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GENERALIZED p–ZBĂGANU CONSTANT IN BANACH SPACES

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Abstract. In this paper, we introduce generalized p-Zbăganu constant and discuss its bounds. Also, we obtain its relation with the generalized von Neumann-Jordan constant, the generalized Ptolemy constant, and the modulus of convexity. Furthermore, using generalized p-Zbăganu constant, we characterize the uniformly non-square and the weak normal structure and obtain that a Banach space has the fixed point property.

1. Introduction

The study of geometric constants is a quantitative analysis of the geometry of space, and it also plays an essential role in investigating several other geometrical problems related to functional analysis. Numerous geometric constants are associated with Banach spaces. There are innumerable connections among various geometric constants, and these constants are mathematically beautiful. Many scientists have defined various specific constants to explore and study some abstract aspects of Banach spaces.

In 1936, Clarkson [4] proposed the concept of convexity modules $\delta_{\mathscr{X}}(\varepsilon)$ for describing uniformly convex spaces. The modulus of convexity measures how far the middle point of the segment joining two points must be from $S_{\mathscr{X}}$. Later motivated by the outstanding work on the characterization of inner product spaces by Jordan and von Neumann [12], Clarkson [5] defined the von Neumann-Jordan constant $C_{NJ}(\mathscr{X})$.

Zbăganu [19] introduced a new constant $\mathscr{C}_z(\mathscr{X})$ and conjectured that it coincide with $\mathscr{C}_{NJ}(\mathscr{X})$ in a normed space. Later, Alonso and Martin [1] provided a counterexample, stating that $\mathscr{C}_{NJ}(\mathscr{X}) \neq \mathscr{C}_z(\mathscr{X})$ in general and they discussed its relationship with other renowned constants. In 2021, Liu et al. [17] introduced a new geometric constant $\mathscr{C}_z(\lambda,\mu,\mathscr{X})$ in a Banach space $\mathscr{X}$ which used to describe the generalizations of the parallelogram law.

Cui et al. [3] introduced the generalized von Neumann-Jordan constant $\mathscr{C}^p_{NJ}(\mathscr{X})$ and discussed its bounds in uniformly non-square Banach spaces. Recently, Ni et al. [16] introduced the skew generalized von Neumann-Jordan constant $\mathscr{C}^p_{NJ}(\lambda,\mu,\mathscr{X})$ in Banach spaces.

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The space $(\mathscr{X}, \|.\|)$ has the fixed point property if every non-expansive self-mapping of each nonempty, bounded, closed, convex, subset C of $\mathscr{X}$ has a fixed point. According to the result of Kirk [13], the reflexive Banach spaces with normal structure have the fixed point property.

To check that a given Banach space has normal structure is not an easy task. Whether or not a Banach space has normal structure depends on the geometry of a unit sphere. The purpose of this study is to provide sufficient criteria for normal structure in reflexive Banach spaces, specifically using the generalized p-Zbăganu constant.

Various constants associated with Banach spaces such as the characteristic of convexity $\varepsilon_0(\mathscr{X})$, the von Neumann-Jordan constant $C_{NJ}(\mathscr{X})$, among others have been useful in providing sufficient conditions for fixed point property or normal structure in reflexive Banach spaces.

Motivated by these works, we introduce another geometric constant called generalized p-Zbăganu constant (denoted by $\mathscr{C}_z^p(\lambda, \mu, \mathscr{X})$) and discuss its interesting geometric properties in Banach spaces. Also, we provide a characterization of uniformly non-square Banach spaces using a condition on generalized p-Zbăganu constant.

It is to noting that every uniformly non-square Banach space has the fixed point property (see [7]). So, one can solve some fixed point problems using generalized p-Zbăganu constant in Banach spaces.

The article's structure follows: After the introductory Section 1, we proceed to Section 2, providing some fundamental definitions, notations and results. Section 3, has been divided into three subsections. In the first subsection, we introduce the generalized p-Zbăganu constant $\mathscr{C}_z^p(\lambda, \mu, \mathscr{X})$. We also investigate its bounds in Hilbert spaces. In the next subsection, we establish its relation with renowned geometric constants viz. von Neumann-Jordan constant, generalized von Neumann-Jordan constant, generalized Ptolemy constant, and modulus of convexity. In the last, we use the generalized p-Zbăganu constant to characterize uniformly non-square Banach spaces. In addition, we give some necessary conditions for weak normal structure using generalized p-Zbăganu constant.

2. Preliminaries

The following are some notations and definitions that will be utilized in the subsequent sections.

Let $\mathscr{X}$ be a Banach space and $\mathscr{S}_{\mathscr{X}}$ be a unit sphere of a Banach space $\mathscr{X}$.

DEFINITION 1. Let $\mathscr{X}$ be a Banach space.

(i) $\mathscr{X}$ is called uniformly non-square [10] if there exists a $\delta > 0$ such that $x, y \in \mathscr{S}_{\mathscr{X}}$, either

$$\left\|\frac{1}{2}(x+y)\right\| < 1-\delta \text{ or } \left\|\frac{1}{2}(x-y)\right\| < 1-\delta.$$

(ii) The James constant [10] $\mathscr{J}(\mathscr{X})$, is defined as

$$\mathscr{J}(\mathscr{X}) = \sup\left\{ \min\{\|x+y\|, \|x-y\|\} : x,y \in \mathscr{S}_{\mathscr{X}} \right\}.$$

(iii) The von Neumann-Jordan constant [5] $\mathscr{C}_{NJ}(\mathscr{X})$, is defined as

$$\mathscr{C}_{NJ}(\mathscr{X}) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x,y \in \mathscr{X}, (x,y) \neq (0,0) \right\}.$$

(iv) The generalized von Neumann-Jordan constant [3] $\mathscr{C}_{NJ}(\mathscr{X})$, is defined as

$$\mathscr{C}_{NJ}^{p}(\mathscr{X}) = \sup\left\{ \frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p + \|y\|^p)} : x,y \in \mathscr{X}, (x,y) \neq (0,0) \right\},$$

where $1 \leqslant p < \infty$.

