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POSITIVE DEFINITE MATRIX–VALUED KERNELS AND THEIR SCALAR VALUED PROJECTIONS: COUNTEREXAMPLES

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Abstract. In this paper we show that the strictly positive definite matrix valued isotropic kernels in the circle and the dot product kernels in Euclidean spaces are not well behaved with respect to its scalar valued projections. We generalize the counterexamples that we obtained to an abstract setting by using the concepts of unitarily invariant kernels and adjointly invariant kernels, provided the existence of an aperiodic invariant function.

1. Introduction

A matrix valued kernel $K : X \times X \to M_\ell(\mathbb{C})$ is positive definite if for every finite quantity of distinct points $x_1, x_2, \dots, x_n$ in X and vectors $v_1, v_2, \dots, v_n \in \mathbb{C}^\ell$, we have

$$\sum_{i,j=1}^{n} \langle K(x_i, x_j) v_i, v_j \rangle \geqslant 0.$$

In addition, if the above inequalities are strict whenever at least one of the vectors v_i is nonzero, then the kernel is termed strictly positive definite.

When $\ell = 1$, the previous definition is the standard notion of a positive definite and strictly positive definite kernel. A very simple connection between the matrix valued and the scalar valued cases is obtained by the scalar valued projections of the kernel, which are the scalar valued kernels $K_v : X \times X \to \mathbb{C}$, $v \in \mathbb{C}^\ell \setminus \{0\}$, given by

$$K_v(x,y) = \langle K(x,y)v, v \rangle, \quad x, y \in X. \tag{1}$$

If the matrix valued kernel $K : X \times X \to M_\ell(\mathbb{C})$ is a positive definite (strictly), then all of its scalar valued projections kernels are positive definite (strictly). Indeed, for distinct points $x_1, x_2, \dots, x_n$ in X, scalars $c_1, c_2, \dots, c_n \in \mathbb{C}$ and a vector $v \in \mathbb{C}^\ell \setminus \{0\}$, we have that

$$\sum_{i,j=1}^{n} c_i \overline{c_j} K_v(x_i, x_j) = \sum_{i,j=1}^{n} \langle K(x_i, x_j) c_i v_i, c_j v_j \rangle \geqslant 0.$$

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Additionally, if K is strictly positive definite then K_v is strictly positive definite as well because for $1 \leqslant i \leqslant n$ all vectors $c_i v_i$ would be zero, but this can only occur if all scalars c_i, $1 \leqslant i \leqslant n$, are zero.

However, the converse of this property is not true in general. For instance, given two functions $f : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$, all scalar valued projections of the kernel

$$K(x,y) = \begin{pmatrix} 2f(x)f(y) & f(x)h(y)+h(x)f(y) \\ f(x)h(y)+h(x)f(y) & 2h(x)h(y) \end{pmatrix},$$

are positive definite. Suitable choice of the functions f and h lead to a counterexample, like $f(x_1) = 1$, $f(x_2) = 1$, $h(x_1) = 1$ and $h(x_2) = 2$, for arbitrary distinct points x_1, x_2 in X. Note that if the set X has a topology for which the space real valued continuous functions defined on X, that is $C(X,\mathbb{R})$, has dimension at least two, the above method generates a counterexample for the converse of the scalar valued projections when the kernel K is continuous.

However, several important classes of kernels are well behaved with respect its the scalar valued projections, precisely, that the fact that all scalar valued projections of a matrix valued kernel are positive definite (strictly) is equivalent that the matrix valued kernel is positive definite (strictly).

Below, we present several of those classes and describe which ones are well behaved with respect to its scalar valued projections:

(1) **Isotropic kernels in real spheres** $X = S^d$, the unit sphere in $\mathbb{R}^{d+1}$, K continuous and also isotropic in the sense that

$$K(Qx,Qy) = K(x,y), \quad x,y \in S^d, \quad Q \in \mathscr{O}(d+1),$$

where $\mathscr{O}(d+1)$ is the set of all orthogonal transformations on $\mathbb{R}^{d+1}$. The characterization of the positive definite scalar valued kernels fulfilling this condition was achieved in [27], while the strictly positive definite case was proved in [9] ($d \geqslant 2$) and [21] ($d = 1$). The positive definite matrix valued case traces back to [33], while the strictly positive definite case was proved in [17] except for the case $d = 1$, which remained open. Except for the strictly positive definite kernels in S^1, all of them are well behaved with respect to the scalar valued projections.

There is also the limiting case, of kernels defined on S^∞, the unit sphere of an infinity dimensional real Hilbert space, K is continuous and the isotropy is the invariance of the kernel for all linear isometries in it. The previous references contain the analysis for this case and it is well behaved with respect to the scalar valued projections property.

(2) **Isotropic kernels in two-point compact homogeneous spaces** $X = M^d$, where M^d is a compact connected two point homogeneous space, but not a real sphere in $\mathbb{R}^{d+1}$, K continuous and also isotropic in the sense that

$$K(Qx,Qy) = K(x,y), \quad x,y \in S^d, \quad Q \in ISO(M^d),$$

where $ISO(M^d)$ is the set of all isometries in M^d. The two-point compact homogeneous spaces were classified in [31], while the characterization of the positive definite isotropic kernels were characterized in [12]. The strictly positive definite case was proved in [1], and the matrix valued results were obtained in [8]. Unlike the real sphere, the characterizations for the infinite dimensional case (for the real, complex and quaternionic projective spaces) came as a corollary of a different paper [14]. Although there is no paper that explicitly describes the matrix valued case in the infinite dimensional setting, it is well behaved with respect to the scalar valued projections as well as the other cases.

(3) **Radial kernels** $X = \mathbb{R}^d$, K continuous and also radial in the sense that

$$K(Qx+z, Qy+z) = K(x,y), \quad x,y,z \in \mathbb{R}^d, \quad Q \in \mathscr{O}(d).$$

The characterization of the positive definite scalar valued kernels fulfilling this condition was achieved in [27], while the strictly positive definite case was proved in [29] ($d \geqslant 2$). The positive definite matrix valued case was characterized in [32], while the strictly positive definite case was proved in [16] except for the case $d = 1$, which remained open.

Like the real sphere, the previous papers contain proofs for the infinite dimensional case.

With the exception of strictly positive definite kernels in R^1, all of them are well behaved with respect to the scalar valued projections.

(4) **Translation invariant kernels** $X = \mathbb{R}^d$, K is continuous and translation invariant, in the sense that

$$K(x+z, y+z) = K(x,y), \quad x,y,z \in \mathbb{R}^d.$$

The characterization of all positive definite kernels that fulfill this property is in the classical paper [7], only sufficient conditions of when this kernel is strictly positive definite are known and an operator valued version of these kernels can be found in [23]. Only the positive definite case is well behaved with respect to the scalar valued projections.

