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SOME PROPERTIES OF NORMALIZED REMAINDERS OF THE MACLAURIN EXPANSION FOR A FUNCTION ORIGINATING FROM AN INTEGRAL REPRESENTATION OF THE RECIPROCAL OF THE GAMMA FUNCTION

JUAN ZHANG AND FENG QI*

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Abstract. It is well known that the classical Euler gamma function $\Gamma(z)$ has had very extensive applications in mathematical sciences, including physics and engineering, in the past centuries. In this study, the authors introduce the normalized remainder $T_{2n+1}[\Phi(\theta)]$ of the Maclaurin expansion of the function $\Phi(\theta)=1-\frac{\theta}{\tan\theta}+\ln\frac{\theta}{\sin\theta}$ for $\theta\in(-\pi,\pi)$, which is contained in an integral representation of the reciprocal $\frac{1}{\Gamma(z)}$. In light of the increasing property of two sequences involving the ratio of two non-zero Bernoulli numbers and with the aid of the monotonicity rule for the ratio of two Maclaurin series, they present the logarithmic convexity of the normalized remainder $T_{2n+1}[\Phi(\theta)]$ for $n\in\mathbb{N}_0=\{0,1,2,\dots\}$ in $\theta\in(-\pi,\pi)$ and discuss the monotonicity of the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ for $n\in\mathbb{N}_0$ in $\theta\in(-\pi,0)\cup(0,\pi)$.

1. A short survey on normalized remainders and motivations

We start out by recalling the following definition of normalized remainders of the Maclaurin expansions of functions, which has been created since April 2023 and can be found in [2, Section 5], [16, Section 1], [18, Sections 1.9 and 1.10], [25, Remarks 2 and 4], and [31, Section 1].

DEFINITION 1. Let G be a real infinitely differentiable function on an interval $I\subseteq\mathbb{R}$ such that the origin 0 is an interior point of I. If $G^{(n+1)}(0)\ne0$ for some $n\in\mathbb{N}_0=\{0\}\cup\mathbb{N}=\{0,1,2,\dots\}$, then we call the function

$$T_n[G(u)]=\begin{cases}\dfrac{1}{G^{(n+1)}(0)}\dfrac{(n+1)!}{u^{n+1}}\left[G(u)-\displaystyle\sum_{j=0}^{n}G^{(j)}(0)\frac{u^j}{j!}\right], & u\ne0\\ 1, & u=0\end{cases}\tag{1}$$

for $u\in I$ the nth normalized remainder or the nth normalized tail of the Maclaurin expansion of the function G.

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* Corresponding author.

The notion of normalized remainders $T_n[G(u)]$ is an intrinsic modification of the Lagrange remainders, the Cauchy remainders, the Schlömilch remainders, and the Rouché remainders stated in [4, p. 19]. For $G(u)=\sin u$ in (1), the normalized remainder $T_{2n}[\sin u]$ was considered in [6, Remark 7] and [10, 12, 26, 33]. For $G(u)=\cos u$, the normalized remainder $T_{2n-1}[\cos u]$ was studied in [6, Remark 7] and [7, 12, 24, 33]. For

$$G(u)=\ln T_1[\cos u]=\begin{cases}\ln\dfrac{2(1-\cos u)}{u^2}, & 0<|u|<2\pi\\ 0, & u=0\end{cases}$$

in (1), the normalized remainder $T_{2n-1}[\ln T_1[\cos u]]$ was explored in [13]. For the case $G(u)=e^u$ in (1), the normalized remainder $T_n[e^u]$ was investigated in [2, 8, 16, 30]. For $G(u)=\tan u$, $G(u)=\tan^2 u$, and $G(u)=\sec^2 u$, three normalized remainders $T_{2n}[\tan u]$, $T_{2n-1}\left[\tan^2 u\right]$, and $T_{2n-1}\left[\sec^2 u\right]$ were explored in [9, 19, 31]. For $G(u)=\frac{u}{e^u-1}$, the normalized remainder $T_{2n-1}\left[\frac{u}{e^u-1}\right]=T_{2n-1}\left[\frac{1}{T_0[e^u]}\right]$ was examined in [32]. For $G(u)=\ln\sec u$, equivalently $G(u)=\ln\cos u$, the normalized remainder $T_{2n-1}[\ln\sec u]$ was studied in [29]. In [16, Section 1], [18, Section 1], and [31, Section 1], from several different angles, the second author and his coworkers reviewed, surveyed, described, depicted, and retrospected their motivations, ideas, and thoughts to introduce and invent the notion of normalized remainders $T_n[G(u)]$ in details.

