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THREE GEOMETRIC CONSTANTS FOR TOTAL MORREY SPACES

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Abstract. In this paper, we calculate the von Neumann–Jordan constant, the James constant and the Dunkl–Williams constant for total Morrey spaces. As a corollary, the aforementioned constants for modified Morrey spaces are established.

1. Introduction

For a Banach space $(X, \|\cdot\|)$, the von Neumann–Jordan constant $C_{\mathrm{NJ}}(X)$ (see [8]), the James constant $C_{\mathrm{J}}(X)$ (see [7]), and the Dunkl–Williams constant $C_{\mathrm{DW}}(X)$ (see [3]) are given by

$$C_{\mathrm{NJ}}(X) := \sup\left\{\frac{\|x+y\|_X^2 + \|x-y\|_X^2}{2\left(\|x\|_X^2 + \|y\|_X^2\right)} : x, y \in X \backslash \{0\}\right\},$$

$$C_{\mathrm{J}}(X) := \sup\{\min\{\|x+y\|_X, \|x-y\|_X\} : x, y \in X, \|x\|_X = \|y\|_X = 1\},$$

$$C_{\mathrm{DW}}(X) := \sup\left\{\frac{\|x\|_X + \|y\|_X}{\|x-y\|_X}\left\|\frac{x}{\|x\|_X} - \frac{y}{\|y\|_X}\right\|_X : x, y \in X \backslash \{0\}, x \neq y\right\},$$

respectively. These constants measure the uniform nonsquareness of X. For the von Neumann–Jordan constant $C_{\mathrm{NJ}}(X)$, it is well known that $1 \leqslant C_{\mathrm{NJ}}(X) \leqslant 2$, and $C_{\mathrm{NJ}}(X) = 1$ if and only if X is a Hilbert space (see [1]). It was shown in [4] that $\sqrt{2} \leqslant C_{\mathrm{J}}(X) \leqslant 2$, and $C_{\mathrm{J}}(X) = \sqrt{2}$ if X is a Hilbert space. Besides, from [3], we know that $2 \leqslant C_{\mathrm{DW}}(X) \leqslant 4$, and that $C_{\mathrm{DW}}(X) = 2$ if and only if X is a Hilbert space.

For some concrete Banach spaces, the geometric constants can be precisely given. For instance, the geometric constants for Lebesgue spaces and discrete Lebesgue spaces were established in [4]. Recently, Gunawan et al. [6, 9] calculated the aforementioned geometric constants for Morrey spaces and small Morrey spaces. In this paper, we shall consider these three constants for total Morrey spaces, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. Now we recall the definition of the total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. Let $0 < p < \infty$, $0 \leqslant \lambda, \mu \leqslant n$, $[t]_1 =$

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$\min\{1,t\}, t > 0$. The total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$ is the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L^{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at x with radius $t > 0$. A particular case of $L^{p,\lambda,\mu}(\mathbb{R}^n)$ is the modified Morrey space denoted by $\widetilde{L}^{p,\lambda}(\mathbb{R}^n) = L^{p,\lambda,0}(\mathbb{R}^n)$.

It was proven in [5] that $L^{p,\lambda,\mu}(\mathbb{R}^n)$ is a Banach space for $1 \leqslant p < \infty$. As a consequence, we have

$$C_{\mathrm{NJ}}(L^{p,\lambda,\mu}(\mathbb{R}^n)) \leqslant 2,\ C_{\mathrm{J}}(L^{p,\lambda,\mu}(\mathbb{R}^n)) \leqslant 2,\ C_{\mathrm{DW}}(L^{p,\lambda,\mu}(\mathbb{R}^n)) \leqslant 4$$

for $1 \leqslant p < \infty$ and $0 < \lambda \leqslant n$, $0 \leqslant \mu \leqslant n$. Our aim in this paper is to establish the precise geometric constants for total Morrey spaces. In fact, we shall prove that the inequalities given above are exactly equalities.

2. Main result

Our main result can be stated as follows.

