3^*|^2 \right\|}.$$

We deduce the desired result after taking supremum over all unit vectors $x \in \mathbb{H}$.

The second inequality follows from the following observation

$$\begin{aligned}
|\langle (T_1T_2 \circ T_3T_4)x, x\rangle| &\leqslant \sqrt{\left\langle \left(|T_2|^2 \circ |T_4|^2\right)x, x\right\rangle \left\langle \left(|T_1^*|^2 \circ |T_3^*|^2\right)x, x\right\rangle} \\
&\leqslant \frac{1}{2}\left(\left\langle \left(|T_2|^2 \circ |T_4|^2\right)x, x\right\rangle + \left\langle \left(|T_1^*|^2 \circ |T_3^*|^2\right)x, x\right\rangle \right) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{2}\left\langle \left(\left(|T_2|^2 \circ |T_4|^2\right) + \left(|T_1^*|^2 \circ |T_3^*|^2\right) \right)x, x\right\rangle \\
&\leqslant \frac{1}{2}\left\| \left(|T_2|^2 \circ |T_4|^2\right) + \left(|T_1^*|^2 \circ |T_3^*|^2\right) \right\|.
\end{aligned}$$

That is,

$$|\langle (T_1T_2 \circ T_3T_4)x, x\rangle| \leqslant \frac{1}{2}\left\| \left(|T_2|^2 \circ |T_4|^2\right) + \left(|T_1^*|^2 \circ |T_3^*|^2\right) \right\|.$$

We conclude the desired result after taking supremum over all unit vectors $x \in \mathbb{H}$. □

This entails the following simple bound for the Hadamard product of two operators.

COROLLARY 10. *Let* $A, B \in \mathbb{B}(\mathbb{H})$ *and let* $0 \leqslant t \leqslant 1$. *Then*

$$\omega(A \circ B) \leqslant \sqrt{\left\| |A|^{2t} \circ |B|^{2(1-t)} \right\| \left\| |A^*|^{2(1-t)} \circ |B^*|^{2t} \right\|}$$

and

$$\omega(A \circ B) \leqslant \frac{1}{2} \left\| |A|^{2t} \circ |B|^{2(1-t)} + |A^*|^{2(1-t)} \circ |B^*|^{2t} \right\|.$$

Proof. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B, respectively. Letting $T_1 = U|A|^{1-t}$, $T_2 = |A|^t$, $T_3 = V|B|^t$, and $T_4 = |B|^{1-t}$, in Theorem 5, we deduce the desired result. □

THEOREM 6. *Let* $T_1, T_2, T_3, T_4 \in \mathbb{B}(\mathbb{H})$ *and let* $0 \leqslant t \leqslant 1$. *Then*

$$\begin{aligned} &\omega((T_1 \circ T_2) + (T_3 \circ T_4)) \\ &\leqslant \frac{1}{2} \left\| \left(|T_1|^{2t} \circ |T_2|^{2(1-t)}\right) + \left(|T_3|^{2t} \circ |T_4|^{2(1-t)}\right) + \left(|T_1^*|^{2(1-t)} \circ |T_2^*|^{2t}\right) + \left(|T_3^*|^{2(1-t)} \circ |T_4^*|^{2t}\right) \right\|. \end{aligned}$$

Proof. It follows from (11) that

$$\begin{aligned} &|\langle ((T_1 \circ T_2) + (T_3 \circ T_4))x, x \rangle| \\ &\leqslant \sqrt{\left\langle \left(\left(|T_1|^{2t} \circ |T_2|^{2(1-t)}\right) + \left(|T_3|^{2t} \circ |T_4|^{2(1-t)}\right) \right) x, x \right\rangle} \\ &\quad \times \sqrt{\left\langle \left(\left(|T_1^*|^{2(1-t)} \circ |T_2^*|^{2t}\right) + \left(|T_3^*|^{2(1-t)} \circ |T_4^*|^{2t}\right) \right) x, x \right\rangle} \\ &\leqslant \frac{1}{2} \left\langle \left(\left(|T_1|^{2t} \circ |T_2|^{2(1-t)}\right) + \left(|T_3|^{2t} \circ |T_4|^{2(1-t)}\right) \right. \right. \\ &\quad \left. \left. + \left(|T_1^*|^{2(1-t)} \circ |T_2^*|^{2t}\right) + \left(|T_3^*|^{2(1-t)} \circ |T_4^*|^{2t}\right) \right) x, x \right\rangle \\ &\leqslant \frac{1}{2} \left\| \left(|T_1|^{2t} \circ |T_2|^{2(1-t)}\right) + \left(|T_3|^{2t} \circ |T_4|^{2(1-t)}\right) \right. \\ &\quad \left. + \left(|T_1^*|^{2(1-t)} \circ |T_2^*|^{2t}\right) + \left(|T_3^*|^{2(1-t)} \circ |T_4^*|^{2t}\right) \right\|. \end{aligned}$$