In the above-mentioned papers, the second author and his coworkers mainly explored the following properties of the normalized remainder $T_n[G(u)]$:

1. Positivity of the normalized remainder $T_n[G(u)]$; see [6, Remark 7], [18, Section 1], and [12, 16, 32, 33].

2. Monotonicity of the normalized remainder $T_n[G(u)]$; see [2, 12, 24, 32, 33], [6, Remark 7], and [18, Section 1].

3. Convexity of the normalized remainder $T_n[G(u)]$; see [6, Remark 7], [18, Section 1], [32, Remark 5], and [33].

4. Logarithmic convexity of the normalized remainder $T_n[G(u)]$; see [2, 24, 31], [6, Remark 7], and [18, Sections 1 and 5].

5. Absolute monotonicity of the normalized remainder $T_n[G(u)]$; see [16], [18, Section 1], and [25, Remarks 2 and 4].

6. Maclaurin series of the logarithm $\ln T_n[G(u)]$; see [2, 7, 9, 10, 24, 26, 31], [6, Remark 7], and [18, Sections 1 and 7].

7. Monotonicity of the ratio $\frac{T_{n+1}[G(u)]}{T_n[G(u)]}$; see [11, 12, 16, 32] and [18, Section 1].

8. Monotonicity of the ratio $\frac{\ln T_{n+1}[G(u)]}{\ln T_n[G(u)]}$; see [2, 7, 10, 24, 26], [6, Remark 7], and [18, Section 1].

9. Connections between the normalized remainder $T_n[G(u)]$ with hypergeometric functions; see [6, Remark 7], [18, Section 1], and [12, 26, 33].

10. Inequalities for the function $G(u)$ and related ones; see [2, 11, 12, 16, 25, 32, 33] and [18, Sections 1 and 5].

11. General properties of the normalized remainder $T_n[G(u)]$; see [18, Section 1.10].

In [8, Section 1] and [25, Remark 2], the normalized remainder $T_n[\mathrm{e}^u]$ considered in [2, 8, 16, 30, 32] was applied to unify the generating functions of the Bernoulli numbers and polynomials, the Stirling numbers of the second kind, and the Carlitz–Howard numbers and polynomials.

The classical Euler gamma function $\Gamma(z)$ can be defined [23, Chapter 3] by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

It is general knowledge for scientists that the gamma function $\Gamma(z)$ has had very extensive applications in mathematical sciences, including physics and engineering, in the past centuries. In [23, p. 71, Eq. (3.38)], we find the integral representation

$$\frac{1}{\Gamma(z)}=\frac{\mathrm{e}^z z^{1-z}}{\pi}\int_0^\pi \mathrm{e}^{-z\Phi(\theta)}\,\mathrm{d}\theta,\quad \Re(z)\geqslant 0, \tag{2}$$

where

$$\begin{aligned}\Phi(\theta)&=1-\frac{\theta}{\tan\theta}+\ln\frac{\theta}{\sin\theta}\\ &=\sum_{j=1}^{\infty}\frac{2j+1}{2j}|B_{2j}|\frac{(2\theta)^{2j}}{(2j)!}\\ &=\frac{1}{2}\theta^2+\frac{1}{36}\theta^4+\frac{1}{405}\theta^6+\frac{1}{4200}\theta^8+\frac{1}{42525}\theta^{10}+\dots\end{aligned} \tag{3}$$

for $|\theta|<\pi$ and B_{2j} stands for the classical Bernoulli numbers which are generated in [23, p. 3] by

$$\frac{1}{T_0[\mathrm{e}^\theta]}=\frac{\theta}{\mathrm{e}^\theta-1}=\sum_{j=0}^{\infty}B_j\frac{\theta^j}{j!}=1-\frac{\theta}{2}+\sum_{j=1}^{\infty}B_{2j}\frac{\theta^{2j}}{(2j)!},\quad |\theta|<2\pi.$$

As stated in [23, p. 71], the integral representation (2) is very useful when one wants to evaluate the gamma function $\Gamma(z)$ by means of a simple quadrature rule.

In this paper, we concentrate our attention on the normalized remainder

$$T_{2n+1}[\Phi(\theta)]=\begin{cases}\dfrac{2n+2}{2n+3}\dfrac{1}{|B_{2n+2}|}\dfrac{(2n+2)!}{(2\theta)^{2n+2}}\left[\Phi(\theta)-\displaystyle\sum_{j=1}^{n}\frac{2j+1}{2j}|B_{2j}|\frac{(2\theta)^{2j}}{(2j)!}\right], & \theta\ne0\\ 1, & \theta=0\end{cases}$$

for $n\in\mathbb{N}_0$ and $\theta\in(-\pi,\pi)$, which is obviously an even function of $\theta\in(-\pi,\pi)$.