(v) The modulus of convexity [4] $\delta_{\mathscr{X}}(\varepsilon)$, is defined as

$$\delta_{\mathscr{X}}(\varepsilon) = \inf\left\{ 1 - \frac{\|x+y\|}{2} : x,y \in \mathscr{S}_{\mathscr{X}}, \|x-y\| = \varepsilon \right\}, \quad \text{where } \varepsilon \in [0,2].$$

(vi) The Zbăganu constant [19] $\mathscr{C}_z(\mathscr{X})$ is defined as

$$\mathscr{C}_z(\mathscr{X}) = \sup\left\{ \frac{\|x+y\| \, \|x-y\|}{(\|x\|^2 + \|y\|^2)} : x,y \in \mathscr{X}, (x,y) \neq (0,0) \right\}.$$

(vii) The generalized Zbăganu constant [19] $\mathscr{C}_z(\lambda,\mu,\mathscr{X})$ is defined as, for $\lambda,\mu > 0$,

$$\mathscr{C}_z(\lambda,\mu,\mathscr{X}) = \sup\left\{ \frac{2\|\lambda x + \mu y\| \, \|\mu x - \lambda y\|}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)} : x,y \in \mathscr{X}, (x,y) \neq (0,0) \right\}.$$

(viii) The Ptolemy constant [15] is defined as

$$\mathscr{C}_{\mathscr{P}}(\mathscr{X}) := \sup_{x \neq y, y \neq w} \left\{ \frac{\|x-y\| \|w\|}{(\|x-w\| \|y\| + \|w-y\| \|x\|)} : x,y,w \in \mathscr{X} \backslash \{0\} \right\}.$$

(ix) The characteristic of convexity [11] is defined as

$$\varepsilon_0(\mathscr{X}) := \sup\{\varepsilon \in [0,2] : \delta_{\mathscr{X}}(\varepsilon) = 0\}.$$

Alonso and Fuster [2] introduced a new constant by using the geometric mean of the lengths of the non-constant sides of the triangles inscribed in $\mathscr{S}_{\mathscr{X}}$ with vertices $-x$, x and y. They defined it as

$$\mathscr{C}_G(\mathscr{X}) := \sup\{\sqrt{\|x+y\| \|x-y\|} : x,y \in \mathscr{S}_{\mathscr{X}}\}.$$

More about the above-defined constants can be found in Dinarvand [6] and Li et al. [14].

DEFINITION 2. Let $\mathscr{M}$ be a subset of a Banach space $(\mathscr{X}, \|.\|)$. A mapping $T : \mathscr{M} \to \mathscr{X}$ is called non-expansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in \mathscr{M}$.

DEFINITION 3. (see [9]) A non-empty convex and bounded convex subset $\mathscr{M}$ of a Banach space $\mathscr{X}$ is said to have normal structure if for every convex subset C of $\mathscr{M}$ that contains more than one point, there exists a point $x_0 \in C$ such that

$$\sup\{\|x_0 - y\| : y \in C\} < \sup\{\|x - y\| : x, y \in C\}.$$

DEFINITION 4. (see [9]) A Banach space $\mathscr{X}$ is said to have weak normal structure if each weakly compact convex subset $\mathscr{M}$ of $\mathscr{X}$ that contains more than one point has normal structure.

For a reflexive Banach space, normal structure and weak normal structure coincide.

The following results will be used in the sequel.

LEMMA 1. *If $x, y > 0$, $1 \leqslant p < \infty$, then $(x+y)^p \leqslant 2^{p-1}(x^p + y^p)$.*

If a Banach space does not have weak normal structure, then there exists an inscribed hexagon in $\mathscr{S}_{\mathscr{X}}$ with a length of each side arbitrarily closed to 1 and with at least four sides whose distance to $\mathscr{S}_{\mathscr{X}}$ are arbitrarily small.

LEMMA 2. [8] *Let $\mathscr{X}$ be a Banach space without weak normal structure, then for any ε, $0 < \varepsilon < 1$, there exist x, y, w in $\mathscr{S}_{\mathscr{X}}$ satisfying*

(a) $y - w = ax$ *with* $|a - 1| < \varepsilon$;

(b) $|\|x - y\| - 1|, |\|w - (-x)\| - 1| < \varepsilon$; *and*

(c) $\|\frac{x+y}{2}\|$, $\|\frac{w-x}{2}\| > 1 - \varepsilon$.

3. Main results

First, we introduce the generalized p-Zbăganu constant and discuss its bounds.

3.1. Generalized p-Zbăganu constant and its bounds

First, we introduce a new geometric constant and its different equivalent forms.

DEFINITION 5. Let $(\mathscr{X}, \|.\|)$ be a Banach space of dimension at least 2. Then for $\lambda, \mu > 0$ and $p \geqslant 1$, the number

$$\mathscr{C}_z^p(\lambda, \mu, \mathscr{X}) := \sup\left\{ \frac{\|\lambda x + \mu y\|^p \, \|\mu x - \lambda y\|^p}{2^{2p-3}(\lambda^{2p} + \mu^{2p})(\|x\|^{2p} + \|y\|^{2p})} : x, y \in \mathscr{X}, (x,y) \neq (0,0) \right\} \tag{1}$$

is said to be a generalized p-Zbăganu constant.

Now, we give another form of $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$ for $x,y\in\mathscr{S}_{\mathscr{X}}$ as follows:

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) := \sup\left\{\frac{\|\lambda x+\mu ty\|^p\|\mu x-\lambda ty\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(1+t^{2p})} : x,y\in\mathscr{S}_{\mathscr{X}}, 0\leqslant t\leqslant 1\right\}. \quad (2)$$

For $\lambda=\mu$ in (1), we define

$$\mathscr{C}_z^p(\mathscr{X}) = \mathscr{C}_z^p(\lambda,\lambda,\mathscr{X}) := \sup\left\{\frac{\|x+y\|^p\|x-y\|^p}{2^{2p-2}(\|x\|^{2p}+\|y\|^{2p})} : x,y\in\mathscr{X}, (x,y)\neq(0,0)\right\}. \quad (3)$$

REMARK 1. *(a) For $p=1$ in (1), we get $\mathscr{C}_z^1(\lambda,\mu,\mathscr{X})=\mathscr{C}_z(\lambda,\mu,\mathscr{X})$, introduced by Liu et al.* [17].

(b) In particular, $p=1$ in (3), we have $\mathscr{C}_z^1(\mathscr{X})=\mathscr{C}_z(\mathscr{X})$, introduced by Zbăganu [19].

We observe that the following is one possible form of $\mathscr{C}_z^p(\mathscr{X})$:

$$\mathscr{C}_z^p(\mathscr{X}) = \sup\left\{\frac{2^{2p}\|x\|^p\|y\|^p}{2^{2p-2}(\|x+y\|^{2p}+\|x-y\|^{2p})} : x,y\in\mathscr{X}, (x,y)\neq(0,0)\right\}. \quad (4)$$

Next, we discuss about the lower and upper bounds of the constant $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$.