Thus, we have shown that

$$\begin{aligned} &|\langle ((T_1 \circ T_2) + (T_3 \circ T_4))x, x \rangle| \\ &\leqslant \frac{1}{2} \left\| \left(|T_1|^{2t} \circ |T_2|^{2(1-t)}\right) + \left(|T_3|^{2t} \circ |T_4|^{2(1-t)}\right) \right. \\ &\quad \left. + \left(|T_1^*|^{2(1-t)} \circ |T_2^*|^{2t}\right) + \left(|T_3^*|^{2(1-t)} \circ |T_4^*|^{2t}\right) \right\|, \end{aligned}$$

which implies the expected result after taking supremum over all unit vectors $x \in \mathbb{H}$. □

COROLLARY 11. *Let* $A,B \in \mathbb{B}(\mathbb{H})$ *and let* $0 \leqslant t \leqslant 1$. *Then*

$$\|\Re(A\circ B)\| \leqslant \frac{1}{2}\Big\| \left(|\Re A|^{2t}\circ|\Re B|^{2(1-t)}\right)+\left(|\Im A|^{2t}\circ|\Im B|^{2(1-t)}\right) + \left(|\Re A|^{2(1-t)}\circ|\Re B|^{2t}\right)+\left(|\Im A|^{2(1-t)}\circ|\Im B|^{2t}\right)\Big\|.$$

In particular,

$$\|\Re(A\circ B)\| \leqslant \|(|\Re A|\circ|\Re B|)+(|\Im A|\circ|\Im B|)\|.$$

Proof. Letting $T_1=\Re A$, $T_2=\Re B$, $T_3=\Im A$, and $T_4=\Im B$, in Theorem 6, we have by (2) that

$$\begin{aligned}\|\Re(A\circ B)\| &= \omega(\Re(A\circ B))\\ &= \omega((\Re A\circ\Re B)+(\Im A\circ\Im B))\\ &\leqslant \frac{1}{2}\Big\| \left(|\Re A|^{2t}\circ|\Re B|^{2(1-t)}\right)+\left(|\Im A|^{2t}\circ|\Im B|^{2(1-t)}\right)\\ &\quad + \left(|\Re A|^{2(1-t)}\circ|\Re B|^{2t}\right)+\left(|\Im A|^{2(1-t)}\circ|\Im B|^{2t}\right)\Big\|,\end{aligned}$$

as required. □

We have seen in Corollary 9 how the real part can be sub-multiplicative over the Hadamard product, up to a certain constant, when dealing with sectorial matrices. In the following, we show a stronger result for the class of accretive-dissipative operators.

COROLLARY 12. *Let* $A,B \in \mathbb{B}(\mathbb{H})$ *be two accretive-dissipative operators. Then*

$$\|\Re(A\circ B)\| \leqslant \|\Re A\circ\Re B\|.$$

We conclude this work by presenting upper bounds for the numerical and spectral radius of operator matrices involving the Hadamard product.

