Mathematical
Inequalities
& Applications
Volume 28, Number 2 (2025), 299–306 doi:10.7153/mia-2025-28-20

CONTINUOUS RANKIN BOUND FOR HILBERT AND BANACH SPACES

K. MAHESH KRISHNA

(*Communicated by I. Perić*)

Abstract. Let (Ω,μ) be a finite measure space and $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a normalized continuous Bessel family for a real Hilbert space $\mathscr{H}$. If the diagonal $\Delta := \{(\alpha,\alpha) : \alpha \in \Omega\}$ is measurable in the measure space $\Omega\times\Omega$, then we show that

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} \langle \tau_\alpha, \tau_\beta\rangle \geqslant \frac{-(\mu\times\mu)(\Delta)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}. \tag{1}$$

We call Inequality (1) as continuous Rankin bound. It improves 77 years old result of Rankin [*Ann. of Math., 1947*]. It also answers one of the questions asked by K. M. Krishna in the paper [Continuous Welch bounds with applications, *Commun. Korean Math. Soc., 2023*]. We also derive Banach space version of Inequality (1).

1. Introduction

In 1947, Rankin derived following result for a collection of unit vectors in $\mathbb{R}^d$.

THEOREM 1. (Rankin Bound) [12, 13] (Theorem 7.10 [17]) *If $\{\tau_j\}_{j=1}^n$ is a collection of unit vectors in $\mathbb{R}^d$, then*

$$\max_{1\leqslant j,k\leqslant n, j\neq k} \langle \tau_j, \tau_k\rangle \geqslant \frac{-1}{n-1}. \tag{2}$$

In particular,

$$\min_{1\leqslant j,k\leqslant n, j\neq k} \|\tau_j - \tau_k\|^2 \leqslant \frac{2n}{n-1}. \tag{3}$$

Striking feature of Inequalities (2) and (3) is that they do not depend upon the dimension d. Inequalities (2) and (3) play important roles in the study of packings of lines (which motivated to study the packings of planes) [6, 5], Kepler conjecture [8, 15], sphere packings [3, 18] and the geometry of numbers [4].

After the derivation of continuous Welch bounds in most general form, author of the paper [11] asked what is the version of Rankin bound for collections indexed by measure spaces. We are going to answer this in this paper.

Mathematics subject classification (2020): 42C15.
Keywords and phrases: Rankin bound, continuous Bessel family, Hilbert space, Banach space.

2. Continuous Rankin bound

We start by recalling the notion of continuous frames which are introduced independently by Ali, Antoine and Gazeau [1] and Kaiser [9]. In the paper, $\mathscr{H}$ denotes a real Hilbert space (need not be finite dimensional).

DEFINITION 1. [1, 9, 2, 7] Let (Ω,μ) be a measure space. A collection $\{\tau_\alpha\}_{\alpha\in\Omega}$ in a Hilbert space $\mathscr{H}$ is said to be a *continuous frame* (or generalized frame) for $\mathscr{H}$ if the following holds.

(i) For each $h\in\mathscr{H}$, the map $\Omega\ni\alpha\mapsto\langle h,\tau_\alpha\rangle\in\mathbb{K}$ is measurable.

(ii) There are $a,b>0$ such that

$$a\|h\|^2\leqslant\int_\Omega|\langle h,\tau_\alpha\rangle|^2\,d\mu(\alpha)\leqslant b\|h\|^2,\quad\forall h\in\mathscr{H}.$$

If we do not demand the first inequality in (ii), then we say it is a *continuous Bessel family* for $\mathscr{H}$. A continuous Bessel family $\{\tau_\alpha\}_{\alpha\in\Omega}$ is said to be normalized or unit norm if $\|\tau_\alpha\|=1$, $\forall\alpha\in\Omega$.