(5) **Real Dot product kernels** $X = \mathscr{H}$, a real Hilbert space, K is continuous and adjointly invariant, in the sense that

$$K(Ax, y) = K(x, A^t y), \quad x,y \in \mathscr{H}, \quad A \in \mathscr{L}(\mathscr{H})$$

where A^t is the adjoint operator of A and $\mathscr{L}(\mathscr{H})$ is the vector space of continuous linear operators from $\mathscr{H}$ to itself. If a kernel fulfills this property then there exists a continuous function $h : \mathbb{R} \to \mathbb{R}$ such that $K(x,y) = h(\langle x,y \rangle)$ (we prove this affirmation in Lemma 6). The characterization of which functions h generates a positive definite kernel is obtained as a corollary of a result in [27] when $\dim(\mathscr{H}) = \infty$, and generalized to when $\dim(\mathscr{H}) \geqslant 2$ in [20]. The characterization for when $\dim(\mathscr{H}) = 1$ can be obtained as Corollary of Remark 3.9 in [5] at page 161. The strictly positive definite case was proved in [25] for $\dim(\mathscr{H}) \geqslant 2$. We are not aware of a characterization for the strictly positive definite case when $\dim(\mathscr{H}) = 1$. Only the positive definite case is well behaved with respect to the scalar valued projections.

In the following example we close the gaps in the families (1), (4) and (5) by providing an example of a positive definite matrix valued kernel that is not strictly positive definite but whose all scalar valued projections are strictly positive definite:

EXAMPLE 1. Consider the kernels

$$(\cos\theta,\sin\theta),(\cos\vartheta,\sin\vartheta)\in S^1\times S^1\to\begin{bmatrix} e^{\cos(\theta-\vartheta)} & e^{\cos(\theta+\rho-\vartheta)} \\ e^{\cos(\theta-\vartheta-\rho)} & e^{\cos(\theta-\vartheta)}\end{bmatrix},$$

$$\rho\in[0,2\pi),\ \ \rho\notin\mathbb{Q}\pi,$$

$$(x,y)\in\mathbb{R}^m\times\mathbb{R}^m\to\begin{bmatrix} e^{-\sigma\|x-y\|^2} & e^{-\sigma\|x+z-y\|^2} \\ e^{-\sigma\|x-y-z\|^2} & e^{-\sigma\|x-y\|^2}\end{bmatrix},\quad z\in\mathbb{R}^m\setminus\{0\},$$

$$(x,y)\in\mathbb{R}^m\times\mathbb{R}^m\to\begin{bmatrix} e^{r^2\langle x,y\rangle}+1 & e^{r\langle x,y\rangle} \\ e^{r\langle x,y\rangle} & e^{\langle x,y\rangle}+1\end{bmatrix},\quad r\in\mathbb{R}\setminus\{0,1,-1\},$$

which are, respectively, an isotropic kernel in S^1, a translation invariant kernel in $\mathbb{R}^m$ and a real dot product kernel in $\mathbb{R}^m$. They are positive definite but not strictly positive definite (which is a consequence of Theorem 1) and all scalar valued projections are strictly positive definite.

In Section 2 we generalize the first two examples to a broader setting, of kernels unitarily invariant by a family (more precisely a semigroup) of functions, where the criteria is based on the existence of an aperiodic function in the center of the semigroup. With this result, we are able to fully characterize in Subsection 2.1 which locally compact Abelian groups G the group invariant continuous matrix valued strictly positive definite kernels are well behaved with respect to the scalar valued projections. On Subsection 2.2 we show that the strictly positive definite isotropic kernels on complex spheres are not well behaved with respect to its scalar valued projections.

In Section 3 we obtain similar results of Section 2, but for kernels that are adjointly invariant by a semigroup of functions with involution, provided the existence of an injective aperiodic function in the center of the semigroup. We prove in Subsection 3.1 a similar representation as presented in the class of kernels (5) for the continuous complex dot product spaces. By making an adjustment on Theorem 5, we are able to prove that the continuous matrix valued strictly positive definite real and complex dot product kernels are not well behaved with respect to its scalar valued projections

It is important to point out that the results we present can be easily adapted for the operator valued kernels, see [23]. Also, as in general the most common classes of scalar valued positive definite kernels with an invariance (either unitarily or adjointly) are uniquely described by a specific set of functions, then for those classes the positive definite matrix valued kernels are well behaved with respect to the scalar valued projections (usually the matrix valued case is characterized using this property, see the references mentioned in the classes (1), (2), (3), (4) and (5)).

We reemphasize that all known methods that characterize matrix valued strictly positive definite kernels with a symmetry (either unitarily or adjointly), occur by the scalar valued projections. Hence, for the 3 examples presented, a new and complete different method will be required to characterize these matrix valued strictly positive definite kernels.

2. Unitarily invariant kernels

We recall that a semigroup $(S,\circ)$ is a set S together with an operation $\circ : S\times S \to S$ that is associative, that is, $((a\circ b)\circ c) = (a\circ(b\circ c))$, for every $a,b,c \in S$. A semigroup $(S,\circ)$ has an unity if there exist $e\in S$ for which $e\circ a = a\circ e = a$ for all $a \in S$.

For a topological space X, we define $C(X,X)$ as the set of all continuous functions from X to X, which is a semigroup with the standard composition of functions, and the unit is the identity function.

The classes of kernels $(1),(2),(3)$ and (4), presented in Section 1, have something in common, these kernels satisfy the following property:

DEFINITION 1. Let X be a topological space and $S \subset C(X,X)$ be a family of continuous functions. We say that a continuous matrix valued kernel $K : X\times X \to M_\ell(\mathbb{C})$ is unitarily invariant the family of functions S if

$$K(\phi(x),\phi(y)) = K(x,y)$$

for all $x,y \in X$ and $\phi \in S$.

We assume that the space X has a topology and that the kernel K is continuous, only by its importance rather than a condition itself. The term unitarily is inspired by the standard definition of unitary matrix, which occurs when $X = \mathbb{R}^m$ and $K(x,y) = \langle x,y\rangle$.

Although we have made no assumption about the set S in the above definition, we can assume without loss of generality that it is a semigroup of continuous functions with an unity. Indeed, the identity function $i : X \to X$ satisfies the required symmetry relation, and given two functions $\phi_1,\phi_2 \in S$, then $\phi_2\phi_1$ also satisfies, because

$$K(x,y) = K(\phi_1(x),\phi_1(y)) = K(\phi_2(\phi_1(x)),\phi_2(\phi_1(y))), \quad x,y \in X.$$

If a continuous matrix valued kernel K is unitarily invariant by a semigroup of continuous functions with unity S and is positive definite, we use the notation $K \in P(X,S,\mathbb{C}^\ell)$. If in addition the kernel is strictly positive definite we use the notation $K \in P^+(X,S,\mathbb{C}^\ell)$. Similarly, if all of the scalar valued projections of a continuous matrix valued kernel K that is unitarily invariant by a semigroup of continuous functions with unity S are scalar valued positive definite kernels, we use the notation $K \in P_{proj}(X,S,\mathbb{C}^\ell)$. Likewise if in addition all of the scalar valued projections of the kernel K are scalar valued strictly positive definite kernels, we use the notation $K \in P^+_{proj}(X,S,\mathbb{C}^\ell)$. When $\ell = 1$ we use the simplified notation $P(X,S)$ and $P^+(X,S)$.

We reemphasize that, as shown at the Introduction, the inclusions

$$P(X,S,\mathbb{C}^\ell) \subset P_{proj}(X,S,\mathbb{C}^\ell) \tag{2}$$

$$P^+(X,S,\mathbb{C}^\ell) \subset P^+_{proj}(X,S,\mathbb{C}^\ell) \tag{3}$$

always holds and that under the conditions of Theorem 2 we show that the inclusion $P^+(X,S,\mathbb{C}^\ell) \subset P^+_{proj}(X,S,\mathbb{C}^\ell)$ is strict.