THEOREM 1. *Let $1 \leqslant p < \infty$, $0 < \lambda \leqslant n$, $0 \leqslant \mu \leqslant n$. Then we have $C_{\mathrm{NJ}}(L^{p,\lambda,\mu}(\mathbb{R}^n)) = C_{\mathrm{J}}(L^{p,\lambda,\mu}(\mathbb{R}^n)) = 2$ and $C_{\mathrm{DW}}(L^{p,\lambda,\mu}(\mathbb{R}^n)) = 4$.*

Proof. We borrow some ideas from [2, 6, 9]. For sufficiently small $\varepsilon, \delta \in (0,1)$, we define some useful functions as follows:

$$f(x) = \chi_{(0,1)}(|x|)\,|x|^{\frac{\lambda-n}{p}};$$
$$g(x) = \chi_{(0,\varepsilon)}(|x|)\,f(x) = \chi_{(0,\varepsilon)}(|x|)\,|x|^{\frac{\lambda-n}{p}};$$
$$h(x) = f(x) - g(x) = \chi_{(\varepsilon,1)}(|x|)\,|x|^{\frac{\lambda-n}{p}};$$
$$k(x) = -f(x) + 2g(x) = \big(2\chi_{(0,\varepsilon)}(|x|) - \chi_{(0,1)}(|x|)\big)\,|x|^{\frac{\lambda-n}{p}};$$
$$l(x) = (1+\delta)\,g(x) + (1-\delta)\,h(x) = \big(2\delta\chi_{(0,\varepsilon)}(|x|) - \delta + 1\big)\,\chi_{(0,1)}(|x|)\,|x|^{\frac{\lambda-n}{p}}.$$

By using the definition of total Morrey spaces, we have

$$\begin{aligned}\|f\|_{L^{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x,t))} \\ &= \sup_{x\in\mathbb{R}^n, t>0} (\min\{t,1\})^{-\frac{\lambda}{p}} (\min\{1/t,1\})^{\frac{\mu}{p}} \|f\|_{L^p(B(x,t))} \\ &= \max\left\{ \sup_{x\in\mathbb{R}^n, t\geqslant 1} t^{-\frac{\mu}{p}} \|f\|_{L^p(B(x,t))},\ \sup_{x\in\mathbb{R}^n, 0<t<1} t^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,t))} \right\}.\end{aligned}$$

Let ω_n be the area of the unit sphere in $\mathbb{R}^n$. By a simple calculation, we get

$$\begin{aligned}\sup_{x\in\mathbb{R}^n,t\geqslant 1} t^{-\frac{\mu}{p}}\|f\|_{L^p(B(x,t))} &= \sup_{x\in\mathbb{R}^n,t\geqslant 1} t^{-\frac{\mu}{p}}\left(\int_{B(x,t)}\chi_{(0,1)}(|y|)\,|y|^{\lambda-n}dy\right)^{\frac{1}{p}}\\ &= \left(\int_{B(0,1)}|x|^{\lambda-n}dx\right)^{\frac{1}{p}}\\ &= \omega_n^{\frac{1}{p}}\left(\int_0^1 r^{\lambda-1}dr\right)^{\frac{1}{p}}\\ &= \left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}},\end{aligned}$$

and

$$\begin{aligned}\sup_{x\in\mathbb{R}^n,0<t<1} t^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,t))} &= \sup_{0<t<1} t^{-\frac{\lambda}{p}}\left(\int_{B(0,t)}|x|^{\lambda-n}dx\right)^{\frac{1}{p}}\\ &= \omega_n^{\frac{1}{p}}\sup_{0<t<1} t^{-\frac{\lambda}{p}}\left(\int_0^t r^{\lambda-1}dr\right)^{\frac{1}{p}}\\ &= \left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}}.\end{aligned}$$

Consequently, $\|f\|_{L^{p,\lambda,\mu}} = \left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}}$. In a similar way, we obtain

$$\begin{aligned}\sup_{x\in\mathbb{R}^n,t\geqslant 1} t^{-\frac{\mu}{p}}\|g\|_{L^p(B(x,t))} &= \left(\int_{B(0,\varepsilon)}|x|^{\lambda-n}dx\right)^{\frac{1}{p}}\\ &= \left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}}\varepsilon^{\frac{\lambda}{p}},\end{aligned}$$