Given a continuous Bessel family, the analysis operator

$$\theta_\tau:\mathscr{H}\ni h\mapsto\theta_\tau h\in\mathscr{L}^2(\Omega);\quad\theta_\tau h:\Omega\ni\alpha\mapsto\langle h,\tau_\alpha\rangle\in\mathbb{K}$$

is a well-defined bounded linear operator. Its adjoint, the synthesis operator is given by

$$\theta_\tau^*:\mathscr{L}^2(\Omega)\ni f\mapsto\int_\Omega f(\alpha)\tau_\alpha\,d\mu(\alpha)\in\mathscr{H}.$$

By combining analysis and synthesis operators, we get the frame operator, defined as

$$S_\tau:=\theta_\tau^*\theta_\tau:\mathscr{H}\ni h\mapsto\int_\Omega\langle h,\tau_\alpha\rangle\tau_\alpha\,d\mu(\alpha)\in\mathscr{H}.$$

Note that the integrals are weak integrals (Pettis integrals [16]). With this machinery, we generalize Theorem 1.

THEOREM 2. (Continuous Rankin Bound) *Let (Ω,μ) be a finite measure space and $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a normalized continuous Bessel family for a real Hilbert space $\mathscr{H}$. If the diagonal $\Delta:=\{(\alpha,\alpha):\alpha\in\Omega\}$ is measurable in the measure space $\Omega\times\Omega$, then*

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta}\langle\tau_\alpha,\tau_\beta\rangle\geqslant\frac{-(\mu\times\mu)(\Delta)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}.\tag{4}$$

In particular,

$$\inf_{\alpha,\beta\in\Omega,\alpha\neq\beta}\|\tau_\alpha-\tau_\beta\|^2\leqslant 2\left(1+\frac{(\mu\times\mu)(\Delta)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\right).\tag{5}$$

Proof. Since $\mu(\Omega) < \infty$, $\chi_\Omega \in \mathscr{L}^2(\Omega)$ and

$$\int\limits_{(\Omega\times\Omega)\setminus\Delta} |\langle \tau_\alpha, \tau_\beta\rangle| \, d(\mu\times\mu)(\alpha,\beta) \leqslant \int\limits_{(\Omega\times\Omega)\setminus\Delta} \|\tau_\alpha\|\|\tau_\beta\| \, d(\mu\times\mu)(\alpha,\beta)$$
$$= (\mu\times\mu)((\Omega\times\Omega)\setminus\Delta) < \infty.$$

Now by using Fubini's theorem, we get

$$\begin{aligned}
0 &\leqslant \|\theta_\tau^*\chi_\Omega\|^2 = \langle \theta_\tau^*\chi_\Omega, \theta_\tau^*\chi_\Omega\rangle \\
&= \left\langle \int\limits_\Omega \chi_\Omega(\alpha)\tau_\alpha \, d\mu(\alpha), \int\limits_\Omega \chi_\Omega(\beta)\tau_\beta \, d\mu(\beta) \right\rangle = \left\langle \int\limits_\Omega \tau_\alpha \, d\mu(\alpha), \int\limits_\Omega \tau_\beta \, d\mu(\beta) \right\rangle \\
&= \int\limits_\Omega \int\limits_\Omega \langle \tau_\alpha, \tau_\beta\rangle \, d\mu(\alpha)\, d\mu(\beta) = \int\limits_{\Omega\times\Omega} \langle \tau_\alpha, \tau_\beta\rangle \, d(\mu\times\mu)(\alpha,\beta) \\
&= \int\limits_\Delta \langle \tau_\alpha, \tau_\beta\rangle \, d(\mu\times\mu)(\alpha,\beta) + \int\limits_{(\Omega\times\Omega)\setminus\Delta} \langle \tau_\alpha, \tau_\beta\rangle \, d(\mu\times\mu)(\alpha,\beta) \\
&= \int\limits_\Delta \langle \tau_\alpha, \tau_\alpha\rangle \, d(\mu\times\mu)(\alpha,\beta) + \int\limits_{(\Omega\times\Omega)\setminus\Delta} \langle \tau_\alpha, \tau_\beta\rangle \, d(\mu\times\mu)(\alpha,\beta) \\
&= (\mu\times\mu)(\Delta) + \int\limits_{(\Omega\times\Omega)\setminus\Delta} \langle \tau_\alpha, \tau_\beta\rangle \, d(\mu\times\mu)(\alpha,\beta) \\
&\leqslant (\mu\times\mu)(\Delta) + \left(\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} \langle \tau_\alpha, \tau_\beta\rangle \right) (\mu\times\mu)((\Omega\times\Omega)\setminus\Delta).
\end{aligned}$$