The following simple Lemma explain why we focus only on $M_2(\mathbb{C})$ valued kernels.

LEMMA 1. *Let X be a topological space, S a semigroup of continuous functions on X. Then $P^{+}_{proj}(X,S,\mathbb{C}^{\ell}) = P^{+}(X,S,\mathbb{C}^{\ell})$ for some $\ell \geqslant 2$ if and only if $P^{+}_{proj}(X,S,\mathbb{C}^{m}) = P^{+}(X,S,\mathbb{C}^{m})$ for every $2 \leqslant m \leqslant \ell$.*

Proof. The converse is immediate. If it holds for an ℓ, then for any $2 \leqslant m \leqslant \ell$ and kernel K in $P^{+}_{proj}(X,S,\mathbb{C}^{m})$ we extend it to have $M_{\ell}(\mathbb{C})$ values by adding 0 in the remaining $\ell^2 - m^2 + (\ell - m)$ off diagonal coordinates and on the remaining $\ell - m$ diagonal coordinates we put an arbitrary $P^{+}(X,S)$ kernel (like $\langle Ke_1, e_1\rangle$). This new kernel is in $P^{+}_{proj}(X,S,\mathbb{C}^{\ell})$, hence in $P^{+}(X,S,\mathbb{C}^{\ell})$, and from that we obtain that the original kernel K is in $P^{+}(X,S,\mathbb{C}^{m})$. $\square$

Another simple statement that we use, which is independent of the setting of kernels unitarily invariant by a semigroup of functions, but it will simplify the construction of the counterexample is the following

LEMMA 2. *Let $K : X \times X \to M_2(\mathbb{C})$ be a kernel. Then K is (strictly) positive definite if and only if for every finite quantity of distinct points $x_1, x_2, \ldots, x_n \in X$, the following matrix in $M_{2n}(\mathbb{C})$*

$$\begin{bmatrix} [K_{11}(x_i,x_j)]_{i,j=1}^{n} & [K_{12}(x_i,x_j)]_{i,j=1}^{n} \\ [K_{21}(x_i,x_j)]_{i,j=1}^{n} & [K_{22}(x_i,x_j)]_{i,j=1}^{n} \end{bmatrix},$$

is positive semidefinite (definite).

We denote by $Z(S)$ the center of the semigroup S, that is the abelian semigroup

$$Z(S) := \{\phi \in S : \phi\psi = \psi\phi, \text{ for all } \psi \in S\}.$$

Note that $Z(S)$ is never empty, being the identity function $i : X \to X$ an example.

THEOREM 1. *Let $k \in P(X,S)$ and $\phi \in Z(S)$. The matrix valued kernel $K : X \times X \to M_2(\mathbb{C})$ given by*

$$K(x,y) = [K_{ij}]_{i,j=1}^{2} = \begin{bmatrix} k(\phi(x),\phi(y)) & k(\phi(x),y) \\ k(x,\phi(y)) & k(x,y) \end{bmatrix}, \quad x,y \in X,$$

belongs to $P(X,S,\mathbb{C}^2) \setminus P^{+}(X,S,\mathbb{C}^2)$. Also, if $P^{+}(X,S)$ is non empty then all functions in S are injective.

Proof. The kernel K is continuous because both the kernel k and the function ϕ are continuous. The kernel K is invariant by the semigroup S because if $x,y \in X$ and $\psi \in S$, we have that

$$K_{12}(\psi(x),\psi(y)) = k(\phi(\psi(x)),\psi(y)) = k(\psi(\phi(x)),\psi(y)) = k(\phi(x),y) = K_{12}(x,y).$$

In the second equality we have used the fact that $\phi \in Z(S)$ and in the third one that k is invariant by the semigroup S. Similarly for the other 3 entries of K. Next, let us verify the positive definiteness of K. For distinct points $x_1, \ldots, x_n \in X$, we have that

$$\begin{bmatrix} [k(\phi(x_i), \phi(x_j))]_{i,j=1}^n & [k(\phi(x_i), x_j)]_{i,j=1}^n \\ [k(x_i, \phi(x_j))]_{i,j=1}^n & [k(x_i, x_j)]_{i,j=1}^n \end{bmatrix} = [k(y_i, y_j)]_{i,j=1}^{2n},$$

where

$$y_i = \begin{cases} \phi(x_i) & \text{if } i = 1, \ldots, n \\ x_{i-n} & \text{if } i = n+1, \ldots, 2n \end{cases}$$

Thus, by Lemma 2 the positive definiteness of K follows from that of k. In order to see that K is not strictly positive definite, we consider two cases. Given an arbitrary $x \in X$, either $\phi(x) = x$, and then

$$K(x,x) = \begin{bmatrix} k(x,x) & k(x,x) \\ k(x,x) & k(x,x) \end{bmatrix}$$

and K is obviously not strictly positive definite, or $\phi(x) \neq x$, defining $x_1 = x$ and $x_2 = \phi(x)$, we have that by Lemma 2, the matrices

$$[K(x_i, x_j)]_{i,j=1}^2, \quad \begin{bmatrix} k(\phi(x), \phi(x)) & k(\phi(x), \phi^2(x)) & k(\phi(x), x) & k(\phi(x), \phi(x) \\ k(\phi^2(x), \phi(x)) & k(\phi^2(x), \phi^2(x)) & k(\phi^2(x), x) & k(\phi^2(x), \phi(x)) \\ k(x, \phi(x)) & k(x, \phi^2(x)) & k(x,x) & k(x, \phi(x)) \\ K(\phi(x), \phi(x)) & k(\phi(x), \phi^2(x)) & k(\phi(x), x) & k(\phi(x), \phi(x)) \end{bmatrix}$$

have rank < 4.

For the second part, if by an absurd there is an $k \in P^+(X, S, \mathbb{C})$ but there is an $\phi \in S$ that is non injective, taking distinct $x_1, x_2 \in X$ such that $\phi(x_1) = \phi(x_2) = z$, the matrix

$$\begin{bmatrix} k(x_1,x_1) & k(x_1,x_2) \\ k(x_2,x_1) & k(x_2,x_2) \end{bmatrix} = \begin{bmatrix} k(\phi(x_1), \phi(x_1)) & k(\phi(x_1), \phi(x_2)) \\ k(\phi(x_2), \phi(x_1)) & k(\phi(x_2), \phi(x_2)) \end{bmatrix} = \begin{bmatrix} k(z,z) & k(z,z) \\ k(z,z) & k(z,z) \end{bmatrix}$$

is non invertible, which is an absurd. □

Several results in dynamical system theory are related to the periodic behavior (or lack of) for the iterations of a function $\sigma : X \to X$, for instance on Sharkovskii's Theorem and the logistic function behavior in [18]. To proceed, we introduce a new definition, based on this terminology.

DEFINITION 2. Let X be a topological space. A function $\phi \in C(X,X)$ is said to be aperiodic if $\phi^m(x) \neq x$ for all $x \in X$ and $m \in \mathbb{N}$.

Note that the existence of an aperiodic function is more restrictive than demanding that the semigroup S is non torsion, that is, that the functions $\phi^m \neq i_X$ for every $m \in \mathbb{N}$.

The next lemma and its corollary will help us understand how the set

$$\{y_1, y_2, \ldots, y_{2n}\} := \{\phi(x_1), \phi(x_2), \ldots \phi(x_n), x_1, x_2, \ldots x_n\}.$$

that appeared on the proof of Theorem 1 behaves if ϕ is an aperiodic injective function in $Z(S)$.

