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# ZYGMUND–TYPE INTEGRAL INEQUALITIES FOR POLYNOMIALS NOT VANISHING IN A DISC

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*Abstract.* The study of inequalities in various norms for polynomials and their derivatives in the plane is fundamental to geometric function theory. This paper focuses on Zygmund-type norm estimates for polynomials that do not vanish in a positive-radius disc. We establish integral norm estimates for the growth of higher-order derivatives of a polynomial in the plane, including extensions of several important inequalities of approximation theory and related inequalities established by Jain [*Turk. J. Math.* **31** (2007), 89–94].

## 1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree $n$, and let $P'(z)$ be its derivative. For any polynomial $P \in \mathcal{P}_n$, we set $M(P,R) := \max\limits_{|z|=R} |P(z)|$, the uniform norm of $P$ on the disc $|z| = R$, $R > 0$, and $m := \min\limits_{|z|=k} |P(z)|$.

The integral norm of $P$ over the unit disc $|z| = 1$ is expressed as

$$||P||_\gamma := \left( \frac{1}{2\pi} \int_0^{2\pi} |P(\mathrm{e}^{\mathrm{i}\theta})|^\gamma \, \mathrm{d}\theta \right)^{1/\gamma},$$

for any $\gamma > 0$ and $0 \leqslant \theta < 2\pi$. According to a well-known result in analysis [20], it holds that:

$$\lim_{\gamma \to \infty} \left( \frac{1}{2\pi} \int_0^{2\pi} |P(\mathrm{e}^{\mathrm{i}\theta})|^\gamma \, \mathrm{d}\theta \right)^{1/\gamma} := ||P||_\infty,$$

where

$$||P||_\infty = \max_{|z|=1} |P(z)|.$$

The study of comparison inequalities that relate the norm of polynomials on a disc is a well-known topic in analysis, particularly due to its numerous applications in geometric function theory. Bernstein-type inequalities relating norm estimates, which generalize

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classical estimates for polynomials, are the most familiar in the study (see, for example, [1], [8], [9], [14], [16]). These inequalities are important in the literature for proving the inverse theorems in approximation theory and, of course, have their own intrinsic interest. These inequalities exist for various norms and for many classes of functions, such as polynomials with various constraints and on various regions of the complex plane. If $P \in \mathcal{P}_n$, then

$$||P'||_\infty \leqslant n||P||_\infty \tag{1.1}$$

and for every $\gamma \geqslant 1$,

$$||P'||_\gamma \leqslant n||P||_\gamma. \tag{1.2}$$

Inequality (1.1) is a classical result by Bernstein [5], while as its integral norm analogue in the form of inequality (1.2) was established by Zygmund in [21] for all trigonometric polynomials of degree $n$, not just for those of the form $P(\mathrm{e}^{\mathrm{i}\theta})$. The inequalities (1.1) and (1.2) demonstrate how quickly a polynomial of degree $n$ or its derivative can change, which is crucial in approximation theory. Since the establishment of these inequalities, a large number of papers on polynomial approximation theory have been published. These works rapidly advanced the theory of extremal problems of analytic functions, particularly polynomials, as well as its implications in other fields. In [4], Arestov showed that the inequality (1.2) remains true for $0 < \gamma < 1$ as well. Taking the limit as $\gamma \to \infty$ in (1.2) yields inequality (1.1). Inequalities (1.1) and (1.2) can be strengthened for polynomials without zeros inside the unit disc $|z| = 1$. Specifically, if $P(z) \neq 0$ in $|z| < 1$, then inequalities (1.1) and (1.2) can be replaced by:

$$||P'||_\infty \leqslant \frac{n}{2}||P||_\infty \tag{1.3}$$

and

$$||P'||_\gamma \leqslant \frac{n}{||1+z||_\gamma}||P||_\gamma. \tag{1.4}$$

Erdős first proposed inequality (1.3), which was later confirmed by Lax in [15], whereas inequality (1.4) was found out by De-Bruijn in [7] for $\gamma \geqslant 1$. Rahman and Schmeisser in [19] extended the validity of inequality (1.4) to the range $0 < \gamma < 1$. Taking the limit as $\gamma \to \infty$ in (1.4) yields inequality (1.3).