Now writing inner product using norm, we get

$$\begin{aligned}
\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} \langle \tau_\alpha, \tau_\beta\rangle &= \sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} \left(\frac{\|\tau_\alpha\|^2 + \|\tau_\beta\|^2 - \|\tau_\alpha - \tau_\beta\|^2}{2} \right) \\
&= \sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} \left(\frac{2 - \|\tau_\alpha - \tau_\beta\|^2}{2} \right) \\
&= 1 - \frac{\inf_{\alpha,\beta\in\Omega,\alpha\neq\beta} \|\tau_\alpha - \tau_\beta\|^2}{2}.
\end{aligned}$$

Therefore

$$1 - \frac{\inf_{\alpha,\beta\in\Omega,\alpha\neq\beta} \|\tau_\alpha - \tau_\beta\|^2}{2} \geqslant \frac{-(\mu\times\mu)(\Delta)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}$$

which gives

$$\frac{\inf_{\alpha,\beta\in\Omega,\alpha\neq\beta} \|\tau_\alpha - \tau_\beta\|^2}{2} \leqslant 1 + \frac{(\mu\times\mu)(\Delta)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}. \quad \square$$

COROLLARY 1. *Theorem 1 follows from Theorem 2.*

Proof. Take $\Omega = \{1,\dots,n\}$ and μ as the counting measure. □

EXAMPLE 1. Let $\Omega := [0,2\pi]$ and μ be the Lebesgue measure on Ω. Define

$$\tau_\alpha := (\cos\alpha, \sin\alpha), \quad \forall \alpha \in \Omega.$$

Then

$$\begin{aligned}\int_\Omega |\langle (x,y), \tau_\alpha\rangle|^2 \, d\alpha &= \int_0^{2\pi} |\langle (x,y), (\cos\alpha, \sin\alpha)\rangle|^2 \, d\alpha \\ &= \int_0^{2\pi} (x\cos\alpha + y\sin\alpha)^2 \, d\alpha \\ &= \int_0^{2\pi} (x^2\cos^2\alpha + y^2\sin^2\alpha + 2xy\sin\alpha\cos\alpha) \, d\alpha \\ &= \pi(x^2+y^2) \\ &= \pi \|(x,y)\|^2, \quad \forall (x,y) \in \mathbb{R}^2.\end{aligned}$$

Therefore $\{\tau_\alpha\}_{\alpha\in\Omega}$ is a normalized continuous frame for $\mathbb{R}^2$. In particular, it is a normalized continuous Bessel family for $\mathbb{R}^2$. We now have

$$\begin{aligned}\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} \langle \tau_\alpha, \tau_\beta\rangle &= \sup_{\alpha,\beta\in[0,2\pi],\alpha\neq\beta} \langle (\cos\alpha, \sin\alpha), (\cos\beta, \sin\beta)\rangle \\ &= \sup_{\alpha,\beta\in[0,2\pi],\alpha\neq\beta} (\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= \sup_{\alpha,\beta\in[0,2\pi],\alpha\neq\beta} \cos(\alpha-\beta) \\ &= 1 > \frac{0}{4\pi^2} = \frac{-(\mu\times\mu)(\Delta)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}.\end{aligned}$$

EXAMPLE 2. Let (Ω, μ) be a measure space. Let $\mathscr{H}$ be a reproducing kernel Hilbert space on Ω with kernel

$$K : \Omega\times\Omega \to \mathbb{R}, \quad K(\alpha,\beta) := \langle K_\beta, K_\alpha\rangle = K_\beta(\alpha), \quad \forall \alpha,\beta \in \Omega.$$