and

$$\begin{aligned}&\sup_{x\in\mathbb{R}^n,0<t<1} t^{-\frac{\lambda}{p}}\|g\|_{L^p(B(x,t))} = \sup_{0<t<1} t^{-\frac{\lambda}{p}}\left(\int_{B(0,t)}\chi_{(0,\varepsilon)}(|x|)\,|x|^{\lambda-n}dx\right)^{\frac{1}{p}}\\ &= \max\left\{\sup_{0<t<\varepsilon} t^{-\frac{\lambda}{p}}\left(\int_{B(0,t)}\chi_{(0,\varepsilon)}(|x|)\,|x|^{\lambda-n}dx\right)^{\frac{1}{p}},\right.\\ &\qquad\left.\sup_{\varepsilon<t<1} t^{-\frac{\lambda}{p}}\left(\int_{B(0,t)}\chi_{(0,\varepsilon)}(|x|)\,|x|^{\lambda-n}dx\right)^{\frac{1}{p}}\right\}\\ &= \omega_n^{1/p}\max\left\{\sup_{0<t<\varepsilon} t^{-\frac{\lambda}{p}}\left(\int_0^t r^{\lambda-1}dr\right)^{\frac{1}{p}}, \sup_{\varepsilon<t<1} t^{-\frac{\lambda}{p}}\left(\int_0^\varepsilon r^{\lambda-1}dr\right)^{\frac{1}{p}}\right\}\\ &= \omega_n^{1/p}\max\left\{\sup_{0<t<\varepsilon} t^{-\frac{\lambda}{p}}\left(\frac{t^\lambda}{\lambda}\right)^{\frac{1}{p}}, \sup_{\varepsilon<t<1}\left(\frac{1}{\lambda}\right)^{\frac{1}{p}}\left(\frac{\varepsilon}{t}\right)^{\frac{\lambda}{p}}\right\} = \left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}},\end{aligned}$$

which yield $\|g\|_{L^{p,\lambda,\mu}} = \max\left\{ \left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}} \varepsilon^{\frac{\lambda}{p}}, \left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}} \right\} = \left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}} = \|f\|_{L^{p,\lambda,\mu}}$.

For h, we have the following estimate:

$$\begin{aligned}\|h\|_{L^{p,\lambda,\mu}} \geqslant \|h\|_{L^p(B(0,1))} &= \left(\int_{B(0,1)} \chi_{(\varepsilon,1)}(|x|)\,|x|^{\lambda-n}dx\right)^{\frac{1}{p}} \\ &= \omega_n^{1/p}\left(\int_\varepsilon^1 r^{\lambda-1}dr\right)^{\frac{1}{p}} = \left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}}\left(1-\varepsilon^\lambda\right)^{\frac{1}{p}} = \left(1-\varepsilon^\lambda\right)^{\frac{1}{p}}\|f\|_{L^{p,\lambda,\mu}}.\end{aligned}$$

Noting that $|k| = |f|$, we get $\|k\|_{L^{p,\lambda,\mu}} = \|f\|_{L^{p,\lambda,\mu}}$.

As for l, we have

$$\begin{aligned} l(x) &= \left(2\delta\chi_{(0,\varepsilon)}(|x|) - \delta + 1\right)\chi_{(0,1)}(|x|)\,|x|^{\frac{\lambda-n}{p}} \\ &= \begin{cases} (1+\delta)\,|x|^{\frac{\lambda-n}{p}}, & \text{if} \quad 0<|x|<\varepsilon; \\ (1-\delta)\,|x|^{\frac{\lambda-n}{p}}, & \text{if} \quad \varepsilon \leqslant |x| < 1; \\ 0, & \text{if} \quad |x| \geqslant 1. \end{cases}\end{aligned}$$

Now we calculate the total Morrey norm of l:

$$\begin{aligned}\|l\|_{L^{p,\lambda,\mu}} &= \max\left\{ \sup_{x\in\mathbb{R}^n, t\geqslant 1} t^{-\frac{\mu}{p}}\|l\|_{L^p(B(x,t))}, \sup_{x\in\mathbb{R}^n, 0<t<1} t^{-\frac{\lambda}{p}}\|l\|_{L^p(B(x,t))} \right\} \\ &= \max\left\{ \left(\int_{B(0,1)}|l(x)|^p dx\right)^{\frac{1}{p}}, \sup_{0<t<\varepsilon} t^{-\frac{\lambda}{p}}\left(\int_{B(0,t)}|l(x)|^p dx\right)^{\frac{1}{p}}, \right. \\ &\qquad \left. \sup_{\varepsilon<t<1} t^{-\frac{\lambda}{p}}\left(\int_{B(0,t)}|l(x)|^p dx\right)^{\frac{1}{p}} \right\} \\ &=: \max\{I_1, I_2, I_3\}.\end{aligned}$$

For I_2, there holds

$$\begin{aligned}\sup_{0<t<\varepsilon} t^{-\frac{\lambda}{p}}\left(\int_{B(0,t)}|l(x)|^p dx\right)^{\frac{1}{p}} &= \sup_{0<t<\varepsilon} t^{-\frac{\lambda}{p}}\left(\int_{B(0,t)}(1+\delta)^p\,|x|^{\lambda-n}dx\right)^{\frac{1}{p}} \\ &= (1+\delta)\,\|f\|_{L^{p,\lambda,\mu}}.\end{aligned}$$