As a generalization of (1.3) Malik in [17] proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then

$$||P'||_\infty \leqslant \frac{n}{1+k}||P||_\infty,$$

whereas under the same hypothesis, Govil and Rahman in [19] extended this inequality by showing that

$$||P'||_\gamma \leqslant \frac{n}{||k+z||_\gamma}||P||_\gamma, \quad \gamma \geqslant 1.$$

It was shown by Gardner and Weems in [10] that the last inequality also holds for $0 < \gamma < 1$.

Govil and Rahman in [11] generalized Malik's inequality [17] in a different direction and proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$ and $1 \leqslant s < n$, then

$$||P^{(s)}||_\infty \leqslant \frac{n(n-1)\cdots(n-s+1)}{1+k^s}||P||_\infty. \tag{1.5}$$

As an extension of the inequality (1.5), Aziz and Shah in [3] proved that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then for $1 \leqslant s < n$ and for each $\gamma > 0$,

$$||P^{(s)}||_\gamma \leqslant \frac{n(n-1)\cdots(n-s+1)}{||k^s+z||_\gamma}||P||_\gamma. \tag{1.6}$$

Under the same hypothesis, Aziz and Rather in [2] refined inequality (1.6) using the coefficients of the polynomial $P(z)$ in the form of the following inequality:

$$||P^{(s)}||_\gamma \leqslant \frac{n(n-1)\cdots(n-s+1)}{||\delta(k,s)+z||_\gamma}||P||_\gamma, \tag{1.7}$$

where $\delta(k,s)$ is defined as

$$\delta(k,s) = \frac{C(n,s)|a_0|k^{s+1}+|a_s|k^{2s}}{C(n,s)|a_0|+|a_s|k^{s+1}}, \quad C(n,s) = \binom{n}{s}. \tag{1.8}$$

As a consequence of the maximum modulus principle, the following inequality holds for any polynomial $P \in \mathcal{P}_n$ and for each $R \geqslant 1$:

$$M(P,R) \leqslant R^n||P||_\infty, \tag{1.9}$$

with equality holds for $P(z) = \lambda z^n$.

Hardy in [12] established the following integral mean extension of (1.9), for each $R \geqslant 1$, we have

$$||P(Rz)||_\gamma \leqslant R^n||P||_\gamma, \quad \gamma > 0. \tag{1.10}$$

As a generalization of the aforementioned inequality of Malik [17], Dewan and Bidkham in [6] proved that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then for $0 < r \leqslant R \leqslant k$,

$$M(P',R) \leqslant \frac{n(R+k)^{n-1}}{(r+k)^n}M(P,r). \tag{1.11}$$

Jain in [13] established the following result for $0 \leqslant s < n$ and $1 \leqslant R \leqslant k$ when generalizing (1.11) (for $r = 1$) to the $s^{\text{th}}$ derivative of the polynomial under the same assumptions:

$$M(P^{(s)},R) \leqslant \frac{1}{R^s+k^s}\left[\left\{\frac{\mathrm{d}^s}{\mathrm{d}x^s}(1+x^n)\right\}_{x=1}\right]\left(\frac{R+k}{1+k}\right)^n||P||_\infty. \tag{1.12}$$

Mir in [18] extended and refined (1.12) by involving the coefficients of the polynomial $P(z)$. He proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 \leqslant s < n$ and $0 < r \leqslant R \leqslant k$,

$$\begin{aligned} M(P^{(s)},R) \leqslant & \frac{C(n,s)|a_0|R+|a_s|k^{s+1}}{C(n,s)\left(k^{s+1}+R^{s+1}\right)|a_0|+\left(k^{s+1}R^s+k^{2s}R\right)|a_s|} \\ & \times \left[\left\{\frac{\mathrm{d}^s}{\mathrm{d}x^s}(1+x^n)\right\}_{x=1}\right]\left(\frac{R+k}{r+k}\right)^n M(p,r). \end{aligned} \tag{1.13}$$

In the same paper [18] and under the same hypothesis, Mir improved (1.13) by deriving the following inequality applicable for $1 \leqslant s < n$ and $0 < r \leqslant R \leqslant k$:

$$M(P^{(s)}, R) \leqslant \frac{C(n,s)(|a_0| - m)R + |a_s|k^{s+1}}{C(n,s)\left(k^{s+1} + R^{s+1}\right)(|a_0| - m) + \left(k^{s+1}R^s + k^{2s}R\right)|a_s|} \times \left[\left\{\frac{\mathrm{d}^s}{\mathrm{d}x^s}(1+x^n)\right\}_{x=1}\right]\left(\frac{R+k}{r+k}\right)^n (M(p,r) - m). \tag{1.14}$$

Given the importance of the problem in approximation theory of determining the growth of analytic functions with respect to various norms, particularly polynomials, it is natural to investigate integral norm estimates for these functions. This paper establishes various $L_\gamma$-norm inequalities of Zygmund-type for polynomials not vanishing in a positive radius disc. The obtained results will include various generalizations and refinements of the aforementioned inequalities, as well as other related inequalities.