Then $\{K_\alpha\}_{\alpha\in\Omega}$ is a continuous Parseval frame for $\mathscr{L}^2(\Omega,\mu)$. Let Ω_1 be a measurable subset of Ω such that the family $\{K_\alpha\}_{\alpha\in\Omega_1}$ is bounded below on Ω_1. Then $\{K_\alpha\}_{\alpha\in\Omega_1}$ is a continuous Bessel family for $\mathscr{L}^2(\Omega,\mu)$. Let Δ_1 be the diagonal of $\Omega_1\times\Omega_1$. Inequality (4) then gives

$$\sup_{\alpha,\beta\in\Omega_1,\alpha\neq\beta} \langle K_\alpha, K_\beta\rangle \geqslant \frac{-(\mu\times\mu)(\Delta_1)}{(\mu\times\mu)((\Omega_1\times\Omega_1)\setminus\Delta_1)}.$$

A remarkable feature of Inequality (4) is that it allows to derive Inequality (5). We can not do this by using first order continuous Welch bound [11].

Given a measure space (Ω,μ) with measurable diagonal and a normalized continuous Bessel family $\{\tau_\alpha\}_{\alpha\in\Omega}$ for a real Hilbert space $\mathscr{H}$, we define

$$\mathscr{M}(\{\tau_\alpha\}_{\alpha\in\Omega}) := \sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} \langle \tau_\alpha, \tau_\beta\rangle$$

and

$$\mathscr{N}(\{\tau_\alpha\}_{\alpha\in\Omega}) := \inf_{\alpha,\beta\in\Omega,\alpha\neq\beta} \|\tau_\alpha - \tau_\beta\|^2.$$

Similar to the problem of Grassmannian frames (see [14]), we propose following problem.

QUESTION 1. Given a measure space (Ω,μ) with measurable diagonal and a real Hilbert space $\mathscr{H}$, find normalized continuous Bessel family $\{\tau_\alpha\}_{\alpha\in\Omega}$ for $\mathscr{H}$, such that

$$\mathscr{M}(\{\tau_\alpha\}_{\alpha\in\Omega}) = \inf\{\mathscr{M}(\{\omega_\alpha\}_{\alpha\in\Omega}) : \{\omega_\alpha\}_{\alpha\in\Omega} \text{ is a normalized continuous Bessel family for } \mathscr{H}\}. \tag{6}$$

Equivalently, find normalized continuous Bessel family $\{\tau_\alpha\}_{\alpha\in\Omega}$ for $\mathscr{H}$, such that

$$\mathscr{N}(\{\tau_\alpha\}_{\alpha\in\Omega}) = \sup\{\mathscr{N}(\{\omega_\alpha\}_{\alpha\in\Omega}) : \{\omega_\alpha\}_{\alpha\in\Omega} \text{ is a normalized continuous Bessel family for } \mathscr{H}\}.$$

Further, for which measure spaces (Ω,μ) and real Hilbert spaces $\mathscr{H}$, solution to (6) exists?

3. Continuous Rankin bound for Banach spaces

In this section, we derive continuous Rankin bound for Banach spaces. First we need a notion.

DEFINITION 2. [10] Let (Ω,μ) be a measure space and $p\in[1,\infty)$. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a collection in a Banach space $\mathscr{X}$ and $\{f_\alpha\}_{\alpha\in\Omega}$ be a collection in $\mathscr{X}^*$. The pair $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be a *continuous p-Bessel family* for $\mathscr{X}$ if the following conditions are satisfied.

(i) For each $x\in\mathscr{X}$, the map $\Omega\ni\alpha\mapsto f_\alpha(x)\in\mathbb{K}$ is measurable.

(ii) For each $u\in\mathscr{L}^p(\Omega,\mu)$, the map $\Omega\ni\alpha\mapsto u(\alpha)\tau_\alpha\in\mathscr{X}$ is measurable.

(iii) The map (*continuous analysis operator*)

$$\theta_f : \mathscr{X}\ni x\mapsto \theta_f \in \mathscr{L}^p(\Omega,\mu); \quad \theta_f x : \Omega\ni\alpha\mapsto(\theta_f x)(\alpha) := f_\alpha(x)\in\mathbb{K}$$

is a well-defined bounded linear operator.