For I_1, we obtain the following estimates:

$$\begin{aligned}\left(\int_{B(0,1)}|l(x)|^p dx\right)^{\frac{1}{p}} &= \omega_n^{1/p}\left(\int_0^\varepsilon (1+\delta)^p\, r^{\lambda-1}dr + \int_\varepsilon^1 (1-\delta)^p\, r^{\lambda-1}dr\right)^{\frac{1}{p}} \\ &\leqslant \omega_n^{1/p}\left(\int_0^1 (1+\delta)^p\, r^{\lambda-1}dr\right)^{\frac{1}{p}} \\ &= (1+\delta)\left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}} = (1+\delta)\,\|f\|_{L^{p,\lambda,\mu}}.\end{aligned}$$

For I_3, we have

$$\begin{aligned}
&\sup_{\varepsilon<t<1} t^{-\frac{\lambda}{p}}\left(\int_{B(0,t)}|l(x)|^p dx\right)^{\frac{1}{p}}\\
&=\omega_n^{1/p}\sup_{\varepsilon<t<1} t^{-\frac{\lambda}{p}}\left(\int_0^{\varepsilon}(1+\delta)^p r^{\lambda-1}dr+\int_{\varepsilon}^{t}(1-\delta)^p r^{\lambda-1}dr\right)^{\frac{1}{p}}\\
&\leqslant \omega_n^{1/p}\sup_{\varepsilon<t<1} t^{-\frac{\lambda}{p}}\left(\int_0^{t}(1+\delta)^p r^{\lambda-1}dr\right)^{\frac{1}{p}}\\
&=(1+\delta)\left(\frac{\omega_n}{\lambda}\right)^{\frac{1}{p}}=(1+\delta)\,\|f\|_{L^{p,\lambda,\mu}}.
\end{aligned}$$

Combining the estimates for I_1, I_2 and I_3, we get $\|l\|_{L^{p,\lambda,\mu}}=(1+\delta)\,\|f\|_{L^{p,\lambda,\mu}}$.

For the von Neumann–Jordan constant $C_{\mathrm{NJ}}(L^{p,\lambda,\mu}(\mathbb{R}^n))$, we have

$$\begin{aligned}
C_{\mathrm{NJ}}(L^{p,\lambda,\mu}(\mathbb{R}^n)) &\geqslant \frac{\|f+k\|^2_{L^{p,\lambda,\mu}}+\|f-k\|^2_{L^{p,\lambda,\mu}}}{2\left(\|f\|^2_{L^{p,\lambda,\mu}}+\|k\|^2_{L^{p,\lambda,\mu}}\right)}\\
&=\frac{\|2g\|^2_{L^{p,\lambda,\mu}}+\|2h\|^2_{L^{p,\lambda,\mu}}}{2\left(\|f\|^2_{L^{p,\lambda,\mu}}+\|f\|^2_{L^{p,\lambda,\mu}}\right)}\\
&\geqslant \frac{\|f\|^2_{L^{p,\lambda,\mu}}+\left(1-\varepsilon^{\lambda}\right)^{\frac{2}{p}}\|f\|^2_{L^{p,\lambda,\mu}}}{\|f\|^2_{L^{p,\lambda,\mu}}}\\
&=1+\left(1-\varepsilon^{\lambda}\right)^{\frac{2}{p}}.
\end{aligned}$$

Letting $\varepsilon\to 0^+$, we obtain $C_{\mathrm{NJ}}(L^{p,\lambda,\mu}(\mathbb{R}^n))\geqslant 2$. Since $C_{\mathrm{NJ}}(L^{p,\lambda,\mu}(\mathbb{R}^n))\leqslant 2$, we have $C_{\mathrm{NJ}}(L^{p,\lambda,\mu}(\mathbb{R}^n))=2$.

For the James constant $C_{\mathrm{J}}(L^{p,\lambda,\mu}(\mathbb{R}^n))$, one has

$$\begin{aligned}
C_{\mathrm{J}}(L^{p,\lambda,\mu}(\mathbb{R}^n)) &\geqslant \min\left\{\left\|\frac{f}{\|f\|_{L^{p,\lambda,\mu}}}+\frac{k}{\|k\|_{L^{p,\lambda,\mu}}}\right\|_{L^{p,\lambda,\mu}},\left\|\frac{f}{\|f\|_{L^{p,\lambda,\mu}}}-\frac{k}{\|k\|_{L^{p,\lambda,\mu}}}\right\|_{L^{p,\lambda,\mu}}\right\}\\
&=\min\left\{\frac{\|2g\|_{L^{p,\lambda,\mu}}}{\|f\|_{L^{p,\lambda,\mu}}},\frac{\|2h\|_{L^{p,\lambda,\mu}}}{\|f\|_{L^{p,\lambda,\mu}}}\right\}\\
&\geqslant \min\left\{2,2\left(1-\varepsilon^{\lambda}\right)^{\frac{1}{p}}\right\}=2\left(1-\varepsilon^{\lambda}\right)^{\frac{1}{p}}.
\end{aligned}$$