## 2. Main results

Without loss of generality, we use the following notations in our results:

$$\left.\begin{aligned} S_0 &= \frac{C(n,s)|a_0|k^{s+1} + |a_s|k^{2s}R}{C(n,s)|a_0|R^{s+1} + |a_s|k^{s+1}R^s}, \quad S_1 = \frac{C(n,s)|a_0|k^{s+1} + |a_s|k^{2s}R}{C(n,s)|a_0|R + |a_s|k^{s+1}} \\ S_2 &= \frac{C(n,s)(|a_0| - |\delta|m)k^{s+1} + |a_s|k^{2s}R}{C(n,s)(|a_0| - |\delta|m)R + |a_s|k^{s+1}}, \end{aligned}\right\} \tag{2.1}$$

as well as

$$F(r) \equiv F(r,n,s) = \left\{\frac{\mathrm{d}^s}{\mathrm{d}x^s}(1+x^n)\right\}_{x=r} = \begin{cases} 1 + r^n, & s = 0, \\ r^{n-s}s!\dbinom{n}{s}, & 1 \leqslant s \leqslant n. \end{cases} \tag{2.2}$$

Note that $S_0 = S_1/R^s$ and for $\delta = 0$, $S_1 = S_2$.

We begin with a generalization of (1.7), which further generalizes and refines related results.

THEOREM 1. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then for $1 \leqslant R \leqslant k$, for each $\gamma > 0$ and $1 \leqslant s < n$,*

$$||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(R)}{||S_0 + z||_\gamma} ||P||_\gamma, \tag{2.3}$$

*where $S_0$ and $F$ are defined in (2.1) and (2.2), respectively.*

REMARK 1. For $R = 1$ in (2.3), we get (1.7).

The following result immediately follows by taking the limit $\gamma \to \infty$ in (2.3).

COROLLARY 1. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then for $1 \leqslant R \leqslant k$ and $1 \leqslant s < n$,*

$$M(P^{(s)}, R) \leqslant \frac{F(R)}{S_0 + 1} ||P||_\infty. \tag{2.4}$$

If we take $R = 1$ in (2.4), we get the following refinement of (1.5) and note that it was also established by Aziz and Rather [2].

COROLLARY 2. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then for $1 \leqslant s < n$,*

$$||P^{(s)}||_\infty \leqslant \frac{F(1)}{\delta(k,s) + 1} ||P||_\infty,$$

*where $\delta(k,s)$ is defined as in* (1.8).

REMARK 2. If $P(z) \neq 0$ in $|z| < k$, $k > 0$, the polynomial $P(Rz) \neq 0$ in $|z| \leqslant k/R$, $k/R \geqslant 1$, $0 < R \leqslant k$. Hence applying inequality (3.2) of Lemma 1 (stated in next section) to $P(Rz)$, we get for $0 \leqslant s < n$,

$$\frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| R^s \left( \frac{k}{R} \right)^s \leqslant 1,$$

i.e.,

$$\left| \frac{a_s}{a_0} \right| k^s \leqslant C(n,s). \tag{2.5}$$

Also $k \geqslant R$, therefore

$$\left| \frac{a_s}{a_0} \right| k^s (k - R) \leqslant (k - R) C(n,s),$$

which implies

$$C(n,s) R + \left| \frac{a_s}{a_0} \right| k^{s+1} \leqslant R \left| \frac{a_s}{a_0} \right| k^s + C(n,s) k.$$

This gives $S_1 \geqslant k^s$, i.e., $S_1/R^s \geqslant (k/R)^s$, which gives

$$\frac{1}{1 + S_1/R^s} \leqslant \frac{1}{1 + (k/R)^s}.$$

As $k/R \geqslant 1$, therefore the previous inequality implies for every $\gamma > 0$,

$$\frac{1}{\left\| z + \frac{S_1}{R^s} \right\|_\gamma} \leqslant \frac{1}{\left\| z + \left( \frac{k}{R} \right)^s \right\|_\gamma}.$$

Equivalently,

$$\frac{1}{||R^s z + S_1||_\gamma} \leqslant \frac{1}{||R^s z + k^s||_\gamma}, \quad \gamma > 0. \tag{2.6}$$

Inequality (2.6) is also equivalent to

$$\frac{1}{||S_0+z||_\gamma} \leqslant \frac{R^s}{||R^s z+k^s||_\gamma}, \quad \gamma>0, \tag{2.7}$$

where $S_0$ and $S_1$ are as defined in (2.1).