PROPOSITION 1. *For a Banach space $\mathscr{X}$, $\frac{1}{2^{2p-2}}\leqslant\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})\leqslant 2$.*

Proof. First we assume that $x\neq 0$, $y=0$ with $\lambda=\mu=1$. So, we have

$$\begin{aligned}\frac{\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} &= \frac{\lambda^p\mu^p\|x\|^{2p}}{2^{2p-3}(\lambda^{2p}+\mu^{2p})\|x\|^{2p}}\\ &= \frac{\lambda^p\mu^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})}\\ &= \frac{1}{2^{2p-2}}.\end{aligned}$$

So, by taking supremum over $x,y\in\mathscr{X}$ and $(x,y)\neq(0,0)$, we get

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) \geqslant \frac{1}{2^{2p-2}}.$$

On the other hand,

$$\begin{aligned}
&\frac{\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})}\\
&\leqslant \frac{(\|\lambda x+\mu y\|^{2p}+\|\mu x-\lambda y\|^{2p})}{2\times 2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})}\\
&\leqslant \frac{\left[(\lambda\|x\|+\mu\|y\|)^{2p}+(\mu\|x\|+\lambda\|y\|)^{2p}\right]}{2(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})}\\
&\leqslant \frac{2^{2p-1}(\lambda^{2p}\|x\|^{2p}+\mu^{2p}\|y\|^{2p})+2^{2p-1}(\mu^{2p}\|x\|^{2p}+\lambda^{2p}\|y\|^{2p})}{2^{2p-2}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{Using the Lemma 1}]\\
&= \frac{2^{2p-1}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})}{2^{2p-2}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})}\\
&= 2.
\end{aligned}$$

Therefore, $\frac{1}{2^{2p-2}} \leqslant \mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) \leqslant 2$. □

For $p=1$ and using the Remark 1(a), we get the following result.

COROLLARY 1. (see [17]) *For a Banach space* $\mathscr{X}$, $1 \leqslant \mathscr{C}_z(\lambda,\mu,\mathscr{X}) \leqslant 2$.

Taking $\lambda=\mu$ in Corollary 1, we have the following result.

COROLLARY 2. *For a Banach space* $\mathscr{X}$, $1 \leqslant \mathscr{C}_z(\mathscr{X}) \leqslant 2$.

EXAMPLE 1. We consider a two-dimensional real sequence space $\mathscr{X}=\ell_\infty$ with $(p,q)=\max\{|p|,|q|\}$. Taking $x=(1,1)$ and $y=(1,-1)$ we have $\|x\|=1$ and $\|y\|=1$. Now, $\|\lambda x+\mu y\|=\lambda+\mu$ and $\|\mu x-\lambda y\|=\mu+\lambda$ Let, $\lambda=\mu=2$. Then $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})=\frac{4^p4^p}{2^{2p-3}(2^{2p}+2^{2p})(1+1)}=2$.

To discuss the bounds of the constant $\mathscr{C}_z^p(\rho,\mu,\mathscr{X})$ in a Hilbert space, we need the following theorem as a main tool.

THEOREM 1. *Let* $\mathscr{X}$ *be a real normed linear space, and* $\|.\|$ *derives from an inner product. Then*

$$\|\lambda x+\mu y\|^{2p}+\|\mu x-\lambda y\|^{2p} \leqslant (\lambda+\mu)^{2p}+(|\lambda-\mu|)^{2p} \ \textit{for}\ p \geqslant 1,$$

for all $x,y \in \mathscr{S}_{\mathscr{X}}$.

Proof. Let the norm $\|.\|$ on $\mathscr{X}$ is induced by an inner product $\langle\cdot,\cdot\rangle$ and $x,y \in \mathscr{S}_{\mathscr{X}}$. Consider

$$\begin{aligned}
&\|\lambda x+\mu y\|^{2p}+\|\mu x-\lambda y\|^{2p}\\
&=\{\|\lambda x-\mu y\|^2\}^p+\{\|\mu x-\lambda y\|^2\}^p\\
&=(\lambda^2\|x\|^2+\mu^2\|y\|^2+2\lambda\mu\langle x,y\rangle)^p+(\mu^2\|x\|^2+\lambda^2\|y\|^2-2\lambda\mu\langle x,y\rangle)^p\\
&=(\lambda^2+\mu^2+2\lambda\mu\cos\theta)^p+(\mu^2+\lambda^2-2\lambda\mu\cos\theta)^p,
\end{aligned}$$

where $\cos\theta=\frac{\langle x,y\rangle}{\|x\|\ \|y\|}$, $(x,y)\neq(0,0)$.

Choose $\mathscr{R}(\theta):=(\lambda^2+\mu^2+2\lambda\mu\cos\theta)^p+(\mu^2+\lambda^2-2\lambda\mu\cos\theta)^p$. By taking differentiation, we have

$$\begin{aligned}
\mathscr{R}'(\theta)&=p(\mu^2+\lambda^2+2\lambda\mu\cos\theta)^{p-1}\times(-2\lambda\mu\sin\theta)\\
&\qquad+p(\mu^2+\lambda^2-2\lambda\mu\cos\theta)^{p-1}\times 2\lambda\mu\sin\theta.\\
&=p\Big[(\mu^2+\lambda^2-2\lambda\mu\cos\theta)^{p-1}-(\mu^2+\lambda^2+2\lambda\mu\cos\theta)^{p-1}\Big]\times 2\lambda\mu\sin\theta.
\end{aligned}$$

Therefore, the critical values of $\mathscr{R}(\theta)$, being the roots of $\mathscr{R}'(\theta)=0$, are $\theta=k\pi$, $(2k+1)\frac{\pi}{2}$ for $k=0,\pm1,\pm2,\ldots$.

Again, taking a differentiation, we have

$$\begin{aligned}
\mathscr{R}''(\theta)&=p\Big[(\mu^2+\lambda^2-2\lambda\mu\cos\theta)^{p-1}-(\mu^2+\lambda^2+2\lambda\mu\cos\theta)^{p-1}\Big]\times 2\lambda\mu\cos\theta\\
&\quad+p(p-1)\Big[(\mu^2+\lambda^2-2\lambda\mu\cos\theta)^{p-2}+(\mu^2+\lambda^2+2\lambda\mu\cos\theta)^{p-2}\Big]\\
&\quad\times 2\lambda^2\mu^2\sin^2\theta.
\end{aligned}$$

If $\theta=2k\pi$, then

$$\begin{aligned}
\mathscr{R}''(2k\pi)&=p\Big[(\mu^2+\lambda^2-2\lambda\mu)^{p-1}-(\mu^2+\lambda^2+2\lambda\mu)^{p-1}\Big]\times 2\lambda\mu\\
&\leqslant 0.
\end{aligned}$$

If $\theta=(2k+1)\pi$, then

$$\begin{aligned}
\mathscr{R}''((2k+1)\pi)&=p\Big[(\mu^2+\lambda^2+2\lambda\mu)^{p-1}-(\mu^2+\lambda^2-2\lambda\mu)^{p-1}\Big]\times(-2\lambda\mu)\\
&=p\Big[(\mu^2+\lambda^2-2\lambda\mu)^{p-1}-(\mu^2+\lambda^2+2\lambda\mu)^{p-1}\Big]\times(2\lambda\mu)\\
&\leqslant 0.
\end{aligned}$$

If $\theta = (2k+1)\frac{\pi}{2}$, then

$$\begin{aligned}\mathscr{R}''((2k+1)\frac{\pi}{2}) &= p(p-1)\left[(\mu^2+\lambda^2)^{p-2}+(\mu^2+\lambda^2)^{p-2}\right]\times 2\lambda^2\mu^2 \\ &= p(p-1)\left[2(\mu^2+\lambda^2)^{p-2}\right]\times 2\lambda^2\mu^2 \\ &\geqslant 0.\end{aligned}$$

Now, $\max\{\mathscr{R}(2k\pi), \mathscr{R}(2k+1)\pi\} = (\lambda+\mu)^{2p} + (|\lambda-\mu|)^{2p}$. Therefore,

$$\|\lambda x+\mu y\|^{2p} + \|\mu x-\lambda y\|^{2p} \leqslant (\lambda+\mu)^{2p} + (|\lambda-\mu|)^{2p} \text{ for } p \geqslant 1,$$

for all $x, y \in \mathscr{S}_{\mathscr{X}}$. Hence the result. □

Taking $\lambda = \mu$ in Theorem 1, we have the following result.