By an argument similar to the estimates of $C_{\mathrm{J}}(L^{p,\lambda,\mu}(\mathbb{R}^n))$, we have $C_{\mathrm{J}}(L^{p,\lambda,\mu}(\mathbb{R}^n))=2$.

Finally, we calculate the Dunkl–Williams constant $C_{\mathrm{DW}}(L^{p,\lambda,\mu}(\mathbb{R}^n))$. In fact,

$$\begin{aligned}
C_{\mathrm{DW}}(L^{p,\lambda,\mu}(\mathbb{R}^n)) &\geqslant \frac{\|f\|_{L^{p,\lambda,\mu}}+\|l\|_{L^{p,\lambda,\mu}}}{\|f-l\|_{L^{p,\lambda,\mu}}}\left\|\frac{f}{\|f\|_{L^{p,\lambda,\mu}}}-\frac{l}{\|l\|_{L^{p,\lambda,\mu}}}\right\|_{L^{p,\lambda,\mu}} \\
&= \frac{\|f\|_{L^{p,\lambda,\mu}}+\|l\|_{L^{p,\lambda,\mu}}}{\|\delta k\|_{L^{p,\lambda,\mu}}}\left\|\frac{(1+\delta)f-l}{(1+\delta)\|f\|_{L^{p,\lambda,\mu}}}\right\|_{L^{p,\lambda,\mu}} \\
&= \frac{\|f\|_{L^{p,\lambda,\mu}}+\|l\|_{L^{p,\lambda,\mu}}}{\delta\|f\|_{L^{p,\lambda,\mu}}}\times\frac{1}{(1+\delta)\|f\|_{L^{p,\lambda,\mu}}}\|2\delta h\|_{L^{p,\lambda,\mu}} \\
&\geqslant \frac{2+\delta}{\delta}\cdot\frac{2\delta}{(1+\delta)\|f\|_{L^{p,\lambda,\mu}}}\left(1-\varepsilon^{\lambda}\right)^{\frac{1}{p}}\|f\|_{L^{p,\lambda,\mu}} \\
&= \frac{4+2\delta}{1+\delta}\left(1-\varepsilon^{\lambda}\right)^{\frac{1}{p}}.
\end{aligned}$$

Letting $\varepsilon,\delta\to 0^+$, we have $C_{\mathrm{DW}}(L^{p,\lambda,\mu}(\mathbb{R}^n))\geqslant 4$. By virtue of $C_{\mathrm{DW}}(L^{p,\lambda,\mu}(\mathbb{R}^n))\leqslant 4$, we get $C_{\mathrm{DW}}(L^{p,\lambda,\mu}(\mathbb{R}^n))=4$. The proof is completed. □

By taking $\mu=0$ in Theorem 1, we obtain the von Neumann–Jordan constant, the James constant and the Dunkl–Williams constant for the modified Morrey space $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

COROLLARY 1. *Let $1\leqslant p<\infty$, $0<\lambda\leqslant n$. Then $C_{\mathrm{NJ}}(\widetilde{L}^{p,\lambda}(\mathbb{R}^n))=C_{\mathrm{J}}(\widetilde{L}^{p,\lambda}(\mathbb{R}^n))=2$, $C_{\mathrm{DW}}(\widetilde{L}^{p,\lambda}(\mathbb{R}^n))=4$.*

REMARK 1. Recall that a Banach space X with a norm $\|\cdot\|_X$ is uniformly non-square if and only if there exists a $\delta>0$ such that

$$\min\{\|x+y\|_X,\|x-y\|_X\}\leqslant 2(1-\delta)$$

for all $x,y\in X$ with $\|x\|_X=\|y\|_X=1$. It then follows from the definition of the James constant and $C_{\mathrm{J}}\left(L^{p,\lambda,\mu}(\mathbb{R}^n)\right)=2$ that the total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$ is not uniformly non-square, and hence not uniformly convex (see [7]) for $1\leqslant p<\infty, 0<\lambda\leqslant n, 0\leqslant\mu\leqslant n$.

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