(iv) The map (*continuous synthesis operator*)

$$\theta_\tau : \mathscr{L}^p(\Omega,\mu) \ni u \mapsto \theta_\tau u := \int_\Omega u(\alpha)\tau_\alpha \, d\mu(\alpha) \in \mathscr{X}$$

is a well-defined bounded linear operator.

THEOREM 3. (Functional Continuous Rankin Bound) *Let (Ω,μ) be a finite measure space and $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ be a continuous p-approximate Bessel family for a real Banach space $\mathscr{X}$ satisfying the following.*

(i) $f_\alpha(\tau_\alpha)=1$ *for all* $\alpha\in\Omega$.

(ii) $\|f_\alpha\|\leqslant 1,\ \|\tau_\alpha\|\leqslant 1$ *for all* $1\leqslant\alpha\in\Omega$.

(iii) $\theta_f\theta_\tau\chi_\Omega\geqslant 0$.

If the diagonal $\Delta:=\{(\alpha,\alpha):\alpha\in\Omega\}$ is measurable in the measure space $\Omega\times\Omega$, then

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} f_\alpha(\tau_\beta)\geqslant\frac{-(\mu\times\mu)(\Delta)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}.$$

Proof. Since $\mu(\Omega)<\infty$, we have

$$\begin{aligned}\int_{(\Omega\times\Omega)\setminus\Delta}|f_\alpha(\tau_\beta)|\,d(\mu\times\mu)(\alpha,\beta)&\leqslant\int_{(\Omega\times\Omega)\setminus\Delta}\|f_\alpha\|\|\tau_\beta\|\,d(\mu\times\mu)(\alpha,\beta)\\&\leqslant(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)<\infty.\end{aligned}$$

Now by using Fubini's theorem, we get

$$\begin{aligned}0&\leqslant\int_\Omega(\theta_f\theta_\tau\chi_\Omega)(\alpha)\,d\mu(\alpha)=\int_\Omega f_\alpha(\theta_\tau\chi_\Omega)\,d\mu(\alpha)\\
&=\int_\Omega f_\alpha\left(\int_\Omega\chi_\Omega(\beta)\tau_\beta\,d\mu(\beta)\right)d\mu(\alpha)=\int_\Omega f_\alpha\left(\int_\Omega\tau_\beta\,d\mu(\beta)\right)d\mu(\alpha)\\
&=\int_\Omega\int_\Omega f_\alpha(\tau_\beta)\,d\mu(\beta)\,d\mu(\alpha)=\int_{\Omega\times\Omega}f_\alpha(\tau_\beta)\,d(\mu\times\mu)(\alpha,\beta)\\
&=\int_\Delta f_\alpha(\tau_\beta)\,d(\mu\times\mu)(\alpha,\beta)+\int_{(\Omega\times\Omega)\setminus\Delta}f_\alpha(\tau_\beta)\,d(\mu\times\mu)(\alpha,\beta)\\
&=\int_\Delta f_\alpha(\tau_\alpha)\,d(\mu\times\mu)(\alpha,\beta)+\int_{(\Omega\times\Omega)\setminus\Delta}f_\alpha(\tau_\beta)\,d(\mu\times\mu)(\alpha,\beta)\\
&=(\mu\times\mu)(\Delta)+\int_{(\Omega\times\Omega)\setminus\Delta}f_\alpha(\tau_\beta)\,d(\mu\times\mu)(\alpha,\beta)\\
&\leqslant(\mu\times\mu)(\Delta)+\left(\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta}f_\alpha(\tau_\beta)\right)(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta).\quad\square\end{aligned}$$

COROLLARY 2. *Let* $\{\tau_j\}_{j=1}^n$ *be a collection in a real Banach space* $\mathscr{X}$ *and* $\{f_j\}_{j=1}^n$ *be a collection in* $\mathscr{X}^*$ *satisfying the following.*

(i) $f_j(\tau_j)=1$ *for all* $1\leqslant j\leqslant n$.

(ii) $\|f_j\|\leqslant 1,\ \|\tau_j\|\leqslant 1$ *for all* $1\leqslant j\leqslant n$.