It is clear that

$$T_{2n+1}[\Phi(\theta)]=\frac{2n+2}{2n+3}\frac{(2n+2)!}{|B_{2n+2}|}\sum_{j=0}^{\infty}\frac{2j+2n+3}{2j+2n+2}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}(2\theta)^{2j} \tag{4}$$

for $n \in \mathbb{N}_0$ and $|\theta| < \pi$. Hence, it is easy to see that the normalized remainder $T_{2n+1}[\Phi(\theta)]$ for $n \in \mathbb{N}_0$ is positive in $\theta \in (-\pi,\pi)$, decreasing in $\theta \in (-\pi,0)$, and increasing in $\theta \in (0,\pi)$.

Differentiating twice gains

$$T''_{2n+1}[\Phi(\theta)] = \frac{n+1}{2n+3}\frac{(2n+2)!}{|B_{2n+2}|}\sum_{j=0}^{\infty}\frac{2j+2n+5}{j+n+2}\frac{|B_{2j+2n+4}|}{(2j+2n+4)!}2^{2j+3}(j+1)(2j+1)\theta^{2j}.$$

Therefore, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ for $n \in \mathbb{N}_0$ is convex in $\theta \in (-\pi,\pi)$.

From the series representation (4), it is not difficult to see that the normalized remainder $T_{2n+1}[\Phi(\theta)]$ for $n \in \mathbb{N}_0$ is an absolutely monotonic function in $\theta \in (0,\pi)$ and a completely monotonic function in $\theta \in (-\pi,0)$. For information on absolutely (completely) monotonic functions, please refer to the paper [16] and the monograph [21].

In this work, we aim to present the logarithmic convexity of the normalized remainder $T_{2n+1}[\Phi(\theta)]$ for $n \in \mathbb{N}_0$ in $\theta \in (-\pi,\pi)$ and aim to discuss the monotonicity of the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ for $n \in \mathbb{N}_0$ in $\theta \in (-\pi,0) \cup (0,\pi)$.

2. Lemmas

For smoothly proceeding, we recall and establish the following lemmas.

LEMMA 1. ([13, Lemma 2]) *The sequence*

$$\frac{j}{(j+1)^2(2j+1)}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

is increasing in $j \geqslant 0$.

LEMMA 2. ([3, 27]) *Let* λ_j *and* μ_j *for* $j \in \mathbb{N}_0$ *be real sequences and the Maclaurin power series*

$$P(\vartheta) = \sum_{j=0}^{\infty}\lambda_j\vartheta^j \quad \text{and} \quad Q(\vartheta) = \sum_{j=0}^{\infty}\mu_j\vartheta^j$$

converge on $(-\rho,\rho)$ *for some scalar* $\rho > 0$. *If* $\mu_j > 0$ *and the sequence* $\frac{\lambda_j}{\mu_j}$ *increases in* $j \geqslant 0$, *then the function* $\vartheta \mapsto \frac{P(\vartheta)}{Q(\vartheta)}$ *increases on* $(0,\rho)$.

LEMMA 3. *The sequence*

$$\frac{j(2j+3)}{(j+1)^2(2j+1)^2}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

is increasing in $j \in \mathbb{N}_0$.

Proof. In the proof of [22, Theorem 1.1], the relation

$$\left|\frac{B_{2j+2}}{B_{2j}}\right| = \frac{1}{2\pi^2}\left[(2j+1)(j+1)\frac{\zeta(2j+2)}{\zeta(2j)}\right] \tag{5}$$

for $j \in \mathbb{N}$ was derived, where the Riemann zeta function $\zeta(z)$ can be defined by the series $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ under the condition $\Re(z) > 1$ and by analytic continuation elsewhere. Utilizing the relation (5) yields

$$\frac{j(2j+3)}{(j+1)^2(2j+1)^2}\left|\frac{B_{2j+2}}{B_{2j}}\right| = \frac{1}{2\pi^2}\frac{j(2j+3)}{(j+1)(2j+1)}\frac{\zeta(2j+2)}{\zeta(2j)}$$

for $j \in \mathbb{N}$. Basing on the proof of [13, Lemma 2] (that is, Lemma 1 above-mentioned), we see that the positive sequence $\frac{\zeta(2j+2)}{\zeta(2j)}$ is increasing in $j \in \mathbb{N}$. Meanwhile, the positive sequence $\frac{j(2j+3)}{(j+1)(2j+1)}$ is increasing in $j \in \mathbb{N}_0$. The required result in Lemma 3 is thus proved. □

3. Logarithmic convexity of normalized remainder $T_{2n+1}[\Phi(\theta)]$

In this section, we present a nontrivial fundamental property of the normalized remainder $T_{2n+1}[\Phi(\theta)]$: the logarithmic convexity of $T_{2n+1}[\Phi(\theta)]$ for given $n \in \mathbb{N}_0$ in $\theta \in (-\pi, \pi)$.