LEMMA 3. *Suppose that S is a semigroup of functions in $C(X,X)$ and $\phi \in Z(S)$ is an aperiodic injective function. For any distinct points $x_1, x_2, \dots x_n \in X$, consider the set $F := \{i \in \{1,2,\dots,n\}, \quad \phi(x_i) \in \{x_1, x_2, \dots x_n\}\}$ and the function $\tau : F \to \{1,2,\dots,n\}$ for which $x_{\tau(i)} = \phi(x_i)$. Then the set F and the function τ are well defined and satisfy:*

(i) τ is injective

(ii) $0 \leqslant |F| \leqslant n-1$.

(iii) If $i \in F$ there exists $q \in \mathbb{N}$, which depends on i, where $i, \tau(i), \tau^2(i), \dots \tau^q(i) \in F$, but $\tau^{q+1}(i) \notin F$.

Proof. The function τ is well defined because the points $x_1, \dots, x_n$ are distinct and is injective because ϕ is an injective function.

In order to prove (ii), suppose by contradiction that $|F| = n$. In particular, the function $\tau : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ is bijective. Since the group of bijective functions over $\{1,2,\dots,n\}$ is finite, there will be an $M \in \mathbb{N}$ where $\tau^M = I_n$, the identity function over $\{1,2,\dots,n\}$. Note that $M \neq 1$ because ϕ is aperiodic, so $M \geqslant 2$, but

$$x_1 = x_{\tau^M(1)} = \phi(x_{\tau^{M-1}(1)}) = \dots = \phi^{M-1}(x_{\tau(1)}) = \phi^M(x_1)$$

but then $\phi^M(x_1) = x_1$, which is an absurd because ϕ is aperiodic.

The proof of (iii) is similar to the one we presented in (ii). Suppose by contradiction that exists $i \in F$ for which $\tau^q(i) \in F$ for all $q \in N$. Since the set $\{x_1, \dots, x_n\}$ is finite, there will exist natural numbers $p < q$, where $x_{\tau^p(i)} = x_{\tau^q(i)}$. But,

$$\phi^{q-p}(x_{\tau^p(i)}) = \phi^{q-p}(\phi^p(x_i)) = \phi^q(x_i) = x_{\tau^q(i)} = x_{\tau^p}(i)$$

which is an absurd because ϕ is aperiodic. $\square$

We restate Lemma 3 in a more convenient way for the proof of our main result. The proof is omitted.

COROLLARY 1. *Under the same hypotheses of Lemma 3, there exists $m \in \mathbb{Z}_+$ and $p \in \mathbb{N}$, with $m = |F|$ and $p = n - |F|$, $m + 2p$ distinct points $z_\alpha \in X$, such that:*

(i) $\{\phi(x_1), \phi(x_2), \dots \phi(x_n), x_1, x_2, \dots x_n\} = \{z_1, \dots, z_m, z_{m+1}, \dots, z_{m+p}, z_{m+p+1}, \dots, z_{m+2p}\}$.

(ii) $\{z_1, \dots, z_m\} := \{\phi(x_i), i \in F\} = \{x_{\tau(i)}, i \in F\}$.

(iii) $\{z_{m+1}, \dots, z_{m+p}\} := \{\phi(x_\eta), \eta \notin F\}$.

(iv) $\{z_{m+p+1}, \dots, z_{m+2p}\} := \{x_i, i \notin \tau(F)\}$.

We emphasize that the sets in (iii) and (iv) have the same number of elements because τ is an injective function.

Now we are able to prove the main result of this paper.

THEOREM 2. *Suppose that S is a semigroup of functions in $C(X,X)$, there exists an aperiodic function $\phi \in Z(S)$ and that $P^+(X,S)$ is non empty. Then, for any $\ell \geqslant 2$*

$$P^+(X,S,\mathbb{C}^\ell) \subsetneq P^+_{Proj}(X,S,\mathbb{C}^\ell)$$

Proof. By Lemma 1, we only have to show an example for $\ell = 2$.

Let $k \in P^+(X,S)$ and $\phi \in Z(S)$ for which $\phi^m(x) \neq x$ for all $m \in \mathbb{N}$ and $x \in X$. Theorem 1 asserts that the kernel $K : X \times X \to M_2(\mathbb{C})$ given by

$$K(x,y) = \begin{bmatrix} k(\phi(x),\phi(y)) & k(\phi(x),y) \\ k(x,\phi(y)) & k(x,y) \end{bmatrix}, \quad x,y \in X,$$

belongs to $P(X,S,\mathbb{C}^2) \setminus P^+(X,S,\mathbb{C}^2)$ and that ϕ is injective. Next, we show that $K \in P^+_{Proj}(X,S,\mathbb{C}^2)$. Fix $v \in \mathbb{C}^2 \setminus \{(0,0)\}$ and arbitrary distinct points $x_1,\dots,x_n$ in X. Let $y_1,\dots,y_{2n}$ as in the proof of Theorem 1. We will show $[K_v(x_i,x_j)]_{i,j=1}^n$ is positive definite. If $c_1,\dots,c_n \in \mathbb{C}$ and

$$0 = \sum_{i,j=1}^{n} c_i\overline{c_j}K_v(x_i,x_j),$$

then, we have that

$$\begin{aligned} 0 &= \sum_{i,j=1}^{n} c_i\overline{c_j}[v_1\overline{v_1}k(\phi(x_i),\phi(x_j)) + v_1\overline{v_2}k(\phi(x_i),x_j) + v_2\overline{v_1}k(x_i,\phi(x_j)) + v_2\overline{v_2}k(x_i,x_j)] \\ &= \sum_{i,j=1}^{2n} d_i\overline{d_j}k(y_i,y_j), \end{aligned}$$

where $d_i = v_1c_i$ and $d_{i+n} = v_2c_i$ for $1 \leqslant i \leqslant n$. Taking into account Lemma 3 and Corollary 1, we can rewrite this double sum as:

$$\sum_{\alpha,\beta=1}^{m+2p} e_\alpha\overline{e_\beta}k(z_\alpha,z_\beta) = 0,$$

where:

(I) $1 \leqslant \alpha \leqslant m$, $e_\alpha = v_1c_i + v_2c_{\tau(i)}$, for some $i \in F$.

(II) $m+1 \leqslant \alpha \leqslant m+p$, $e_\alpha = v_1c_i$, for some $i \notin F$.

(III) $m+p+1 \leqslant \alpha \leqslant m+2p$, $e_\alpha = v_2c_i$, for some $i \notin \tau(F)$.

But the kernel k is strictly positive definite and the $m+2p$ points z_α are distinct, so $e_\alpha = 0$ for all $1 \leqslant \alpha \leqslant m+2p$. We separete the proof in the case that $v_1 \neq 0$ and the case that $v_1 = 0$.