COROLLARY 3. *For a norm space $\mathscr{X}$, $\|.\|$ derives from an inner product if and only if for all $x, y \in \mathscr{S}_{\mathscr{X}}$ and $p \geqslant 1$, we have*

$$\|x+y\|^{2p} + \|x-y\|^{2p} \leqslant 2^{2p}.$$

THEOREM 2. *For a Hilbert space $\mathscr{H}$, $\frac{1}{2^{2p-2}} \leqslant \mathscr{C}_z^p(\lambda,\mu,\mathscr{H}) \leqslant 1 + \frac{(|\lambda-\mu|)^{2p}}{2^{2p-1}(\lambda^{2p}+\mu^{2p})}$, where $x, y \in \mathscr{S}_{\mathscr{H}}$.*

Proof. For $p \geqslant 1$, using (1) and Theorem 1, we have

$$\begin{aligned}\frac{\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} &\leqslant \frac{\|\lambda x+\mu y\|^{2p}+\|\mu x-\lambda y\|^{2p}}{2\times 2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\ &\leqslant \frac{(\lambda+\mu)^{2p}+(|\lambda-\mu|)^{2p}}{2^{2p-2}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\ &\leqslant \frac{2^{2p-1}(\lambda^{2p}+\mu^{2p})+(|\lambda-\mu|)^{2p}}{2^{2p-2}(\lambda^{2p}+\mu^{2p})(1+1)} \\ &= 1+\frac{(|\lambda-\mu|)^{2p}}{2^{2p-1}(\lambda^{2p}+\mu^{2p})}.\end{aligned}$$

By taking the supremum over $x, y \in \mathscr{S}_{\mathscr{H}}$, we get

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{H}) \leqslant 1+\frac{(|\lambda-\mu|)^{2p}}{2^{2p-1}(\lambda^{2p}+\mu^{2p})}.$$

Also, by Proposition 1, $\frac{1}{2^{2p-2}} \leqslant \mathscr{C}_z^p(\lambda,\mu,\mathscr{H})$. Therefore,

$$\frac{1}{2^{2p-2}} \leqslant \mathscr{C}_z^p(\lambda,\mu,\mathscr{H}) \leqslant 1+\frac{(|\lambda-\mu|)^{2p}}{2^{2p-1}(\lambda^{2p}+\mu^{2p})}. \quad \square$$

Taking $\lambda = \mu$ in Theorem 2, we have

COROLLARY 4. *For a Hilbert space* $\mathscr{H}$, $\frac{1}{2^{2p-2}} \leqslant \mathscr{C}_z^p(\mathscr{H}) \leqslant 1$, *where* $x, y \in \mathscr{S}_{\mathscr{H}}$.

REMARK 2. *It is easy to see that, for* $x, y \in \mathscr{S}_{\mathscr{H}}$ *in Hilbert space* $\mathscr{H}$, $0 \leqslant \lim_{p\to\infty} \mathscr{C}_z^p(\mathscr{H}) \leqslant 1$.

EXAMPLE 2. Consider a two-dimensional real sequence space $\mathscr{H} = \ell_2$ with $(p,q) = \sqrt{|p|^2+|q|^2}$. Taking $x = (1,2)$ and $y = (1,0)$ we have $\|x\| = \sqrt{5}$ and $\|y\| = 1$. Now, $\|x+y\| = \sqrt{8}$ and $\|x-y\| = 2$. Then,

$$\frac{\|x+y\|^p\|x-y\|^p}{2^{2p-2}(\|x\|^{2p}+\|y\|^{2p})} = \frac{8^{\frac{p}{2}} \times 2^p}{2^{2p-2}(5^p+1)} = \frac{4 \times 2^{\frac{p}{2}}}{5^p+1}.$$

Then $\mathscr{C}_z^p(\mathscr{H}) \leqslant \frac{4\sqrt{2}}{6} < 1$.

3.2. Relationship with other geometric constants

In this section, we discuss the relationship of $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$ with other renowned geometric constants.

First, we provide a relation of $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$ with constants like $\mathscr{C}_{NJ}^p(\mathscr{X})$ and $\mathscr{C}_z^p(\mathscr{X})$.

PROPOSITION 2. *For a Banach space* $\mathscr{X}$,

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) \leqslant \frac{[(\lambda+\mu)(|\lambda-\mu|)]^p}{(\lambda^{2p}+\mu^{2p})}\mathscr{C}_{NJ}^{2p}(\mathscr{X}) + \frac{2^{2p-1}\left[\left(\frac{(\lambda+\mu)}{2}\right)^{2p}+\left(\frac{(|\lambda-\mu|)}{2}\right)^{2p}\right]}{(\lambda^{2p}+\mu^{2p})}\mathscr{C}_z^p(\mathscr{X}).$$

Proof. Consider

$$\begin{aligned}
&\frac{\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\
&= \frac{\|\frac{\lambda+\mu}{2}(x+y)+\frac{\lambda-\mu}{2}(x-y)\|^p\|\frac{\mu-\lambda}{2}(x+y)+\frac{\mu+\lambda}{2}(x-y)\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\
&\leqslant \frac{\left(\frac{\lambda+\mu}{2}\|(x+y)\|+\frac{|\lambda-\mu|}{2}\|(x-y)\|\right)^p\left(\frac{|\lambda-\mu|}{2}\|(x+y)\|+\frac{\mu+\lambda}{2}\|(x-y)\|\right)^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\
&\leqslant 2^{p-1}\left[\left(\frac{\lambda+\mu}{2}\right)^p\|(x+y)\|^p+\left(\frac{|\lambda-\mu|}{2}\right)^p\|(x-y)\|^p\right] \\
&\qquad \times \frac{2^{p-1}\left[\left(\frac{|\lambda-\mu|}{2}\right)^p\|(x+y)\|^p+\left(\frac{\mu+\lambda}{2}\right)^p\|(x-y)\|^p\right]}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})}
\end{aligned}$$

$$= \frac{2^{2p-2}\left\{ \left(\frac{(\lambda+\mu)(|\lambda-\mu|)}{4}\right)^p \left[\|(x+y)\|^{2p}+\|(x-y)\|^{2p}\right] + \left[\left(\frac{(\lambda+\mu)}{2}\right)^{2p}+\left(\frac{(|\lambda-\mu|)}{2}\right)^{2p}\right]\|(x+y)\|^p\|(x-y)\|^p \right\}}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})}$$