THEOREM 1. *For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ is logarithmically convex in $\theta \in (-\pi, \pi)$.*

Proof. Directly computing yields

$$\begin{aligned}
\frac{\mathrm{d}\ln T_{2n+1}[\Phi(\theta)]}{\mathrm{d}\theta} &= \frac{T'_{2n+1}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]} \\
&= \frac{\sum_{j=1}^{\infty} \frac{2j+2n+3}{2j+2n+2}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}2^{2j}(2j)\theta^{2j-1}}{\sum_{j=0}^{\infty} \frac{2j+2n+3}{2j+2n+2}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}2^{2j}\theta^{2j}} \\
&= 8\theta\frac{\sum_{j=0}^{\infty} \frac{2j+2n+5}{j+n+2}\frac{|B_{2j+2n+4}|}{(2j+2n+4)!}2^{2j}(j+1)\theta^{2j}}{\sum_{j=0}^{\infty} \frac{2j+2n+3}{j+n+1}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}2^{2j}\theta^{2j}},
\end{aligned}$$

where we used the series representation (4). Furthermore, the ratio between the corresponding coefficients of the factors θ^{2j} of two series in the last fraction is

$$\begin{aligned}
\frac{\frac{2j+2n+5}{j+n+2}\frac{|B_{2j+2n+4}|}{(2j+2n+4)!}2^{2j}(j+1)}{\frac{2j+2n+3}{j+n+1}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}2^{2j}} &= \frac{(j+1)(j+n+1)(2j+2n+5)}{2(j+n+2)^2(2j+2n+3)^2}\left|\frac{B_{2j+2n+4}}{B_{2j+2n+2}}\right| \\
&= \frac{(j+1)(2j+2n+5)}{2(2j+2n+3)}\frac{j+n+1}{(j+n+2)^2(2j+2n+3)}\left|\frac{B_{2(j+n+2)}}{B_{2(j+n+1)}}\right|, \quad j,n \in \mathbb{N}_0.
\end{aligned} \tag{6}$$

Making use of Lemma 1, we see that the sequence

$$\frac{j+n+1}{(j+n+2)^2(2j+2n+3)}\left|\frac{B_{2(j+n+2)}}{B_{2(j+n+1)}}\right|$$

for given $n\in\mathbb{N}_0$ is increasing in $j\in\mathbb{N}_0$. It is not difficult to verify that the positive sequence $\frac{(j+1)(2j+2n+5)}{2(2j+2n+3)}$ for given $n\in\mathbb{N}_0$ is increasing in $j\in\mathbb{N}_0$ too. Therefore, the sequence in (6) is increasing in $j\in\mathbb{N}_0$ for given $n\in\mathbb{N}_0$. Consequently, by virtue of Lemma 2, we derive that the derivative

$$\frac{\mathrm{d}\ln T_{2n+1}[\Phi(\theta)]}{\mathrm{d}\theta}=\frac{T'_{2n+1}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$$

for given $n\in\mathbb{N}_0$ is increasing in $\theta\in(0,\pi)$. This means that the second derivative

$$\frac{\mathrm{d}^2\ln T_{2n+1}[\Phi(\theta)]}{\mathrm{d}\theta^2}=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{T'_{2n+1}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}\right)$$

for given $n\in\mathbb{N}_0$ is positive in $\theta\in(0,\pi)$. As a result, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ for given $n\in\mathbb{N}_0$ is logarithmically convex in $\theta\in(0,\pi)$. Considering the evenness of $T_{2n+1}[\Phi(\theta)]$ for $n\in\mathbb{N}_0$ in $\theta\in(-\pi,\pi)$, we conclude that the normalized remainder $T_{2n+1}[\Phi(\theta)]$ for given $n\in\mathbb{N}_0$ is logarithmically convex in $\theta\in(-\pi,\pi)$ too. Theorem 1 is thus proved. □

COROLLARY 1. *For $n\in\mathbb{N}_0$ and $\theta\in(-\pi,\pi)$, the inequality*

$$\frac{T'_{2n+1}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}\geqslant 4\theta\frac{(n+1)(2n+5)}{(n+2)^2(2n+3)^2}\left|\frac{B_{2n+4}}{B_{2n+2}}\right| \tag{7}$$

is sound. The equality in (7) is valid if and only if $\theta=0$.