If $v_1 \neq 0$, since $p \geqslant 1$, equation (II) implies that $c_i = 0$ for all $i \notin F$. If $i \in F$, by the relation (iii) in Lemma 3, there exist $q \in \mathbb{Z}_+$, which depends on i, for which $i, \tau(i), \tau^2(i), \dots, \tau^q(i) \in F$, but $\tau^{q+1} \notin F$. In particular, equation (I) implies that

$$0 = v_1c_{\tau^q(i)} + v_2c_{\tau^{q+1}(i)} = v_1c_{\tau^q(i)},$$

so $c_{\tau^q(i)} = 0$. In case $q > 1$, we use equation (I) again in order to obtain

$$0 = v_1c_{\tau^{q-1}(i)} + v_2c_{\tau^q(i)} = v_1c_{\tau^{q-1}(i)},$$

so $c_{\tau^{q-1}(i)} = 0$. After finitely many similar steps we conclude that $c_i = 0$, and so, the kernel K_v is strict.

If $v_1 = 0$, then $v_2 \neq 0$. Equation (I) imply that $0 = v_2 c_\eta$, for all $\eta \in \tau(F)$ while equation (III) imply that $0 = v_2 c_\eta$ for all $\eta \notin \tau(F)$, and again, $c_\eta = 0$ for all η and the kernel K_v is strictly positive definite. $\square$

REMARK 1. If the function $\phi \in Z(S)$ in Theorem 1 is not aperiodic or is not injective, then it can be proved that $K \notin P^+_{Proj}(X,S,\mathbb{C}^2)$.

The fact that $\phi \in Z(S)$ is an essential requirement. This can be seen on a very familiar example, the set of kernels defined on $\mathbb{R}$ and invariant by the group of functions $\psi_{i,w}(x) := (-1)^i x + w$, $i \in \{0,1\}$ and $w \in \mathbb{R}$, which by the coments made at the introduction, are the radial kernels in $\mathbb{R}^1$. The center of this group only contains the identity function $\psi_{0,0}$. Furthermore, fixed a function $\psi_{i,w} \in S$, $w \neq 0$, if a kernel $k : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \in P(\mathbb{R},S)$ is such that

$$k(x, \psi_{i,w}(y)) = k(\phi(x), \psi_{i,w}(\phi(y)))$$

for all $\phi \in S$ and $x, y \in \mathbb{R}$ (hence the kernel in Theorem 1 would be well defined even if $\psi_{i,w}$ is not in $Z(S)$), then this kernel is a nonnegative constant function.

2.1. Abelian groups

In this subsection we completely characterize the locally compact Abelian groups G together with the group of functions $S = \{\phi_g : G \to G,\ \phi(x) = gx,\ g \in G\}$, for which $P^+_{proj}(G,S,\mathbb{C}^\ell) = P^+(G,S,\mathbb{C}^\ell)$, for any $\ell \in \mathbb{N}$. These kernels are usually called on the literature as translational invariant. It is worth mentioning that $P_{proj}(G,S,\mathbb{C}^\ell) = P(G,S,\mathbb{C}^\ell)$, any $\ell \in \mathbb{N}$ and every locally compact Abelian group G, this result is a consequence of Theorem III.3 page 20 in [23].

Except for the finite Abelian groups, we will not need the complete description of the positive definite kernels $P(G,S)$ or the harmonic analysis on locally compact Abelian groups. The interested reader may look at [26] or [30] for more information on them.

In order to prove the characterization, first note that all functions $\phi_g \in S$ are injective and $Z(S) = S$. Moreover, $\phi_g^m(x) = g^m x$, so it exists an aperiodic function ϕ_g in S if and only if the group G admits an element of infinite order, that is, if G is not a torsion group. In order to use a notation more common in the literature to present this type of kernels, we omit the S term.

A directly application of Theorem 2 leads to the following result:

COROLLARY 2. *Let G be an Abelian locally compact group that is non torsion and for which $P(G,\mathbb{C})$ is non empty. Then*

$$P^+(G,\mathbb{C}^\ell) \subsetneq P^+_{Proj}(G,\mathbb{C}^\ell),$$

for any $\ell \geqslant 2$.

The d-dimensional torus T_d, defined as

$$T_d := \{x \in \mathbb{R}^d : -\pi \leqslant x_j < \pi; j = 1,2,\ldots,d\},$$

is a locally compact Abelian group and is non torsion. There exist strictly positive definite translational invariant kernels on it like

$$(x,y) \in T_d \times T_d \to \prod_{m=1}^{d} \frac{2}{2 - e^{i(x_m - y_m)}} = \sum_{\alpha \in \mathbb{Z}_+^d} \frac{1}{2^{|\alpha|}} e^{-i\alpha x} \overline{e^{-i\alpha y}} \in \mathbb{C}.$$

The complete characterization of $P(T_d,\mathbb{C})$ can be found in [28], while the characterization for $P^+(T_d,\mathbb{C})$ was given in Theorem 3.5 [11].

In Corollary 2.4 of [11], it was also proved a very interesting result, if G is compact then $P^+(G,\mathbb{C})$ is non empty if and only if G is metrizable.

The d dimensional Euclidean space $\mathbb{R}^d$ is also non torsion (any nonzero element is an example), there exists a strictly positive definite translation invariant kernel on it, like the Gaussian kernel

$$(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \to e^{-\|x-y\|^2},$$

see Section 3 in [13] for this and other properties for this kernel. Then, the kernel

$$(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \to \begin{bmatrix} e^{-\|x-y\|^2} & e^{-\|x+z-y\|^2} \\ e^{-\|x-y-z\|^2} & e^{-\|x-y\|^2} \end{bmatrix}, \quad z \in \mathbb{R}^d \setminus \{0\},$$

belongs to $P^+_{Proj}(\mathbb{R}^d,\mathbb{C}^2) \setminus P^+(\mathbb{R}^d,\mathbb{C}^2)$. Note that this kernel is invariant by translation but is not invariant by rotation. As described by the family of kernels (3) at the Introduction, for $m \geqslant 2$, the class of kernels invariant by both operations is well behaved with respect to the scalar valued projections.

Now, we aim to prove the case where G is a torsion group. In this case, given a finite quantity of distinct points $x_1,\ldots,x_n \in G$, the group generated by these elements

$$\langle x_1,\ldots x_n\rangle = \{x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}, \quad p_1,\ldots p_n \in \mathbb{N}\}$$

is finite and Abelian. Given a kernel $K : G \times G \to M_\ell(\mathbb{C})$, it is strictly positive definite if and only if the kernel K restricted to the set $\langle x_1,\ldots x_n\rangle$ is strictly positive definite for every finite quantity of distinct points $x_1,\ldots x_n \in G$. But, by the fundamental Theorem of finite Abelian groups (see Section 1.8 in [19]), a set like $\langle x_1,\ldots x_n\rangle$ is isomorphic to a direct sum of $\mathbb{Z}_q$ groups, that means

$$\langle x_1,\ldots x_n\rangle \simeq \mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_l}.$$

Now we prove a version of Corollary 2 to a group like $G = \mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_l}$, it turns out to be a completely different result, and the general case will follow from this case by the previous comments. Before that, we recall the characterizations of $P(G,\mathbb{C})$ and $P^+(G,\mathbb{C})$, which can be found in [11] at Lemma 2.6.

THEOREM 3. *Let* $G = \mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_l}$ *and* $\psi : G \to \mathbb{C}$. *The kernel* $K(x,y) := \psi(xy^{-1})$ *is positive definite if and only if*

$$\psi(x) = \sum_{g \in G} a_g \xi_g(x),$$

where $a_g \geqslant 0$ *and* $\xi_g(x) = \prod_{r=1}^{l} e^{2\pi i g_r x_r / q_r}$ *for all* $g \in G$. *The representation is unique, moreover, the kernel is strictly positive definite if and only if* $a_g > 0$ *for all* $g \in G$.