(iii) $\sum_{1\leqslant j,k\leqslant n} f_j(\tau_k)\geqslant 0$.

Then

$$\max_{1\leqslant j,k\leqslant n, j\neq k} f_j(\tau_k)\geqslant \frac{-1}{n-1}.$$

Acknowledgements. Author thanks the reviewer for his/her study of the manuscript and suggestions to enhance the presentation and quality. The author was supported by Indian Statistical Institute, Bangalore, through the J. C. Bose Fellowship of Prof. B. V. Rajarama Bhat. He thanks Prof. B. V. Rajarama Bhat for the Post Doc position.

REFERENCES

[1] S. TWAREQUE ALI, J. P. ANTOINE, AND J. P. GAZEAU, *Continuous frames in Hilbert space*, Ann. Physics, **222**, 1 (1993), 1–37.
[2] SYED TWAREQUE ALI, J. P. ANTOINE, AND J. P. GAZEAU, *Coherent states, wavelets, and their generalizations*, Springer, New York, 2014.
[3] BRUCE C. BERNDT, WINFRIED KOHNEN, AND KEN ONO, *The life and work of R. A. Rankin (1915–2001)*, Ramanujan J., **7** (2003), 9–38.
[4] J. W. S. CASSELS, *An introduction to the geometry of numbers*, Classics in Mathematics, Springer-Verlag, Berlin, 1997.
[5] JOHN H. CONWAY, RONALD H. HARDIN, AND NEIL J. A. SLOANE, *Packing lines, planes, etc.: packings in Grassmannian spaces*, Experiment. Math., **5**, 2 (1996), 139–159.
[6] I. S. DHILLON, R. W. HEATH, JR., T. STROHMER, AND J. A. TROPP, *Constructing packings in Grassmannian manifolds via alternating projection*, Experiment. Math., **17**, 1 (2008), 9–35.
[7] JEAN P. GABARDO AND DEGUANG HAN, *Frames associated with measurable spaces*, Adv. Comput. Math., **18** (2003), 127–147.
[8] THOMAS C. HALES, *Historical overview of the Kepler conjecture*, Discrete Comput. Geom., **36**, 1 (2006), 5–20.
[9] GERALD KAISER, *A friendly guide to wavelets*, Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2011.
[10] K. MAHESH KRISHNA, *Feichtinger conjectures,* R_ε*-conjectures and Weaver's conjectures for Banach spaces*, arXiv:2201.00125v1 [math.FA], 1 January, 2022.
[11] K. MAHESH KRISHNA, *Continuous Welch bounds with applications*, Commun. Korean Math. Soc., **38**, 3 (2023), 787–805.
[12] R. A. RANKIN, *On the closest packing of spheres in* n *dimensions*, Ann. of Math. (2), **48** (1947), 1062–1081.
[13] R. A. RANKIN, *The closest packing of spherical caps in* n *dimensions*, Proc. Glasgow Math. Assoc., **2** (1955), 139–144.
[14] THOMAS STROHMER AND ROBERT W. HEATH, JR., *Grassmannian frames with applications to coding and communication*, Appl. Comput. Harmon. Anal., **14**, 3 (2003) 257–275.
[15] GEORGE G. SZPIRO, *Kepler's conjecture: How some of the greatest minds in history helped solve one of the oldest math problems in the world*, John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[16] MICHEL TALAGRAND, *Pettis integral and measure theory*, Mem. Amer. Math. Soc., **51**, 307 (1984), ix+224.
[17] JOEL A. TROPP, *Topics in sparse approximation*, (2004) Thesis (Ph.D.), The University of Texas at Austin.
[18] CHUANMING ZONG, *Sphere packings*, Universitext, Springer-Verlag, New York, 1999.

(Received May 19, 2024)

K. Mahesh Krishna
School of Mathematics and Natural Sciences
Chanakya University Global Campus
NH-648, Haraluru Village, Devanahalli Taluk
Bengaluru Rural District, Karnataka 562 110, India
e-mail: kmaheshak@gmail.com