Proof. From the proof of Theorem 1, we deduce that the even function

$$\frac{1}{8\theta}\frac{T'_{2n+1}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}=\frac{\sum_{j=0}^{\infty}\frac{2j+2n+5}{j+n+2}\frac{|B_{2j+2n+4}|}{(2j+2n+4)!}2^{2j}(j+1)\theta^{2j}}{\sum_{j=0}^{\infty}\frac{2j+2n+3}{j+n+1}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}2^{2j}\theta^{2j}}$$

is increasing in $\theta\in(0,\pi)$ for given $n\in\mathbb{N}_0$. Hence, we acquire the inequality

$$\begin{aligned}\frac{1}{8\theta}\frac{T'_{2n+1}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}&>\lim_{\theta\to0}\frac{\sum_{j=0}^{\infty}\frac{2j+2n+5}{j+n+2}\frac{|B_{2j+2n+4}|}{(2j+2n+4)!}2^{2j}(j+1)\theta^{2j}}{\sum_{j=0}^{\infty}\frac{2j+2n+3}{j+n+1}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}2^{2j}\theta^{2j}}\\&=\frac{(n+1)(2n+5)}{2(n+2)^2(2n+3)^2}\left|\frac{B_{2n+4}}{B_{2n+2}}\right|\end{aligned}$$

for $n\in\mathbb{N}_0$ and $\theta\in(-\pi,0)\cup(0,\pi)$. The proof of Corollary 1 is complete. □

4. Monotonicity of the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$

In this section, we discuss another nontrivial fundamental property of the normalized remainder $T_{2n+1}[\Phi(\theta)]$: the monotonicity of the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ for given $n \in \mathbb{N}_0$ in $\theta \in (-\pi,0) \cup (0,\pi)$.

THEOREM 2. *For $n \in \mathbb{N}_0$, the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ is decreasing in $\theta \in (-\pi,0)$ and increasing in $\theta \in (0,\pi)$. Consequently, the normalized remainders $T_{2n+1}[\Phi(\theta)]$ for $n \in \mathbb{N}_0$ satisfy the inequality*

$$T_{2n+3}[\Phi(\theta)] > T_{2n+1}[\Phi(\theta)], \quad \theta \in (-\pi,0) \cup (0,\pi). \tag{8}$$

Proof. In view of the series representation (4), we obtain

$$\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]} = \frac{\frac{2n+4}{2n+5}\frac{(2n+4)!}{|B_{2n+4}|}\sum_{j=0}^{\infty}\frac{2j+2n+5}{2j+2n+4}\frac{|B_{2j+2n+4}|}{(2j+2n+4)!}(2\theta)^{2j}}{\frac{2n+2}{2n+3}\frac{(2n+2)!}{|B_{2n+2}|}\sum_{j=0}^{\infty}\frac{2j+2n+3}{2j+2n+2}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}(2\theta)^{2j}}. \tag{9}$$

The ratio between the corresponding coefficients of the terms $(2\theta)^{2j}$ of two series in the last fraction is

$$\begin{aligned}\frac{\frac{2j+2n+5}{2j+2n+4}\frac{|B_{2j+2n+4}|}{(2j+2n+4)!}}{\frac{2j+2n+3}{2j+2n+2}\frac{|B_{2j+2n+2}|}{(2j+2n+2)!}} &= \frac{(j+n+1)(2j+2n+5)}{2(j+n+2)^2(2j+2n+3)^2}\left|\frac{B_{2j+2n+4}}{B_{2j+2n+2}}\right| \\ &= \frac{1}{2}\frac{2j+2n+5}{2j+2n+3}\frac{j+n+1}{(j+n+2)^2(2j+2n+3)}\left|\frac{B_{2j+2n+4}}{B_{2j+2n+2}}\right|\end{aligned} \tag{10}$$

for $j,n \in \mathbb{N}_0$. Replacing $j+n+1$ by k in the last sequence gives

$$\frac{1}{2}\frac{k(2k+3)}{(k+1)^2(2k+1)^2}\left|\frac{B_{2k+2}}{B_{2k}}\right|, \quad k \in \mathbb{N}_0.$$

In light of Lemma 3, we are sure that the sequence in (10) for given $n \in \mathbb{N}_0$ is increasing in $j \in \mathbb{N}_0$. By virtue of Lemma 2, we derive that the ratio in (9) for given $n \in \mathbb{N}_0$ is increasing in $\theta \in (0,\pi)$. Due to that the ratio in (9) for given $n \in \mathbb{N}_0$ is even in $\theta \in (-\pi,\pi)$, we deduce that the ratio in (9) for given $n \in \mathbb{N}_0$ is decreasing in $\theta \in (-\pi,0)$.