LEMMA 4. *Let* $q_1, \ldots, q_l \in \{2,3,\ldots\}$, *and* $G := \mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_l}$. *Then* $P^+(G,\mathbb{C})$ *is non empty and*

$$P^+_{Proj}(G,\mathbb{C}^\ell) = P^+(G,\mathbb{C}^\ell),$$

for any $\ell \geqslant 2$.

Proof. Let $\psi : G \to \mathbb{C}$ be such that the kernel $K(x,y) := \psi(xy^{-1})$ is an element of $P^+_{Proj}(G,\mathbb{C}^\ell)$, $\ell \in \mathbb{N}$. By a similar argument as done in Theorem III.3 at page 20 in [23], we have that $K \in P(G,\mathbb{C}^\ell)$ and it has the following decomposition

$$\psi(xy^{-1}) = K(x,y) = \sum_{g \in G} A_g(K) \xi_g(x) \overline{\xi_g(y)},$$

where $A_g(K) \in M_\ell(\mathbb{C})$ is a positive semidefinite matrix. But, since $K \in P^+_{Proj}(G,\mathbb{C}^\ell)$, for every $v \in \mathbb{C}^\ell \setminus \{0\}$, the kernel

$$K_v(x,y) = \langle [\sum_{g \in G} A_g(K) \xi_g(x) \overline{\xi_g(y)}] v, v \rangle = \sum_{g \in G} \langle A_g(K) v, v \rangle \xi_g(x) \overline{\xi_g(y)},$$

is strictly positive definite. Theorem 3 implies that $\langle A_g(K) v, v \rangle > 0$, thus the matrix $A_g(K)$ is positive definite. Hence, if $x_1, \ldots, x_n \in G$ are distinct and $v_1, \ldots, v_n \in \mathbb{C}^\ell$, are such that

$$\begin{aligned} 0 = \sum_{i,j=1}^{n} \langle K(x_i,x_j) v_i, v_j \rangle &= \sum_{i,j=1}^{n} \sum_{g \in G} \langle A_g(K) v_i, v_j \rangle \xi_g(x_i) \overline{\xi_g(x_j)} \\ &= \sum_{g \in G} \langle A_g(K) \sum_{i=1}^{n} v_i \xi_g(x_i), \sum_{j=1}^{n} v_j \xi_g(x_j) \rangle, \end{aligned}$$

then $\sum_{i=1}^{n} v_i \xi_g(x_i) = 0$, for all $g \in G$. After introducing coordinates we obtain that $v_i = 0$ for all i, and then $K \in P^+(G,\mathbb{C}^\ell)$. $\square$

COROLLARY 3. *Let* G *be an Abelian locally compact torsion group. Then*

$$P^+_{Proj}(G,\mathbb{C}^\ell) = P^+(G,\mathbb{C}^\ell),$$

for any $\ell \in \mathbb{N}$.

2.2. Additional examples

Another class of kernels for which Theorem 2 can be applied are the isotropic kernels in complex spheres. Let $X = \Omega^q$, be the unit complex sphere in $\mathbb{C}^q$, the kernel K continuous and also isotropic in the sense that

$$K(Qx, Qy) = K(x,y), \quad x,y \in \Omega^q; Q \in \mathscr{U}(q),$$

where $\mathscr{U}(q)$ is the set of all unitary transformations on $\mathbb{C}^q$. The characterization of the positive definite scalar valued kernels fulfilling this condition was achieved in [22], while the strictly positive definite case was proved in [15] ($q \geqslant 2$) and [21] ($q = 1$). Similar to the real sphere, there is also the limiting case, of kernels defined on Ω^∞, the unit complex sphere of an infinity dimensional complex Hilbert space, K is continuous and the isotropy is the invariance of the kernel for all linear isometries in it and can be found in [10]. The characterization of the positive definite matrix valued case is a consequence of the results in Section 8 at [6].

We can apply Theorem 2 for this type of kernel, because the unitary matrix $e^{i\theta}I$, where $\theta \notin \mathbb{Q}\pi$ and I is the identity matrix in C^q, defines an aperiodic function in Ω^q that is in the center of $\mathscr{U}(q)$.

3. Adjointly invariant kernels

In this section we prove an analogous of Theorem 2 to a class of kernels with a different type of symmetry. Due to the similarities in the arguments, we sometimes omit them.

We recall that as involution on a semigroup S is a function $* : S \to S$ for which $*(*(\phi)) = \phi$, for every $\phi \in Z$, and we write ϕ^* instead of $*(\phi)$. Additionally, for every $\phi, \psi \in S$ we must have that $(\phi\psi)^* = \psi^*\phi^*$.

DEFINITION 3. Let X be a topological space and a semigroup of continuous functions $S \subset C(X,X)$. We say that the kernel K is adjointly invariant by the semigroup S if there exists an involution function $* : S \to S$, for which

$$K(x, \phi(y)) = K(\phi^*(x), y),$$

for all $x, y \in X$ and $\phi \in S$.

Unlike Definition 1, we assume from the beginning in Definition 3 that we have a semigroup of functions because the involution needs to be defined for all elements of the semigroup. Note that for the identity involution we could have used the approach in Definition 1.

Similar to unitarily invariant kernels, given a matrix valued continuous kernel K that is adjointly invariant by a semigroup of continuous functions with involution S, if it is positive definite we use the notation $K \in P^*(X, S, \mathbb{C}^\ell)$. If in addition the kernel is strictly positive definite we use the notation $K \in P^{+,*}(X, S, \mathbb{C}^\ell)$. Similarly, if all of the scalar valued projections of a continuous kernel K that is unitarily invariant by a semigroup with involution of continuous functions with S are scalar valued positive definite kernels, we use the notation $K \in P^{*}_{proj}(X,S,\mathbb{C}^{\ell})$. Likewise if in addition all of the scalar valued projections of the kernel K are scalar valued strictly positive definite kernels, we use the notation $K \in P^{*,+}_{proj}(X,S,\mathbb{C}^{\ell})$.

The following Lemma is a version of Lemma 1 to the context of adjointly invariant kernels. The proof is omitted due to its similarities.

LEMMA 5. *Let X be a topological space, $(S,*)$ be a semigroup of continuous functions on X with an involution. Then $P^{+,*}_{proj}(X,S,\mathbb{C}^{\ell}) = P^{+,*}(X,S,\mathbb{C}^{\ell})$ for some $\ell \geqslant 2$ if and only if $P^{+,*}_{proj}(X,S,\mathbb{C}^{m}) = P^{+,*}(X,S,\mathbb{C}^{m})$ for every $2 \leqslant m \leqslant \ell$.*

The following Theorem is a version of Theorem 1 to the context of adjointly invariant kernels.

THEOREM 4. *Let $k \in P^{*}(X,S,\mathbb{C})$ and $\phi \in Z(S)$. The matrix valued kernel $K : X \times X \to M_2(\mathbb{C})$ given by*

$$K(x,y) = [K_{ij}]_{i,j=1}^{2} = \begin{bmatrix} k(\phi(x),\phi(y)) & k(\phi(x),y) \\ k(x,\phi(y)) & k(x,y) \end{bmatrix}, \quad x,y \in X,$$

belongs to $P^{*}(X,S,\mathbb{C}^2) \setminus P^{+,*}(X,S,\mathbb{C}^2)$.