$$= \frac{2^{(4p-3)}\left(\frac{(\lambda+\mu)(|\lambda-\mu|)}{4}\right)^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})} \frac{\|(x+y)\|^{2p}+\|(x-y)\|^{2p}}{2^{2p-1}(\|x\|^{2p}+\|y\|^{2p})} + \frac{2^{2p-2}\left[\left(\frac{(\lambda+\mu)}{2}\right)^{2p}+\left(\frac{(|\lambda-\mu|)}{2}\right)^{2p}\right]}{2^{-1}(\lambda^{2p}+\mu^{2p})} \times \frac{\|(x+y)\|^p\|(x-y)\|^p}{2^{2p-2}(\|x\|^{2p}+\|y\|^{2p})}$$

$$\leqslant \frac{[(\lambda+\mu)(|\lambda-\mu|)]^p}{(\lambda^{2p}+\mu^{2p})}\mathscr{C}_{NJ}^{2p}(\mathscr{X}) + \frac{2^{2p-1}\left[\left(\frac{(\lambda+\mu)}{2}\right)^{2p}+\left(\frac{(|\lambda-\mu|)}{2}\right)^{2p}\right]}{(\lambda^{2p}+\mu^{2p})}\mathscr{C}_z^p(\mathscr{X}).$$

Therefore,

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) \leqslant \frac{[(\lambda+\mu)(|\lambda-\mu|)]^p}{(\lambda^{2p}+\mu^{2p})}\mathscr{C}_{NJ}^{2p}(\mathscr{X}) + \frac{2^{2p-1}\left[\left(\frac{(\lambda+\mu)}{2}\right)^{2p}+\left(\frac{(|\lambda-\mu|)}{2}\right)^{2p}\right]}{(\lambda^{2p}+\mu^{2p})}\mathscr{C}_z^p(\mathscr{X}). \quad \square$$

For $p=1$ and using Remark 1(a), we have

COROLLARY 5. *For a Banach space $\mathscr{X}$,*

$$\begin{aligned}\mathscr{C}_z(\lambda,\mu,\mathscr{X}) &\leqslant \frac{(\lambda+\mu)|\lambda-\mu|}{2(\lambda^2+\mu^2)}\mathscr{C}_{NJ}^2(\mathscr{X}) + \frac{\left[\left(\frac{(\lambda+\mu)}{2}\right)^2+\left(\frac{(|\lambda-\mu|)}{2}\right)^2\right]}{(\lambda^2+\mu^2)}\mathscr{C}_z^1(\mathscr{X}) \\ &= \frac{(\lambda+\mu)|\lambda-\mu|}{2(\lambda^2+\mu^2)}\mathscr{C}_{NJ}(\mathscr{X}) + \frac{1}{2}\mathscr{C}_z(\mathscr{X}) \\ &= \frac{(\lambda+\mu)|\lambda-\mu|}{(\lambda^2+\mu^2)}\mathscr{C}_{NJ}(\mathscr{X}) + \mathscr{C}_z(\mathscr{X}).\end{aligned}$$

THEOREM 3. *For a Banach space $\mathscr{X}$,*

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) \leqslant \frac{2^{2p}\lambda^{2p}}{(\lambda^{2p}+\mu^{2p})}\mathscr{C}_{NJ}^{2p}(\mathscr{X}) + \frac{2|\mu-\lambda|^{2p}}{(\lambda^{2p}+\mu^{2p})}.$$

Proof. Consider

$$\begin{aligned}
&\frac{\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\
&\leqslant \frac{(\|\lambda x+\mu y\|^{2p}+\|\mu x-\lambda y\|^{2p})}{2\times 2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\
&= \frac{(\|\lambda(x+y)+(\mu-\lambda)y\|)^{2p}+(\|(\mu-\lambda)x+\lambda(x-y)\|)^{2p})}{2^{2p-2}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\
&\leqslant \frac{(\lambda\|(x+y)\|+|\mu-\lambda|\|y\|)^{2p}+(|\mu-\lambda|\|x\|+\lambda\|x-y\|)^{2p})}{2^{2p-2}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\
&\leqslant 2^{2p-1}\times\frac{(\lambda^{2p}\|(x+y)\|^{2p}+|\mu-\lambda|^{2p}\|y\|^{2p})+(|\mu-\lambda|^{2p}\|x\|^{2p}+\lambda^{2p}\|x-y\|^{2p})}{2^{2p-2}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\
&= \frac{\lambda^{2p}(\|(x+y)\|^{2p}+\|x-y\|^{2p})+|\mu-\lambda|^{2p}(\|x\|^{2p}+\|y\|^{2p})}{2^{-1}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\
&= \frac{2^{2p}\lambda^{2p}}{(\lambda^{2p}+\mu^{2p})}\cdot\frac{\|(x+y)\|^{2p}+\|x-y\|^{2p}}{2^{2p-1}(\|x\|^{2p}+\|y\|^{2p})}+\frac{2|\mu-\lambda|^{2p}}{(\lambda^{2p}+\mu^{2p})} \\
&\leqslant \frac{2^{2p}\lambda^{2p}}{(\lambda^{2p}+\mu^{2p})}\mathscr{C}_{NJ}^{2p}(\mathscr{X})+\frac{2|\mu-\lambda|^{2p}}{(\lambda^{2p}+\mu^{2p})}.
\end{aligned}$$

Therefore, $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})\leqslant\frac{2^{2p}\lambda^{2p}}{(\lambda^{2p}+\mu^{2p})}\mathscr{C}_{NJ}^{2p}(\mathscr{X})+\frac{2\times|\mu-\lambda|^{2p}}{(\lambda^{2p}+\mu^{2p})}$. □

Now, we introduce a generalized version of the Ptolemy constant and the constant $\mathscr{C}_G(\mathscr{X})$. Also, we find the lower and upper bound of $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$ using the generalized Ptolemy constant.

DEFINITION 6. For a Banach space $\mathscr{X}$, the real number

$$\mathscr{C}_{\mathscr{P}}^p(\mathscr{X}):=\sup_{x\neq y,y\neq w}\left\{\frac{\|x-y\|^p\|w\|^p}{(\|x-w\|^p\|y\|^p+\|w-y\|^p\|x\|^p)}:x,y,w\in\mathscr{X}\backslash\{0\}\right\}$$

is called a generalized Ptolemy constant.