The inequality (8) follows from the monotonicity of the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ and the value of the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ at $\theta = 0$ is equal to 1. The proof of Theorem 2 is complete. □

5. Remarks

In this section, we list several remarks as follows.

REMARK 1. In [4, p. 55, Entry 1.518], we find the Maclaurin expansion

$$\ln\sin\theta = \ln\theta + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-1}}{j} B_{2j} \frac{\theta^{2j}}{(2j)!}, \quad 0 < \theta < \pi. \tag{11}$$

The power series expansion (11) is equivalent to

$$\ln\frac{\theta}{\sin\theta} = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{2^{2j-1}}{j} B_{2j} \frac{\theta^{2j}}{(2j)!}, \quad 0 < |\theta| < \pi. \tag{12}$$

In [1, p. 75, Entry 4.3.70], we find the Laurent series expansion

$$\cot\theta = \frac{1}{\theta} - \sum_{j=1}^{\infty} (-1)^{j-1} 2^{2j} B_{2j} \frac{\theta^{2j-1}}{(2j)!}, \quad 0 < |\theta| < \pi. \tag{13}$$

The series expansion (13) can also be derived from differentiating on both sides of (11). Conversely, the series expansion (11) can also be derived from integrating on both sides of (13). Therefore, we obtain

$$\begin{aligned}
\Phi(\theta) &= 1 - \theta\left[\frac{1}{\theta} - \sum_{j=1}^{\infty} (-1)^{j-1} 2^{2j} B_{2j} \frac{\theta^{2j-1}}{(2j)!}\right] + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{2^{2j-1}}{j} B_{2j} \frac{\theta^{2j}}{(2j)!} \\
&= \sum_{j=1}^{\infty} (-1)^{j-1} 2^{2j} B_{2j} \frac{\theta^{2j}}{(2j)!} + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{2^{2j-1}}{j} B_{2j} \frac{\theta^{2j}}{(2j)!} \\
&= \sum_{j=1}^{\infty} \frac{2j+1}{2j} |B_{2j}| \frac{(2\theta)^{2j}}{(2j)!}, \quad |\theta| < \pi.
\end{aligned}$$

This gives an alternative proof of the Maclaurin expansion (3).

In the paper [17, Remark 13] and [20], the Maclaurin expansion (12) was discussed and generalized.

REMARK 2. ([5, p. 11] and [17, Theorem 15]) If $\alpha > 0$, the Maclaurin expansion

$$\cos^{\alpha} z = \sum_{k=0}^{\infty} (-1)^k \left[\sum_{\ell=0}^{2k} \frac{(-\alpha)_\ell}{\ell!} \sum_{m=0}^{\ell} \frac{(-1)^m}{2^m} \binom{\ell}{m} \sum_{q=0}^{m} \binom{m}{q} \left(\frac{m}{2} - q\right)^{2k}\right] \frac{(2z)^{2k}}{(2k)!} \tag{14}$$

converges for $z \in \mathbb{C}$; if $\alpha < 0$, the series expansion (14) converges for $|z| < \frac{\pi}{2}$. See also the answer at the website https://math.stackexchange.com/a/4976672 (accessed on 26 September 2024).

REMARK 3. Since

$$\frac{j(2j+3)}{(j+1)^2(2j+1)^2}\left|\frac{B_{2j+2}}{B_{2j}}\right| = \frac{2j+3}{2j+1}\frac{j}{(j+1)^2(2j+1)}\left|\frac{B_{2j+2}}{B_{2j}}\right|, \quad j\in\mathbb{N}_0$$

and the sequence $\frac{2j+3}{2j+1}$ is decreasing in $j\in\mathbb{N}_0$, Lemma 3 is stronger than Lemma 1.