Using (2.7) in (2.3), we get the following extension of (1.6).

COROLLARY 3. *If $P\in\mathcal{P}_n$ and $P(z)\neq 0$ in $|z|<k$, $k\geqslant 1$, then for $1\leqslant R\leqslant k$, for each $\gamma>0$ and $1\leqslant s<n$,*

$$||P^{(s)}(Rz)||_\gamma \leqslant \frac{R^s F(1)}{||R^s z+k^s||_\gamma}\,||P||_\gamma. \tag{2.8}$$

REMARK 3. It is evident that (1.6) is a special case of (2.8), when $R=1$.

Making $\gamma\to\infty$ in (2.8), we get the following generalization of (1.5).

COROLLARY 4. *If $P\in\mathcal{P}_n$ and $P(z)\neq 0$ in $|z|<k$, $k\geqslant 1$, then for $1\leqslant R\leqslant k$ and $1\leqslant s<n$,*

$$M(P^{(s)},R) \leqslant \frac{R^s F(1)}{R^s+k^s}\,||P||_\infty. \tag{2.9}$$

REMARK 4. By taking $R=1$ in (2.9), we get (1.5).

Next, we establish the following integral analogue of (1.13), and it also generalizes the other related results as well.

THEOREM 2. *If $P\in\mathcal{P}_n$ and $P(z)\neq 0$ in $|z|<k$, $k>0$, then for $0\leqslant s<n$, $0<r\leqslant R\leqslant k$ and for each $\gamma>0$,*

$$||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(1)}{||R^s z+S_1||_\gamma}\left(\frac{R+k}{r+k}\right)^n ||P(rz)||_\gamma. \tag{2.10}$$

*where $S_1$ and $F$ are defined in (2.1) and (2.2), respectively.*

REMARK 5. Making the limit as $\gamma\to\infty$ in (2.10), we get (1.13).

Using (2.6) of Remark 2, the following result is a direct consequence of Theorem 2. It generalizes (1.11) and reflects the integral mean extension of a result due to Mir ([18], Corollary 1.3).

COROLLARY 5. *If $P\in\mathcal{P}_n$ and $P(z)\neq 0$ in $|z|<k$, $k>0$, then for $0\leqslant s<n$, $0<r\leqslant R\leqslant k$, and for each $\gamma>0$,*

$$||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(1)}{||R^s z+k^s||_\gamma}\left(\frac{R+k}{r+k}\right)^n ||p(rz)||_\gamma. \tag{2.11}$$

REMARK 6. By taking the limit as $\gamma \to \infty$ in (2.11), we get a result of Mir [18, Corollary 1.3]. In addition to this, if we take $s=1$, it reduces to (1.11).

Taking $s=0$ in (2.11), we get the following result.

COROLLARY 6. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z|<k$, $k>0$, then for $0<r \leqslant R \leqslant k$ and for each $\gamma>0$,*

$$||P(Rz)||_\gamma \leqslant \frac{2}{||1+z||_\gamma}\left(\frac{R+k}{r+k}\right)^n ||P(rz)||_\gamma.$$

Finally, we prove the following theorem which gives a generalization of Theorem 2 and also represents the integral analogue of (1.14).

THEOREM 3. *If $P \in P_n$ and $P(z) \neq 0$ in $|z|<k$, $k>0$, then for $1 \leqslant s<n$, $0<r \leqslant R \leqslant k$, for every complex number $\delta$ with $|\delta| \leqslant 1$ and for each $\gamma>0$,*

$$||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(1)}{||R^s z+S_2||_\gamma}\left(\frac{R+k}{r+k}\right)^n ||P(rz)-\delta m||_\gamma, \tag{2.12}$$

*where $S_2$ and $F$ are defined in* (2.1) *and* (2.2)*, respectively.*

Letting $\gamma \to \infty$ in (2.12), we get the following result:

COROLLARY 7. *If $P \in P_n$ and $P(z) \neq 0$ in $|z|<k$, $k>0$, then for $1 \leqslant s<n$, $0<r \leqslant R \leqslant k$, and for every complex number $\delta$ with $|\delta| \leqslant 1$,*

$$M(P^{(s)},R) \leqslant \frac{F(1)}{R^s+S_2}\left(\frac{R+k}{r+k}\right)^n ||P(rz)-\delta m||_\infty. \tag{2.13}$$

REMARK 7. Suppose $z_0$ on $|z|=1$ be such that

$$\max_{|z|=1}|P(rz)-\delta m| = |P(rz_0)-\delta m|. \tag{2.14}$$

We can write

$$\begin{aligned}|P(rz_0)-\delta m| &= \left||P(rz_0)|e^{i\theta_0} - |\delta|\mathrm{e}^{\mathrm{i}\theta} m\right| \\ &= \left||P(rz_0)| - |\delta|e^{i(\theta-\theta_0)}m\right|.\end{aligned}$$

On choosing the argument of $\delta$ as $\theta=\theta_0$ gives

$$|P(rz_0)-\delta m| = ||P(rz_0)|-|\delta|m|. \tag{2.15}$$

Since $|\delta| \leqslant 1$, therefore

$$|P(rz_0)| \geqslant m \geqslant |\delta|m,$$

which on using in (2.15) gives

$$|P(rz_0) - \delta m| = |P(rz_0)| - |\delta| m. \tag{2.16}$$

From (2.14) and (2.16), we get on using the fact $|P(rz_0)| \leqslant \max_{|z|=1} |P(rz)|$, that

$$\max_{|z|=1} |P(rz) - \delta m| \leqslant \max_{|z|=1} |P(rz)| - |\delta| m = M(P,r) - |\delta| m. \tag{2.17}$$

Combining (2.17) with (2.13), we arrive at the following result:

COROLLARY 8. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k > 0$, then for $1 \leqslant s < n$, $0 < r \leqslant R \leqslant k$, and for every complex number $\delta$ with $|\delta| \leqslant 1$,*

$$M(P^{(s)}, R) \leqslant \frac{F(1)}{R^s + S_2} \left( \frac{R+k}{r+k} \right)^n (M(P,r) - |\delta| m). \tag{2.18}$$

REMARK 8. Letting $|\delta| \to 1$ in (2.18), we get (1.14) due to Mir [18].

## 3. Auxiliary results

To prove our results, we need the following lemmas.

LEMMA 1. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, then for $1 \leqslant s < n$ and $|z| = 1$,*

$$\delta(k,s) |P^{(s)}(z)| \leqslant |Q^{(s)}(z)|, \tag{3.1}$$

*and*

$$\frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^s \leqslant 1, \tag{3.2}$$

*where*

$$\delta(k,s) = \frac{C(n,s)|a_0| k^{s+1} + |a_s| k^{2s}}{C(n,s)|a_0| + |a_s| k^{s+1}}.$$

The above lemma due to Aziz and Rather [2]. It is easy to see that (3.1) and (3.2) holds for $s = 0$ as well. In the same paper, they also proved the following result.

LEMMA 2. *If $P \in \mathcal{P}_n$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, then for each $\alpha$, $0 \leqslant \alpha < 2\pi$ and $\gamma > 0$,*

$$\int_0^{2\pi} \int_0^{2\pi} \left| Q'(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{e}^{i\alpha} P'(\mathrm{e}^{\mathrm{i}\theta}) \right|^\gamma \mathrm{d}\theta \, \mathrm{d}\alpha \leqslant 2\pi n^\gamma \int_0^{2\pi} |P'(\mathrm{e}^{\mathrm{i}\theta})|^\gamma \mathrm{d}\theta. \tag{3.3}$$

Using inequality (1.7) and the fact that $||1+z||_\gamma \leqslant 2$ for $\gamma > 0$, we easily obtain the following:

LEMMA 3. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then for $0 \leqslant s < n$ and for each $\gamma > 0$,*

$$||P^{(s)}||_\gamma \leqslant \frac{F(1)}{||\delta(k,s)+z||_\gamma} ||P||_\gamma.$$