THEOREM 7. *Let* $T_1,T_2,T_3,T_4 \in \mathbb{B}(\mathbb{H})$ *and let* $0 \leqslant t \leqslant 1$. *Then*

$$\omega\left(\begin{bmatrix} O & T_1\circ T_2\\ T_3^*\circ T_4^* & O\end{bmatrix}\right) \leqslant \frac{1}{2}\sqrt{\left\|\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)+\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)\right\|} \times\sqrt{\left\|\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right\|}.$$

Proof. It follows from Theorem 4 that

$$\|(T_1\circ T_2)+(T_3\circ T_4)\| \leqslant \sqrt{\left\|\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)+\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)\right\|} \times\sqrt{\left\|\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right\|}.$$

If we replace T_3 by $e^{i\theta}T_3$, we infer that

$$\left\|(T_1\circ T_2)+e^{i\theta}(T_3\circ T_4)\right\|\leqslant\sqrt{\left\|\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)+\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)\right\|}\times\sqrt{\left\|\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right\|}.$$

Consequently,

$$\begin{aligned}\omega\left(\begin{bmatrix}O & T_1\circ T_2\\ T_3^*\circ T_4^* & O\end{bmatrix}\right)&=\omega\left(\begin{bmatrix}O & T_1\circ T_2\\ (T_3\circ T_4)^* & O\end{bmatrix}\right)\\&=\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|(T_1\circ T_2)+e^{i\theta}(T_3\circ T_4)\right\|\\&\leqslant\frac{1}{2}\sqrt{\left\|\left(|T_1^*|^{2(1-t)}\circ|T_2^*|^{2t}\right)+\left(|T_3^*|^{2(1-t)}\circ|T_4^*|^{2t}\right)\right\|}\\&\quad\times\sqrt{\left\|\left(|T_1|^{2t}\circ|T_2|^{2(1-t)}\right)+\left(|T_3|^{2t}\circ|T_4|^{2(1-t)}\right)\right\|},\end{aligned}$$

where we have used Lemma 3 to obtain the last equality. This completes the proof. □

THEOREM 8. *For $j=1,2$, let $A_{ij},B_{ij}\in\mathbb{B}(\mathbb{H})$, and let $\mathbb{A}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$ and $\mathbb{B}=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}$. Then*

$$r(\mathbb{A}\circ\mathbb{B})\leqslant\frac{1}{2}\Big(\omega(A_{11}\circ B_{11})+\omega(A_{22}\circ B_{22})+\sqrt{(\omega(A_{11}\circ B_{11})-\omega(A_{22}\circ B_{22}))^2+4\|A_{12}\circ B_{12}\|\|A_{21}\circ B_{21}\|}\Big),$$

where $r(\cdot)$ denotes the spectral radius.

Proof. Employing [1, Lemma 2.1], one can write

$$\begin{aligned}r(\mathbb{A}\circ\mathbb{B})&=r\left(\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\circ\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}\right)\\&=r\left(\begin{bmatrix}A_{11}\circ B_{11} & A_{12}\circ B_{12}\\ A_{21}\circ B_{21} & A_{22}\circ B_{22}\end{bmatrix}\right)\\&\leqslant\omega\left(\begin{bmatrix}A_{11}\circ B_{11} & A_{12}\circ B_{12}\\ A_{21}\circ B_{21} & A_{22}\circ B_{22}\end{bmatrix}\right)\\&\leqslant\frac{1}{2}\Big(\omega(A_{11}\circ B_{11})+\omega(A_{22}\circ B_{22})\\&\quad+\sqrt{(\omega(A_{11}\circ B_{11})-\omega(A_{22}\circ B_{22}))^2+4\|A_{12}\circ B_{12}\|\|A_{21}\circ B_{21}\|}\Big),\end{aligned}$$