REMARK 4. Taking $n=0,1,2$ in (7) gives the inequalities

$$9\left(\frac{\theta}{\sin\theta}\right)^2-\left(\theta^2+18\right)\ln\frac{\theta}{\sin\theta}+\theta^2\frac{\theta}{\tan\theta}>9+\theta^2,$$

$$45\left(\frac{\theta}{\sin\theta}\right)^2-4\left(2\theta^2+45\right)\ln\frac{\theta}{\sin\theta}+2\left(4\theta^2+45\right)\frac{\theta}{\tan\theta}>135-37\theta^2-4\theta^4,$$

and

$$5040\left(\frac{\theta}{\sin\theta}\right)^2-108\left(9\theta^2+280\right)\ln\frac{\theta}{\sin\theta}+36\left(27\theta^2+560\right)\frac{\theta}{\tan\theta}$$
$$>25200-9108\theta^2-766\theta^4-27\theta^6$$

for $\theta\in(-\pi,0)\cup(0,\pi)$.

REMARK 5. We guess that the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ for $n\in\mathbb{N}_0$ should be convex, even logarithmically convex, in $\theta\in(-\pi,\pi)$.

For example, when $n=0$, we have

$$\begin{aligned}
\frac{T_3[\Phi(\theta)]}{T_1[\Phi(\theta)]} &= 9\left[\frac{2}{\theta^2}-\frac{1}{\Phi(\theta)}\right]\\
&= 9\left(\frac{2}{\theta^2}-\frac{1}{\frac{\theta^2}{2}+\frac{\theta^4}{36}+\frac{\theta^6}{405}+\frac{\theta^8}{4200}+\frac{\theta^{10}}{42525}+\dots}\right)\\
&= \frac{18}{\theta^2}\left(1-\frac{1}{1+\frac{\theta^2}{18}+\frac{2\theta^4}{405}+\frac{\theta^6}{2100}+\frac{2\theta^8}{42525}+\dots}\right)\\
&= 1+\frac{\theta^2}{30}+\frac{101\theta^4}{56700}+\frac{109\theta^6}{1020600}+\frac{15979\theta^8}{2357586000}+\dots
\end{aligned}$$

for $\theta\in(-\pi,\pi)$, where we used the Maclaurin expansion in (3). This implies that the ratio $\frac{T_3[\Phi(\theta)]}{T_1[\Phi(\theta)]}$ is possibly convex in $\theta\in(-\pi,\pi)$.

6. Conclusions

In this work, we mainly established two conclusions:

1. For $n \in \mathbb{N}_0$, the normalized remainder $T_{2n+1}[\Phi(\theta)]$ is a logarithmically convex function in $\theta \in (-\pi, \pi)$; see Theorem 1.

2. For $n \in \mathbb{N}_0$, the ratio $\frac{T_{2n+3}[\Phi(\theta)]}{T_{2n+1}[\Phi(\theta)]}$ is a decreasing function in $\theta \in (-\pi, 0)$ and an increasing function in $\theta \in (0, \pi)$; see Theorem 2.

The inequality (7) was deduced as a by-product of Theorem 1.

The increasing property of two sequences in Lemmas 1 and 3 are interesting, because these two sequences contain the ratios between two nonzero Bernoulli numbers. For more information and generalizations of the ratios between two nonzero Bernoulli numbers and polynomials, please refer to the papers [14, 15, 22], [32, Proposition 1], and the arXiv preprint [28].

To consider and solve the guess in Remark 5 is much interesting.

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REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.

[2] Z.-H. BAO, R. P. AGARWAL, F. QI, AND W.-S. DU, *Some properties on normalized tails of Maclaurin power series expansion of exponential function*, Symmetry **16** (2024), no. 8, Art. 989, 15 pages; available online at https://doi.org/10.3390/sym16080989.

[3] M. BIERNACKI AND J. KRZYŻ, *On the monotonity of certain functionals in the theory of analytic functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **9** (1955), 135–147 (1957).

[4] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Eighth edition, Revised from the seventh edition, Elsevier/Academic Press, Amsterdam, 2015; available online at https://doi.org/10.1016/B978-0-12-384933-5.00013-8.

[5] C.-Y. HE AND F. QI, *Reformulations and generalizations of Hoffman's and Genčev's combinatorial identities*, Results Math. **79** (2024), no. 4, Paper No. 131, 17 pages; available online at https://doi.org/10.1007/s00025-024-02160-0.

[6] Y.-W. LI AND F. QI, *A new closed-form formula of the Gauss hypergeometric function at specific arguments*, Axioms **13** (2024), no. 5, Art. 317, 24 pages; available online at https://doi.org/10.3390/axioms13050317.

[7] Y.-F. LI AND F. QI, *A series expansion of a logarithmic expression and a decreasing property of the ratio of two logarithmic expressions containing cosine*, Open Math. **21** (2023), no. 1, Paper No. 20230159, 12 pages; available online at https://doi.org/10.1515/math-2023-0159.