Proof. The kernel K is continuous because both the kernel k and the function ϕ are continuous. Note that since $\phi \in Z(S)$ then $\psi^*\phi = \phi\psi^*$ for all $\psi \in S$, and then $\psi\phi^* = (\phi\psi^*) = (\psi^*\phi) = \phi^*\psi$ for all $\psi \in S$, thus $\phi^* \in Z(S)$. The kernel K is adjointly invariant by the semigroup S because if $x,y \in X$ and $\psi \in S$, we have that

$$\begin{aligned} K_{11}(x,\psi(y)) &= k(\phi(x),\phi(\psi(y))) = k(\phi(x),\psi(\phi(y))) = k(\psi^*(\phi(x)),\phi(y)) \\ &= k(\phi(\psi^*(x)),\phi(y)) = K_{11}(\psi^*(x),y). \end{aligned}$$

It follows that the K_{11} is adjointly invariant by the semigroup S. The rest of the arguments are the same as the ones in the proof of Theorem 1. □

Unlike Theorem 1, it is not clear if the non emptiness of $P^{+,*}(X,S)$ implies that all functions in S are injective.

Note that in Lemma 3 and Corollary 1 we did not made any use of the kernel, actually we only used the properties of the functions on the semigroup, being so, they are still valid in this new setting and we immediately have the proof of Theorem 2 to the context of adjointly invariant by a semigroup of functions.

THEOREM 5. *Suppose that $(S,*)$ is a semigroup with involution of functions in $C(X,X)$, there exists an aperiodic injective function $\phi \in Z(S)$ and that $P^{+,*}(X,S)$ is non empty. Then, for any $\ell \geqslant 2$*

$$P^{+,*}(X,S,\mathbb{C}^{\ell}) \subsetneq P^{+,*}_{Proj}(X,S,\mathbb{C}^{\ell}).$$

Other examples for adjointly invariant kernels are:

— $X = \mathbb{R}^m$, $\phi_z(x) = x+z$ and $(\phi_z)^* = \phi_{-z}$. This would lead to Bochner (or translational invariant) kernels in $\mathbb{R}^m$.

— $X = \mathbb{R}^m$, $\phi_z(x) = x+z$ and $(\phi_z)^* = \phi_z$. This would lead to characterizing which functions $h : \mathbb{R}^m \to \mathbb{R}$ the kernel $h(x+y)$ is positive definite, which is proved in Theorem 6.5.11 in [5].

3.1. Dot product kernels

In this subsection we explain in details how dot product kernels can be understood as an adjointly invariant kernel and why we had to add the identity matrix to obtain Example 1 for this class of kernels.

LEMMA 6. *Let $\mathscr{H}$ be a real Hilbert space and a continuous kernel $k : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$. If*

$$k(Ax,y) = k(x,A^t y), \quad x,y \in \mathscr{H}, \quad A \in \mathscr{L}(\mathscr{H})$$

then, there exists a continuous function $h : \mathbb{R} \to \mathbb{C}$ for which $k(x,y) = h(\langle x,y\rangle)$.

Proof. Let x,y be nonzero elements of $\mathscr{H}$. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a complete orthonormal basis for $\mathscr{H}$.

If x,y are linearly dependent, pick an $\mu \in \Lambda$ and operator A for which $A(e_\mu) := x$ and $A(e_\lambda)$ is orthogonal with respect to x for all $\lambda \neq \mu$. Hence, $A^t(y) = \langle x,y\rangle e_\mu$ and then

$$k(x,y) = k(Ae_\mu, y) = k(e_\mu, A^t y) = k(e_\mu, \langle x,y\rangle e_\mu).$$

If x,y are linearly independent, pick distinct $\mu,\eta \in \Lambda$, and an operator A for which $A(e_\mu) = x$, $A(e_\eta) = x - \langle x,y\rangle y/\|y\|^2$ and $A(e_\lambda)$ is orthogonal with respect to x and y for all $\lambda \neq \mu,\eta$. Hence, $A^t(y) = \langle x,y\rangle e_\mu$ and then

$$k(x,y) = k(Ae_\mu, y) = k(e_\mu, A^t y) = k(e_\mu, \langle x,y\rangle e_\mu).$$

By the continuity of k, we may include the cases where either x or y are zero. $\square$

The references and results regarding the real product kernels were given at the class of functions (5) in Section 1.

With similar arguments as Lemma 6 it is possible to prove the following characterization for when $\mathscr{H}$ is a complex Hilbert space.

LEMMA 7. *Let $\mathscr{H}$ be a complex Hilbert space and a continuous kernel $K : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$. If*

$$K(Ax,y) = K(x,A^* y), \quad x,y \in \mathscr{H}, \quad A \in \mathscr{L}(\mathscr{H})$$

then, there exists a continuous function $h : \mathbb{C} \to \mathbb{C}$ for which $K(x,y) = h(\langle x,y\rangle)$.

Regarding complex dot product kernels, the characterization of which functions h generates a positive definite kernel is obtained in [10] when $\dim(H) = \infty$, and generalized to $\dim_{\mathbb{C}}(H) \geqslant 3$ in [24]. We are not aware of a complete characterization for the cases $\dim_{\mathbb{C}} = 1,2$. The strictly positive definite case was also proved in [24] for $\dim_{\mathbb{C}}(H) \geqslant 3$. The positive definite case is well behaved with respect to the scalar valued projections.

Usually the dot product kernels are presented in a different way, as the characterization of which functions $h : \mathbb{R} \to \mathbb{C}$ (or $h : \mathbb{C} \to \mathbb{C}$) are such that for any positive semidefinite real (or complex) matrix A, the matrix $h(A)$, being pointwise defined, is positive semidefinite, that is, which functions preserves positivity. Along this line several important results were achieved in [2], [3] and [4].

By taking k and $\phi(x) = rx$, where r is not a root of unity ($r \neq -1,0,1$ in the real case and $r \neq e^{2\pi i q}$, $q \in \mathbb{Q}$, in the complex case), the counterexample in Theorem 5 can be applied to the set $\mathscr{H} \setminus \{0\}$, as ϕ is aperiodic on this set. If we add the point $0 \in \mathscr{H}$, the matrix valued kernel defined in Theorem 4 does not belong to $P^{+,*}_{proj}(\mathscr{H}, \mathscr{L}(\mathscr{H}), \mathbb{C}^2)$, because the matrix $K(0,0)$ is not invertible.

On the following Lemma, we adapt the proof of Theorem 4 and Theorem 5 to obtain examples that the matrix valued dot product kernels are not well behaved with respect to its scalar valued projections.

LEMMA 8. *Let $k : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ be a continuous strictly positive definite kernel that is adjointly invariant by $\mathscr{L}(\mathscr{H})$, where $\mathscr{H}$ is either a real or complex Hilbert space. Then, if r is not a root of unity and $\phi(x) = rx \in \mathscr{L}(\mathscr{H})$, the matrix valued kernel*

$$K(x,y) + k(0,0)I = \begin{bmatrix} k(rx,ry) + k(0,0) & k(rx,y) \\ k(x,ry) & k(x,y) + k(0,0) \end{bmatrix}, \quad x,y \in \mathscr{H}$$

belongs to $P^*(\mathscr{H}, \mathscr{L}(\mathscr{H}), \mathbb{C}^2) \setminus P^{+,*}(\mathscr{H}, \mathscr{L}(\mathscr{H}), \mathbb{C}^2)$ and also belongs to $P^{+,*}_{proj}(\mathscr{H}, \mathscr{L}(\mathscr{H}), \mathbb{C}^2)$, where K is the kernel defined in Theorem 4.