LEMMA 4. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 < r \leqslant R \leqslant k$ and for each $\gamma > 0$,*

$$||P(Rz)||_\gamma \leqslant \left(\frac{R+k}{r+k}\right)^n ||P(rz)||_\gamma.$$

*Proof.* Since $P(z)$ has all its zeros in $|z| \geqslant k > 0$, we write

$$P(z) = \prod_{j=1}^{n} (z - r_j e^{i\theta_j}),$$

where $r_j \geqslant k,\ j = 1,2,\ldots,n$. Now, for $0 < r \leqslant R \leqslant k$ and $0 \leqslant \theta < 2\pi$, we have

$$\begin{aligned} \left|\frac{P(R\mathrm{e}^{\mathrm{i}\theta})}{P(r\mathrm{e}^{\mathrm{i}\theta})}\right| &= \prod_{j=1}^{n} \left|\frac{R\mathrm{e}^{\mathrm{i}\theta} - r_j \mathrm{e}^{\mathrm{i}\theta_j}}{r\mathrm{e}^{\mathrm{i}\theta} - r_j \mathrm{e}^{\mathrm{i}\theta_j}}\right| \\ &= \prod_{j=1}^{n} \left|\frac{R\mathrm{e}^{\mathrm{i}(\theta-\theta_j)} - r_j}{r\mathrm{e}^{\mathrm{i}(\theta-\theta_j)} - r_j}\right| \\ &\leqslant \prod_{j=1}^{n} \left(\frac{R+k}{r+k}\right) = \left(\frac{R+k}{r+k}\right)^n. \end{aligned}$$

Therefore, for any $\gamma > 0$, we have

$$|P(R\mathrm{e}^{\mathrm{i}\theta})|^\gamma = \left(\frac{R+k}{r+k}\right)^{n\gamma} |P(r\mathrm{e}^{\mathrm{i}\theta})|^\gamma,$$

and hence

$$||P(Rz)||_\gamma \leqslant \left(\frac{R+k}{r+k}\right)^n ||P(rz)||_\gamma. \quad \square$$

## 4. Proofs of the main results

*Proof of Theorem 1.* Let $f(z) = Q(z) + \mathrm{e}^{\mathrm{i}\alpha} P(z)$, where $Q(z) = z^n \overline{P(1/\overline{z})}$, then $F(z)$ is a polynomial of degree $n$, and we have

$$f^{(s)}(z) = Q^{(s)}(z) + \mathrm{e}^{\mathrm{i}\alpha} P^{(s)}(z),$$

which is clearly a polynomial of degree $n-s$, $1 \leqslant s < n$.

Applying inequality (1.10) to $f^{(s)}(z)$, we have for $R \geqslant 1$ and for each $\gamma > 0$,

$$\int_0^{2\pi} \left|Q^{(s)}(R\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{e}^{\mathrm{i}\alpha} P^{(s)}(R\mathrm{e}^{\mathrm{i}\theta})\right|^\gamma \mathrm{d}\theta \leqslant R^{(n-s)\gamma} \int_0^{2\pi} \left|Q^{(s)}(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{e}^{\mathrm{i}\alpha} P^{(s)}(\mathrm{e}^{\mathrm{i}\theta})\right|^\gamma \mathrm{d}\theta.$$

Applying inequality (1.2) repeatedly to the right-hand side of the above inequality (without the factor $R^{(n-s)\gamma}$), it follows for each $\gamma>0$,

$$\begin{aligned}\int_0^{2\pi}&\left|Q^{(s)}(Re^{i\theta})+e^{i\alpha}P^{(s)}(e^{i\theta})\right|^\gamma d\theta\\ &\leqslant (n-s+1)^\gamma\int_0^{2\pi}\left|Q^{(s-1)}(e^{i\theta})+e^{i\alpha}P^{(s-1)}(e^{i\theta})\right|^\gamma d\theta\\ &\vdots\\ &\leqslant (n-s+1)^\gamma(n-s+2)^\gamma\cdots(n-1)^\gamma\int_0^{2\pi}\left|Q'(e^{i\theta})+e^{i\alpha}P'(e^{i\theta})\right|^\gamma d\theta.\end{aligned}$$

In this way, we obtain

$$\begin{aligned}\int_0^{2\pi}&\left|Q^{(s)}(Re^{i\theta})+e^{i\alpha}P^{(s)}(Re^{i\theta})\right|^\gamma d\theta\\ &\leqslant R^{(n-s)\gamma}(n-s+1)^\gamma(n-s+2)^\gamma\cdots(n-1)^\gamma\int_0^{2\pi}\left|Q'(e^{i\theta})+e^{i\alpha}P'(e^{i\theta})\right|^\gamma d\theta.\end{aligned}$$