THEOREM 4. *For a Banach space $\mathscr{X}$, $\mathscr{C}_{\mathscr{P}}^p(\mathscr{X})\geqslant\mathscr{C}_z^p(\mathscr{X})$.*

Proof. If $x,y\in\mathscr{X}$ and $w=x+y$, then

$$\frac{\|x-y\|^p\|w\|^p}{(\|x-w\|^p\|y\|^p+\|w-y\|^p\|x\|^p)}=\frac{\|x-y\|^p\|x+y\|^p}{(\|x\|^{2p}+\|y\|^{2p})}\geqslant\frac{\|x-y\|^p\|x+y\|^p}{2^{2p-2}(\|x\|^{2p}+\|y\|^{2p})}.$$

Therefore, $\mathscr{C}_{\mathscr{P}}^p(\mathscr{X})\geqslant\mathscr{C}_z^p(\mathscr{X})$. □

Consider the following parametrization of the constant $\mathscr{C}_G(\mathscr{X})$: for $\lambda,\mu>0$

$$\mathscr{C}_G^p(\lambda,\mu,\mathscr{X}):=\sup\{\sqrt{\|\lambda x+\mu y\|^p\|\lambda x-\mu y\|^p}:x,y\in\mathscr{S}_{\mathscr{X}}\}. \tag{5}$$

REMARK 3. *For $\lambda = 1$ and $p = 1$ we get, $\mathscr{C}_G^1(1, \mu, \mathscr{X}) = \mathscr{C}_G(\mu, \mathscr{X})$, given by* [15]. *Also, $\mathscr{C}_G^1(1, 1, \mathscr{X}) = \mathscr{C}_G(\mathscr{X})$, given by* [2].

THEOREM 5. *For a Banach space $\mathscr{X}$, $\mathscr{C}_z^p(\lambda, \mu, \mathscr{X}) \geqslant \frac{[\mathscr{C}_G^p(\lambda, \mu, \mathscr{X})]^2}{2^{2p-2}(\lambda^{2p} + \mu^{2p})}$.*

Proof. Given $x, y \in \mathscr{S}_{\mathscr{X}}$ and $\lambda, \mu > 0$, we have

$$\begin{aligned}
\mathscr{C}_z^p(\lambda, \mu, \mathscr{X}) &\geqslant \frac{\|\lambda x + \mu y\|^p \|\mu x - \lambda y\|^p}{2^{2p-3}(\lambda^{2p} + \mu^{2p})(\|x\|^{2p} + \|y\|^{2p})} \\
&= \frac{\|\lambda x + \mu y\|^p \|\mu x - \lambda y\|^p}{2^{2p-2}(\lambda^{2p} + \mu^{2p})}.
\end{aligned}$$

Hence, $\mathscr{C}_z^p(\lambda, \mu, \mathscr{X}) \geqslant \frac{[\mathscr{C}_G^p(\lambda, \mu, \mathscr{X})]^2}{2^{2p-2}(\lambda^{2p} + \mu^{2p})}$. □

THEOREM 6. *For a Banach space $\mathscr{X}$,*

$$\mathscr{C}_z^p(\mathscr{X}) \geqslant \sup\left\{ \frac{[\varepsilon(1 - \delta_{\mathscr{X}}(\varepsilon))]^p}{2^{(p-1)}} : 0 \leqslant \varepsilon \leqslant 2 \right\}.$$

Proof. Suppose that there exist $x, y \in \mathscr{S}_{\mathscr{X}}$, such that $\|x - y\| = \varepsilon$. Consider

$$\begin{aligned}
\mathscr{C}_z^p(\mathscr{X}) &\geqslant \frac{\|x + y\|^p \|x - y\|^p}{2^{2p-2}(\|x\|^{2p} + \|y\|^{2p})} \\
&\geqslant \frac{\|x + y\|^p \varepsilon^p}{2^{2p-2}(1 + 1)}.
\end{aligned}$$

On simplifying, we get

$$\frac{[2^{2p-1}\mathscr{C}_z^p(\mathscr{X})]^{\frac{1}{p}}}{\varepsilon} \geqslant \|x + y\|.$$

Hence

$$1 - \frac{\|x + y\|}{2} \geqslant 1 - \frac{[2^{2p-1}\mathscr{C}_z^p(\mathscr{X})]^{\frac{1}{p}}}{2\varepsilon}.$$

From the definition of $\delta_{\mathscr{X}}(\varepsilon)$, we have

$$\delta_{\mathscr{X}}(\varepsilon) \geqslant 1 - \frac{[2^{2p-1}\mathscr{C}_z^p(\mathscr{X})]^{\frac{1}{p}}}{2\varepsilon}.$$

It implies that

$$2^{2p-1}\mathscr{C}_z^p(\mathscr{X}) \geqslant [2\varepsilon(1 - \delta_{\mathscr{X}}(\varepsilon)]^p.$$

Therefore,

$$\mathscr{C}_z^p(\mathscr{X}) \geqslant \frac{[\varepsilon(1 - \delta_{\mathscr{X}}(\varepsilon))]^p}{2^{(p-1)}}. \tag{6}$$

Hence the result. □

COROLLARY 6. *For a Banach space* $\mathscr{X}$, $\mathscr{C}_z^p(\mathscr{X}) \geqslant \varepsilon_0(\mathscr{X})$.

Proof. Inequality (6) holds for any $\varepsilon \in [0,2]$. It suffices to take in particular $\varepsilon = \varepsilon_0(\mathscr{X})$. □

REMARK 4. *For a given Banach space* $(\mathscr{X}, \|.\|)$, *Theorem* 6 *suggests us to study the function* $\phi_{\mathscr{X}}(\varepsilon) : [0,2] \to [0,2]$ *given by*

$$\phi_{\mathscr{X}}(\varepsilon) = \frac{[\varepsilon(1-\delta_{\mathscr{X}}(\varepsilon))]^p}{2^{(p-1)}}.$$

When $\mathscr{X}$ *is a Hilbert space, then* $\delta_{\mathscr{X}}(\varepsilon) = \frac{2-\sqrt{4-\varepsilon^2}}{2}$. *Therefore, we have*

$$\begin{aligned}\phi_{\mathscr{X}}(\varepsilon) &= \frac{[\varepsilon(1-\delta_{\mathscr{X}}(\varepsilon))]^p}{2^{(p-1)}} \\ &= \frac{\left[\varepsilon\left(1-\frac{2-\sqrt{4-\varepsilon^2}}{2}\right)\right]^p}{2^{(p-1)}} \\ &= \frac{\left[\varepsilon\left(\sqrt{4-\varepsilon^2}\right)\right]^p}{2^{2p-1}}.\end{aligned}$$

It is easy to see that the function ϕ *attains its maximum at* $\varepsilon = \sqrt{2}$, *and that* $\phi(\sqrt{2}) = 1$. *This result indicates that the constant* $\mathscr{C}_z^p(\mathscr{X})$ *has attained the maximum value of 1 in a Hilbert space.*

By restricting $\varepsilon \in (1,2]$, we discuss the positivity of the modulus of convexity with certain conditions using $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$ constant. In addition, we examine the relationship between the modulus of convexity and $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$.