Proof. The kernel $K \in P^*(\mathscr{H}, \mathscr{L}(\mathscr{H}), \mathbb{C}^2)$ by Theorem 4 while the kernel $k(0,0)I \in P^*(\mathscr{H}, \mathscr{L}(\mathscr{H}), \mathbb{C}^2)$ by a simple direct verification, thus their sum is also an element of $P^*(\mathscr{H}, \mathscr{L}(\mathscr{H}), \mathbb{C}^2)$.

The matrix valued kernel does not belong to $P^{+,*}(\mathscr{H}, \mathscr{L}(\mathscr{H}), \mathbb{C}^2)$ because the double sum over the the 3 points $0, x, rx$ (instead of 2 used in Theorem 4) and the corresponding vectors in order $(-1,1), (1,0), (0,-1)$ is zero.

Now, we prove that the matrix valued kernel is an element of $P^{+,*}_{proj}(\mathscr{H}, \mathscr{L}(\mathscr{H}), \mathbb{C}^2)$, and we divide the argument in three parts:

Part 1) The scalar valued kernel $l(x,y) := k(x,y) - k(0,0) \in P^*(\mathscr{H} \setminus \{0\}, \mathscr{L}(\mathscr{H}))$ is strictly positive definite in $\mathscr{H} \setminus \{0\}$:

The invariance is immediate, as for the kernel being strictly positive definite, let $x_1, \ldots, x_n \in \mathscr{H} \setminus \{0\}$ be distinct and the scalars $c_1, \ldots, c_n \in \mathbb{C}$ that are not all zero.

Define $c_0 := -\sum_{i=1}^n c_i$ and $x_0 := 0$, then

$$\sum_{i,j=0}^{n} c_i\overline{c_j}k(x_i,x_j) = c_0\overline{c_0} + c_0\sum_{j=1}^{n}\overline{c_j}k(0,x_j) + \overline{c_0}\sum_{i=1}^{n} c_ik(x_i,0) + \sum_{i,j=1}^{n} c_i\overline{c_j}k(x_i,x_j)$$
$$= -c_0\overline{c_0}k(0,0) + \sum_{i,j=1}^{n} c_i\overline{c_j}k(x_i,x_j) = \sum_{i,j=1}^{n} c_i\overline{c_j}l(x_i,x_j),$$

because $k(x,0) = k(0,x) = k(0,0)$ for every $x \in \mathscr{H}$ by Lemma 6 and Lemma 7. As the kernel k is strictly positive definite in $\mathscr{H}$, the first double sum is a posite number, hence, l is a strictly positive definite kernel as well.

Part 2) The matrix valued kernel

$$L(x,y) := \begin{bmatrix} l(rx,ry) & l(rx,y) \\ l(x,ry) & l(x,y) \end{bmatrix}, \quad x,y \in \mathscr{H}\setminus\{0\}$$

belongs to $P^*(\mathscr{H}\setminus\{0\},\mathscr{L}(\mathscr{H}),\mathbb{C}^2)\setminus P^{+,*}(\mathscr{H}\setminus\{0\},\mathscr{L}(\mathscr{H}),\mathbb{C}^2)$ and to $P^{+,*}_{proj}(\mathscr{H}\setminus\{0\},\mathscr{L}(\mathscr{H}),\mathbb{C}^2)$ when r is not a root of unity:

This is a direct application of Theorem 4 and Theorem 5 because we are dealing with an aperiodic and injective function in $\mathscr{H}\setminus\{0\}$.

Part 3) Conclusion:

Let $v = (v_1,v_2) \neq (0,0)$. Then, for distinct points $x_0,\dots,x_n \in \mathscr{H}$, where $x_0 = 0$, and complex scalars $c_0,\dots,c_n$, the scalar valued kernel $[K+k(0,0)I]_v$ satisfies

$$\begin{aligned} &\sum_{i,j=0}^{n} c_i\overline{c_j}[K+k(0,0)I]_v(x_i,x_j) \\ &\sum_{i,j=0}^{n} c_i\overline{c_j}\left(K_v(x_i,x_j)+k(0,0)\|v\|^2\right) \\ =&\sum_{i,j=0}^{n} c_i\overline{c_j}\left(K_v(x_i,x_j)-k(0,0)|v_1+v_2|^2+k(0,0)|v_1+v_2|^2+k(0,0)\|v\|^2\right) \\ =&\sum_{i,j=1}^{n} c_i\overline{c_j}L_v(x_i,x_j)+\sum_{i,j=0}^{n} c_i\overline{c_j}\left(k(0,0)|v_1+v_2|^2+k(0,0)\|v\|^2\right) \\ =&\sum_{i,j=1}^{n} c_i\overline{c_j}L_v(x_i,x_j)+\left|\sum_{i=0}^{n} c_i\right|^2\left(k(0,0)|v_1+v_2|^2+k(0,0)\|v\|^2\right). \end{aligned}$$

Note that both terms are nonnegative. Since the first term does not include the element 0, Part 2 implies that it is zero if and only if all scalars $c_1,\dots,c_n$ are zero. On the other hand, if all scalars $c_1,\dots,c_n$ are zero the second term implies that $c_0 = 0$, which concludes that $K+k(0,0)I$ is in $P^{+,*}_{proj}(\mathscr{H},\mathscr{L}(\mathscr{H}),\mathbb{C}^2)$. $\square$

REMARK 2. The previous result can be generalized to an abstract setting with the same type of argument. Suppose that $\tilde{X} = X\cup\{0\}$ is a topological space and $(S,*)$ is a semigroup of functions in $C(\tilde{X},\tilde{X})$ with the identity for which:

i) $\phi(0) = 0$ for every $\phi \in S$.

ii) For every $x \in X$ its orbit satisfies $\{\phi(x), \phi \in S\} = X$.

iii) 0 is an accumulation point.

These three properties implies that if $k \in P^*(\tilde{X}, S)$ then $k(0,0) = k(0,x) = k(x,0)$ for any $x \in \tilde{X}$). Indeed, for arbitrary $y, z \in X$, by relation *ii*) there exists an $\psi \in S$ for which $\psi(y) = z$, hence by relation *i*) and the invariance

$$k(0,z) = k(0,\psi(y)) = k(\psi^*(0), y) = k(0,y).$$

The conclusion comes from the continuity of k and the fact that 0 is an accumulation point. By similar arguments as done in Lemma 8, for any $k \in P^*(\tilde{X}, S)$ and $\phi \in S$ the matrix valued kernel

$$(x,y) \in \tilde{X} \times \tilde{X} \to \begin{bmatrix} k(\phi(x), \phi(y)) + k(0,0) & k(\phi(x), y) \\ k(x, \phi(y)) & k(x,y) + k(0,0) \end{bmatrix},$$

belongs to $P^*(\tilde{X}, S, \mathbb{C}^2) \setminus P^{+,*}(\tilde{X}, S, \mathbb{C}^2)$. In addition, if $k \in P^{+,*}(\tilde{X}, S)$ and ϕ is an aperiodic injective function when restricted to X, the previous kernel is an element of $P^{+,*}_{proj}(\tilde{X}, S, \mathbb{C}^2)$.

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