Integrating both sides of this inequality, with respect to $\alpha$ from $0$ to $2\pi$, and using Lemma 2, we get

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left|Q^{(s)}(Re^{i\theta})+e^{i\alpha}P^{(s)}(Re^{i\theta})\right|^\gamma d\theta\, d\alpha\\ &\leqslant R^{(n-s)\gamma}(n-s+1)^\gamma(n-s+2)^\gamma\cdots(n-1)^\gamma\int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta})+e^{i\alpha}P'(e^{i\theta})\right|^\gamma d\theta\, d\alpha\\ &\leqslant 2\pi R^{(n-s)\gamma}(n-s+1)^\gamma(n-s+2)^\gamma\cdots(n-1)^\gamma n^\gamma\int_0^{2\pi}\left|P(e^{i\theta})\right|^\gamma d\theta. \qquad (4.1)\end{aligned}$$

Using (3.1) of Lemma 1 for the polynomial $P(Rz)$ having no zeros in $|z|<k/R$, where $k/R\geqslant 1$, then for $1\leqslant s<n$ and $|z|=1$,

$$S_0|P^{(s)}(Rz)|\leqslant|Q^{(s)}(Rz)|\quad \text{for } |z|=1, \qquad (4.2)$$

where $S_0$ is defined in (2.1). It is easy to verify that $S_0\geqslant 1$.

Now for all real $\alpha$ and $t_1\geqslant t_2\geqslant 1$, it can be easily verified that $\left|t_1+e^{i\alpha}\right|\geqslant\left|t_2+e^{i\alpha}\right|$. Observe that for all $\gamma>0$ and $a,b\in\mathbb{C}$ such that $|a|\geqslant|b|x$, where $x\geqslant 1$, we have

$$\int_0^{2\pi}\left|a+e^{i\alpha}b\right|^\gamma d\alpha\geqslant|b|^\gamma\int_0^{2\pi}\left|x+e^{i\alpha}\right|^\gamma d\alpha. \qquad (4.3)$$

Indeed, if $b=0$, the above inequality is obvious. In case $b\neq 0$, we get

$$\begin{aligned}\int_0^{2\pi}\left|1+e^{i\alpha}\frac{a}{b}\right|^\gamma d\alpha&=\int_0^{2\pi}\left|1+e^{i\alpha}\left|\frac{a}{b}\right|\right|^\gamma d\alpha\\ &\geqslant\int_0^{2\pi}\left|\left|\frac{a}{b}\right|+e^{i\alpha}\right|^\gamma d\alpha\\ &\geqslant\int_0^{2\pi}\left|x+e^{i\alpha}\right|^\gamma d\alpha.\end{aligned}$$

If we take $a = Q^{(s)}(Re^{i\theta})$ and $b = P^{(s)}(Re^{i\theta})$, because $|a| \geqslant |b|S_0$ from (4.2), then we get from (4.3), that

$$\int_0^{2\pi} \left|Q^{(s)}(Re^{i\theta}) + e^{i\alpha}P^{(s)}(Re^{i\theta})\right|^\gamma d\alpha \geqslant |P^{(s)}(Re^{i\theta})|^\gamma \int_0^{2\pi} \left|S_0 + e^{i\alpha}\right|^\gamma d\alpha.$$

Integrating both sides of this inequality with respect to $\theta$ from $0$ to $2\pi$, and using in (4.1), we conclude that

$$\int_0^{2\pi} |P^{(s)}(Re^{i\theta})|^\gamma d\theta \int_0^{2\pi} \left|S_0 + e^{i\alpha}\right|^\gamma d\alpha \leqslant 2\pi R^{(n-s)\gamma}(n-s+1)^\gamma(n-s+2)^\gamma \cdots (n-1)^\gamma n^\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma d\theta,$$

which implies

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |P^{(s)}(Re^{i\theta})|^\gamma d\theta\right\}^{1/\gamma} \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|S_0 + e^{i\alpha}\right|^\gamma d\alpha\right\}^{1/\gamma} \leqslant R^{(n-s)}(n-s+1)(n-s+2)\cdots(n-1)n\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma d\theta\right\}^{1/\gamma},$$

i.e.,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |P^{(s)}(Re^{i\theta})|^\gamma d\theta\right\}^{1/\gamma} \leqslant \frac{R^{(n-s)}(n-s+1)(n-s+2)\cdots(n-1)n\,||P||_\gamma}{||S_0 + z||_\gamma},$$

which is equivalent to (2.3) and this completes the proof of Theorem 1.