as required. □

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REFERENCES

[1] A. ABU OMAR F. KITTANEH, *A numerical radius inequality involving the generalized Aluthge transform*, Studia Math. **216**, 1 (2013), 69–75.
[2] H. ALBADAWI, *Singular value and arithmetic-geometric mean inequalities for operators*, Ann. Funct. Anal. **3**, 1 (2012), 10–18.
[3] A. AL-NATOOR, O. HIRZALLAH AND F. KITTANEH, *Singular value inequalities for convex functions of positive semidefinite matrices*, Ann. Funct. Anal. **14**, 7 (2023).
[4] A. ALUTHGE, *On p-hyponormal operators for $0 < p < 1$*, Integral Equ. Oper. Theory **13** (1990), 307–315.
[5] T. ANDO, *Concavity of certain maps on positive definite matrices and applications to Hadamard prodcts*, Linear Algebra Appl. **26** (1979), 203–241.
[6] M. BAKHERAD, M. S. MOSLEHIAN, *Reverses and variations of Heinz inequality*, Linear Multilinear Algebra **63**, 10 (2015), 1972–1980.
[7] R. BHATIA, *Matrix Analysis*, Graduate Texts in Mathematics, Springer, New York, 1997.
[8] J.-C. BOURIN AND E.-Y. LEE, *On the Russo-Dye theorem for positive linear maps*, Linear Algebra Appl. **571** (2019), 92–102.
[9] J.-C. BOURIN AND E.-Y. LEE, *Diagonal and off-diagonal blocks of positive definite partitioned matrices*, Linear Algebra Appl. **684** (2024), 87–100.
[10] C. CONDE, *Young type inequalities for positive operators*, Ann. Funct. Anal. **4**, 2 (2013), 144–152.
[11] J. FUJII, *The Marcus-Khan theorem for Hilbert space operators*, Math. Jpn. **41** (1995), 531–535.
[12] I. GOHBERG, S. GOLDBERG, AND M. KAASHOEK, *Basic Classes of Linear Operators*, Springer Basel, 2003.
[13] I. C. GOHBERG AND M. G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monogr., vol. 18, Am. Math. Soc. Providence, RI, 1969.
[14] O. HIRZALLAH, F. KITTANEH AND K. SHEBRAWI, *Numerical radius inequalities for certain 2×2 operator matrices*, Integral Equ. Oper. Theory. **71** (2011), 129–147.
[15] T. KATO, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.
[16] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24**, 2 (1988), 283–293.
[17] F. KITTANEH, *On the convexity of the Heinz mean*, Integr. Equ. Oper. Theory **68** (2010) 519–527.
[18] F. KITTANEH, H. R. MORADI AND M. SABABHEH, *Mean inequalities for the numerical radius*, Numer. Funct. Anal. Optim. **44**, 14 (2023), 1523–1537.
[19] F. KITTANEH, M. S. MOSLEHIAN AND M. SABABHEH, *Quadratic interpolation of the Heinz means*, Math. Inequal. Appl. **21**, 3 (2018), 739–757.
[20] J. S. MATHARU, J. S. AUJLA, *Hadamard product versions of the Chebyshev and Kantorovich inequalities*, J. Inequal. Pure Appl. Math. **10**, 2 (2009), Article 51.
[21] H. R. MORADI, W. AUDEH AND M. SABABHEH, *Singular values inequalities via matrix monotone functions*, Anal. Math. Phys. **13**, 71 (2023).

[22] H. R. MORADI AND M. SABABHEH, *New estimates for the numerical radius*, Filomat **35**, 14 (2021), 4957–4962.
[23] M. SABABHEH, *Integral inequalities of the Heinz means as convex functions*, J. Math. Inequal. **10**, 2 (2016), 313–325.
[24] M. SABABHEH AND H. R. MORADI, *New orders among Hilbert space operators*, Math. Inequal. Appl. **26**, 2 (2023), 415–432.
[25] M. SABABHEH AND H. R. MORADI, *Numerical radius of Kronecker product of matrices*, J. Appl. Anal. Comput. **13**, 5 (2023), 2943–2954.
[26] J. ISSAI SCHUR, *Bemerkungen zur Theoric der beschränktcn Bilinearformen mit unendlich vielcn Veränderlichen*, J. Reine Angew. Math. **140** (1911), 1–28.
[27] B. SIMON, *Trace Ideals and Their Applications*, London Math. Soc. Cambridge University press, 1979.
[28] G. STYAN, *Hadamard product and multivariate statistical analysis*, Linear Algebra Appl. **6** (1973), 217–240.
[29] G. VISICK, *A quantitative version of the observation that the Hadamard product is a principal submatrix of the Kronecker product*, Linear Algebra Appl. **304**, 1–3 (2000), 45–68.
[30] L. ZOU, *An arithmetic-geometric mean inequality for singular values and its applications*, Linear Algebra Appl. **528** (2017), 25–32.

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