THEOREM 7. $\delta_{\mathscr{X}}(\varepsilon) > 0$, *for a Banach space* $\mathscr{X}$ *with*

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) < \frac{(\lambda+\mu)^{2p}(\varepsilon-1)^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})},$$

where $1 \leqslant p < \infty$, $\lambda \geqslant \mu$, *and* $\varepsilon \in (1,2]$.

Proof. Suppose $\delta_{\mathscr{X}}(\varepsilon) = 0$, then there exist $x_n, y_n \in S_{\mathscr{X}}$ such that $\|x_n - y_n\| = \varepsilon$ for all $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} \|x_n + y_n\| = 2$ and $x_n \to x$, $y_n \to y$. Consider

$$\begin{aligned}\|\lambda x_n + \mu y_n\| &= \|\lambda(x_n+y_n) - (\lambda-\mu)y_n\| \\ &\geqslant \lambda\|x_n+y_n\| - (\lambda-\mu)\|y_n\| \\ &= 2\lambda - (\lambda-\mu) \\ &= \lambda+\mu,\end{aligned}$$

and

$$\begin{aligned}\|\mu x_n - \lambda y_n\| &= \|(\lambda+\mu)(x_n - y_n) + \mu y_n - \lambda x_n\| \\ &\geqslant (\lambda+\mu)\|x_n - y_n\| - \mu\|y_n\| - \lambda\|x_n\| \\ &= (\lambda+\mu)\varepsilon - \mu - \lambda \\ &= (\lambda+\mu)(\varepsilon - 1).\end{aligned}$$

Further, we have

$$\begin{aligned}\frac{(\lambda+\mu)^{2p}(\varepsilon-1)^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})} &= \frac{(\lambda+\mu)^p\,[(\lambda+\mu)(\varepsilon-1)]^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})} \\ &\leqslant \liminf_{n\to\infty} \frac{\|\lambda x_n + \mu y_n\|^p\|\mu x_n - \lambda y_n\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x_n\|^{2p}+\|y_n\|^{2p})} \\ &\leqslant \limsup_{n\to\infty} \frac{2^p\|\lambda x_n + \mu y_n\|^p\|\mu x_n - \lambda y_n\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x_n\|^{2p}+\|y_n\|^{2p})} \\ &= \mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) \\ &< \frac{(\lambda+\mu)^{2p}(\varepsilon-1)^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})},\end{aligned}$$

which is absurd. Hence the result. □

THEOREM 8. *For a Banach space* $\mathscr{X}$,

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) \geqslant \frac{[(\lambda+\mu)(\varepsilon-1)(\lambda+\mu-2\lambda\delta_{\mathscr{X}}(\varepsilon))]^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})},$$

where $\lambda \geqslant \mu$, *and* $\varepsilon \in (1,2]$.

Proof. Suppose that there exist $x, y \in \mathscr{S}_{\mathscr{X}}$, such that $\|x-y\| = \varepsilon$. Consider

$$\begin{aligned}\|\mu x - \lambda y\| &= \|(\lambda+\mu)(x-y) + \mu y - \lambda x\| \\ &\geqslant (\lambda+\mu)\|x-y\| - \mu\|y\| - \lambda\|x\| \\ &= (\lambda+\mu)\varepsilon - \mu - \lambda \\ &= (\lambda+\mu)(\varepsilon-1).\end{aligned}$$

By using definition, we get

$$\begin{aligned}\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) &\geqslant \frac{\|\lambda x + \mu y\|^p\|\mu x - \lambda y\|^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\ &\geqslant \frac{(\lambda\|x+y\| - (\lambda-\mu)\|y\|)^p[(\lambda+\mu)(\varepsilon-1)]^p}{2^{2p-3}(\lambda^{2p}+\mu^{2p})(\|x\|^{2p}+\|y\|^{2p})} \\ &\geqslant \frac{(\lambda\|x+y\| - (\lambda-\mu)\|y\|)^p[(\lambda+\mu)(\varepsilon-1)]^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})}.\end{aligned}$$

On simplifying, we have

$$\frac{(\lambda-\mu)}{2\lambda}+\frac{[2^{2p-2}\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})(\lambda^{2p}+\mu^{2p})]^{\frac{1}{p}}}{2\lambda(\lambda+\mu)(\varepsilon-1)} \geqslant \|x+y\|.$$

Hence

$$1-\frac{\|x+y\|}{2} \geqslant 1+\frac{(\mu-\lambda)}{2\lambda}-\frac{[2^{2p-2}\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})(\lambda^{2p}+\mu^{2p})]^{\frac{1}{p}}}{2\lambda(\lambda+\mu)(\varepsilon-1)}.$$

From the definition of $\delta_{\mathscr{X}}(\varepsilon)$, we have

$$\delta_{\mathscr{X}}(\varepsilon) \geqslant \frac{1}{2}+\frac{\mu}{2\lambda}-\frac{[2^{2p-2}\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})(\lambda^{2p}+\mu^{2p})]^{\frac{1}{p}}}{2\lambda(\lambda+\mu)(\varepsilon-1)}.$$

It implies that

$$2^{2p-2}\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})(\lambda^{2p}+\mu^{2p}) \geqslant [(\lambda+\mu)(\varepsilon-1)(\lambda+\mu-2\lambda\delta_{\mathscr{X}}(\varepsilon))]^p.$$

Therefore,

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) \geqslant \frac{[(\lambda+\mu)(\varepsilon-1)(\lambda+\mu-2\lambda\delta_{\mathscr{X}}(\varepsilon))]^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})}. \quad \square$$

Taking $\lambda=\mu$ in the Theorem 8, we have

COROLLARY 7. *For a Banach space* $\mathscr{X}$*,*

$$\mathscr{C}_z^p(\mathscr{X}) \geqslant [(\varepsilon-1)(1-\delta_{\mathscr{X}}(\varepsilon))]^p, \textit{ where } \varepsilon \in (1,2].$$

3.3. Uniformly non-square and normal structure

In this section, first, we characterize a uniformly non-square space in terms of the generalized p-Zbăganu constant.