*Proof of Theorem* 2. Recall that $P(z) \neq 0$ in $|z| < k$, $k > 0$, therefore $P(Rz) \neq 0$ in $|z| < k/R$, $k/R \geqslant 1$. On applying Lemma 3 to $P(Rz)$, we have for $0 \leqslant s < n$,

$$R^s||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(1)}{||S_0 + z||_\gamma}||P(Rz)||_\gamma,$$

where $S_0$ and $F$ are defined in (2.1) and (2.2), respectively. Noting that $S_0 = S_1/R^s$, therefore the previous inequality implies

$$||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(1)}{||R^s z + S_1||_\gamma}||P(Rz)||_\gamma,$$

which in conjunction with Lemma 4 gives

$$||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(1)}{||R^s z + S_1||_\gamma}\left(\frac{R+k}{r+k}\right)^n ||P(rz)||_\gamma, \tag{4.4}$$

where $S_1$ is defined in defined in (2.1) and this completes the proof of Theorem 2. $\square$

*Proof of Theorem* 3. By hypothesis $P(z) \neq 0$ in $|z| < k$, $k > 0$. If $P(z)$ has a zero on $|z| = k$, then $m = \min_{|z|=k} |P(z)| = 0$ and the result follows from Theorem 2 in this case. Henceforth, we assume that all the zeros of $P(z)$ lie in $|z| > k$, so that $m > 0$. Now

$$m \leqslant |P(z)| \quad \text{for } |z| = k,$$

therefore if $\delta$ is any complex number such that $|\delta| < 1$, then

$$|\delta m| < m \leqslant |P(z)| \quad \text{for } |z| = k.$$

Since all the zeros of $P(z)$ lie in $|z| > k$, it follows by Rouché's theorem that all the zeros $g(z) = P(z) - \delta m$ also lie in $|z| > k$, $k > 0$. Hence the polynomial $g(Rz) = P(Rz) - \delta m$ has no zeros in $|z| < k/R$, $k/R \geqslant 1$. Applying (4.4) to $g(Rz)$, we have

$$R^s ||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(1)}{||A+z||_\gamma} \left(\frac{R+k}{r+k}\right)^n ||P(rz) - \delta m||_\gamma,$$

where

$$A = \frac{C(n,s)|a_0 - \delta m| k^{s+1} + |a_s| k^{2s} R}{C(n,s)|a_0 - \delta m| R^{s+1} + |a_s| k^{s+1} R^s}.$$

This implies

$$||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(1)}{||R^s z + B||_\gamma} \left(\frac{R+k}{r+k}\right)^n ||P(rz) - \delta m||_\gamma, \tag{4.5}$$

where

$$B = \frac{C(n,s)|a_0 - \delta m| k^{s+1} + |a_s| k^{2s} R}{C(n,s)|a_0 - \delta m| R + |a_s| k^{s+1}}.$$

Also, for every $\delta \in \mathbb{C}$, we have

$$|a_0 - \delta m| \geqslant |a_0| - |\delta| m,$$

and since the function

$$x \mapsto \frac{C(n,s) x k^{s+1} + |a_s| k^{2s} R}{C(n,s) x R + |a_s| k^{s+1}} \quad (x \geqslant 0)$$

is non-decreasing, it follows that

$$\begin{aligned} B &= \frac{C(n,s)|a_0 - \delta m| k^{s+1} + |a_s| k^{2s} R}{C(n,s)|a_0 - \delta m| R + |a_s| k^{s+1}} \\ &\geqslant \frac{C(n,s)(|a_0| - |\delta| m) k^{s+1} + |a_s| k^{2s} R}{C(n,s)(|a_0| - |\delta| m) R + |a_s| k^{s+1}} = S_2. \end{aligned}$$

Using this inequality in (4.5), we have for $1 \leqslant s < n$ and $0 < r \leqslant R \leqslant k$,

$$||P^{(s)}(Rz)||_\gamma \leqslant \frac{F(1)}{||R^s z + S_2||_\gamma} \left(\frac{R+k}{r+k}\right)^n ||P(rz) - \delta m||_\gamma, \tag{4.6}$$

for every complex number $\delta$ with $|\delta| < 1$ and for every $\gamma > 0$. For $\delta$ with $|\delta| = 1$, the inequality (4.6) follows by continuity and this completes the proof of Theorem 3. $\square$

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