THEOREM 9. *A Banach space* $\mathscr{X}$ *with*

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) < \frac{(3\lambda-\mu)^p(\lambda+\mu)^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})},$$

for some $\lambda,\mu>0$*, is uniformly non-square.*

Proof. Without loss of generality, let $\mu \geqslant \lambda$. Suppose $\mathscr{X}$ is not a uniformly non-square space. For all $0<\varepsilon<\lambda^2$, there exist $x,y \in \mathscr{S}_{\mathscr{X}}$, such that both $\|x+y\|$ and

$\|x-y\|$ is strictly greater than $2-\frac{\varepsilon}{4\lambda^2}$. Now, for $x,y \in \mathscr{S}_{\mathscr{X}}$, we have

$$\begin{aligned}
\|\lambda x+\mu y\| &= \|\lambda(x+y)+(\mu-\lambda)y\| \\
&\geqslant \lambda\|x+y\|-(\mu-\lambda)\|y\| \\
&= \lambda\left(2-\frac{\varepsilon}{4\lambda^2}\right)-(\mu-\lambda) \\
&= 3\lambda-\mu-\frac{\varepsilon}{4\lambda},
\end{aligned}$$

and

$$\begin{aligned}
\|\mu x-\lambda y\| &= \|\mu(x-y)+(\mu-\lambda)y-\lambda x\| \\
&\geqslant \mu\|x-y\|-(\mu-\lambda)\|y\| \\
&= \mu\left(2-\frac{\varepsilon}{4\lambda^2}\right)-(\mu-\lambda) \\
&= \lambda+\mu-\frac{\varepsilon\mu}{4\lambda^2}.
\end{aligned}$$

So, $\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p \geqslant \left(3\lambda-\mu-\frac{\varepsilon}{4\lambda}\right)^p\left(\lambda+\mu-\frac{\varepsilon\mu}{4\lambda^2}\right)^p$. Since ε is arbitrarily small, $\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p \geqslant (3\lambda-\mu)^p(\lambda+\mu)^p$. Therefore,

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) \geqslant \frac{\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})} \geqslant \frac{(3\lambda-\mu)^p(\lambda+\mu)^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})},$$

which is a contradiction to the given assumption. Hence the result. □

It is important to note that every uniformly non-square Banach space has the fixed point property [7]. Thus, we conclude the following corollary.

COROLLARY 8. *If $\mathscr{X}$ is a Banach space with $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) < \frac{(3\lambda-\mu)^p(\lambda+\mu)^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})}$, for some $\lambda,\mu>0$, then $\mathscr{X}$ has the fixed point property.*

As every uniformly non-square Banach space is super-reflexive (see [18]), we have the following result.

COROLLARY 9. *If $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) < \frac{(3\lambda-\mu)^p(\lambda+\mu)^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})}$, then $\mathscr{X}$ is a super-reflexive Banach space.*

Taking $\lambda=\mu$ in the Theorem 9, we obtain the following result.

THEOREM 10. *A Banach space $\mathscr{X}$ with $\mathscr{C}_z^p(\mathscr{X})<2$ is uniformly non-square.*

Now, we discuss normal structure of Banach spaces in terms of $\mathscr{C}_z^p(\mathscr{X})$.

THEOREM 11. *A Banach space $\mathscr{X}$ with $\mathscr{C}_z^p(\mathscr{X}) < \frac{2^{(2p+2)}}{1+3^{2p}}$ has normal structure.*

Proof. Assume that $\mathscr{C}_z^p(\mathscr{X}) < \frac{2^{(2p+2)}}{1+3^{2p}}$ is in a Banach space $\mathscr{X}$. Since $\mathscr{C}_z^p(\mathscr{X}) < \frac{2^{(2p+2)}}{1+3^{2p}} < 2$, using Theorem 10, $\mathscr{X}$ is uniformly non-square and so reflexive. In a reflexive Banach space, normal structure and weak normal structure are equivalent. To prove that $\mathscr{X}$ has normal structure, we assume on contrary.

If possible, suppose that $\mathscr{X}$ has not normal structure. Using the Lemma 2 with $a = 1$, substitute $u = \frac{1}{2}(w+y)$ and $v = x + \frac{1}{2}(y-w) = \frac{3}{2}x$ in the definition of $\mathscr{C}_z^p$, we have

$$\begin{aligned}\mathscr{C}_z^p(\mathscr{X}) &\geqslant \frac{\|u+v\|^p\|u-v\|^p}{2^{2p-2}(\|u\|^{2p}+\|v\|^{2p})}\\ &\geqslant \frac{\|x+y\|^p\|w-x\|^p}{2^{2p-2}(\|\frac{w+y}{2}\|^{2p}+\|\frac{3}{2}x\|^{2p})}\\ &> \frac{(2-2\varepsilon)^{2p}}{2^{2p-2}\left(1+(\frac{3}{2})^{2p}\right)}.\end{aligned}$$

Letting $\varepsilon \to 0^+$ we obtain

$$\begin{aligned}\mathscr{C}_z^p(\mathscr{X}) &> \frac{2^{2p}}{2^{2p-2}\left(1+(\frac{3}{2})^{2p}\right)}\\ &= \frac{2^{(2p+2)}}{1+3^{2p}},\end{aligned}$$

which is a contradiction. □

THEOREM 12. *A Banach space $\mathscr{X}$ with*

$$\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) < \frac{(2\mu-\lambda)^p(\lambda+\mu)^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})},$$

for some $\lambda,\mu > 0$, has weak normal structure.

Proof. Let $\mathscr{X}$ be a Banach space without weak normal structure. So, using the Lemma 2, we have

$$\begin{aligned}\|\lambda x+\mu y\| &= \|\lambda(x+y)-(\lambda-\mu)y\|\\ &\geqslant \lambda\|x+y\|-(\lambda-\mu)\|y\|\\ &= \lambda(2-2\varepsilon)-(\lambda-\mu)\\ &= \lambda+\mu-2\lambda\varepsilon,\end{aligned}$$

and

$$
\begin{aligned}
\|\mu x-\lambda y\| &= \|\lambda y-\mu x\| \\
&= \|\lambda y+\mu y-\mu y-\mu x\| \\
&= \|\lambda y+\mu(ax+w)-\mu y-\mu x\| \\
&= \|\mu(w-x)+(\lambda-\mu)y+\mu ax\| \\
&\geqslant \mu(2-2\varepsilon)-(\lambda-\mu+\mu a) \\
&= \mu(2-2\varepsilon)-\lambda-(a-1)\mu \\
&\geqslant (2\mu-\lambda)-2\varepsilon-\delta\mu.
\end{aligned}
$$

So, $\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p \geqslant (\lambda+\mu-2\lambda\varepsilon)^p\,((2\mu-\lambda)-2\varepsilon-\varepsilon\mu)^p$. Since ε is arbitrarily small, $\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p \geqslant (\lambda+\mu)^p\,(2\mu-\lambda)^p$. Therefore,

$$
\begin{aligned}
\mathscr{C}_z^p(\lambda,\mu,\mathscr{X}) &\geqslant \frac{\|\lambda x+\mu y\|^p\|\mu x-\lambda y\|^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})} \\
&\geqslant \frac{(2\mu-\lambda)^p\,(\lambda+\mu)^p}{2^{2p-2}(\lambda^{2p}+\mu^{2p})},
\end{aligned}
$$

which is a contradiction to the given assumption. Hence the result. □

4. Conclusion

In conclusion, the paper presents the generalized p-Zbăganu constant $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$ in Banach space $\mathscr{X}$. We obtain its bound in Banach and Hilbert spaces. Also, we explore the relationship of the generalized p-Zbăganu constant with other renowned geometric constants.

We have the following open question from our observation:

Is it possible to determine the characterisation of the inner product utilising the constant $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$

Which other geometric constants are more closely related to $\mathscr{C}_z^p(\lambda,\mu,\mathscr{X})$ than those mentioned in the study?

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