[8] Y.-W. LI AND F. QI, *Elegant proofs for properties of normalized remainders of Maclaurin power series expansion of exponential function*, Math. Slovaca **75** (2025), accepted on 28 April 2025; available online at https://www.researchgate.net/publication/391245383.

[9] Y.-W. LI, F. QI, AND W.-S. DU, *Two forms for Maclaurin power series expansion of logarithmic expression involving tangent function*, Symmetry **15** (2023), no. 9, Art. 1686, 18 pages; available online at https://doi.org/10.3390/sym15091686.

[10] X.-L. LIU, H.-X. LONG, AND F. QI, *A series expansion of a logarithmic expression and a decreasing property of the ratio of two logarithmic expressions containing sine*, Mathematics **11** (2023), no. 14, Art. 3107, 12 pages; available online at https://doi.org/10.3390/math11143107.

[11] X.-L. LIU AND F. QI, *Monotonicity results of ratio between two normalized remainders of Maclaurin series expansion for square of tangent function*, Math. Slovaca (2025), accepted on 21 January 2025; available online at https://www.researchgate.net/publication/388198214.

[12] D.-W. NIU AND F. QI, *Monotonicity results of ratios between normalized tails of Maclaurin power series expansions of sine and cosine*, Mathematics **12** (2024), no. 12, Art. 1781, 20 pages; available online at https://doi.org/10.3390/math12121781.

[13] W.-J. PEI AND B.-N. GUO, *Monotonicity, convexity, and Maclaurin series expansion of Qi's normalized remainder of Maclaurin series expansion with relation to cosine*, Open Math. **22** (2024), no. 1, Paper No. 20240095, 11 pages; available online at https://doi.org/10.1515/math-2024-0095.

[14] I. PINELIS, *Monotonicity of ratios of Bernoulli polynomials*, Math. Inequal. Appl. **27** (2024), no. 4, 949–953; available online at https://doi.org/10.7153/mia-2024-27-63.

[15] F. QI, *A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers*, J. Comput. Appl. Math. **351** (2019), 1–5; available online at https://doi.org/10.1016/j.cam.2018.10.049.

[16] F. QI, *Absolute monotonicity of normalized tail of power series expansion of exponential function*, Mathematics **12** (2024), no. 18, Art. 2859, 11 pages; available online at https://doi.org/10.3390/math12182859.

[17] F. QI, *Power series expansions of real powers of inverse cosine and sine functions, closed-form formulas of partial Bell polynomials at specific arguments, and series representations of real powers of circular constant*, Symmetry **16** (2024), no. 9, Art. 1145, 21 pages; available online at https://doi.org/10.3390/sym16091145.

[18] F. QI, *Series and connections among central factorial numbers, Stirling numbers, inverse of Vandermonde matrix, and normalized remainders of Maclaurin series expansions*, Mathematics **13** (2025), no. 2, Art. 223, 52 pages; available online at https://doi.org/10.3390/math13020223.

[19] F. QI, R. P. AGARWAL, AND D. LIM, *Decreasing property of ratio of two logarithmic expressions involving tangent function*, Math. Comput. Model. Dyn. Syst. **31** (2025), no. 1, 1–16; available online at https://doi.org/10.1080/13873954.2024.2449322.

[20] F. QI AND P. TAYLOR, *Series expansions for powers of sinc function and closed-form expressions for specific partial Bell polynomials*, Appl. Anal. Discrete Math. **18** (2024), no. 1, 92–115; available online at https://doi.org/10.2298/AADM230902020Q.

[21] R. L. SCHILLING, R. SONG, AND Z. VONDRAČEK, *Bernstein Functions*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; available online at https://doi.org/10.1515/9783110269338.

[22] Y. SHUANG, B.-N. GUO, AND F. QI, *Logarithmic convexity and increasing property of the Bernoulli numbers and their ratios*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 3, Paper No. 135, 12 pages; available online at https://doi.org/10.1007/s13398-021-01071-x.

[23] N. M. TEMME, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996; available online at https://doi.org/10.1002/9781118032572.

[24] A. WAN AND F. QI, *Power series expansion, decreasing property, and concavity related to logarithm of normalized tail of power series expansion of cosine*, Electron. Res. Arch. **32** (2024), no. 5, 3130–3144; available online at https://doi.org/10.3934/era.2024143.

[25] F. WANG AND F. QI, *Absolute monotonicity of four functions involving the second kind of complete elliptic integrals*, J. Math. Inequal. **19** (2025), accepted on 14 April 2025; available online at https://www.researchgate.net/publication/390661160.

[26] F. WANG AND F. QI, *Power series expansion and decreasing property related to normalized remainders of power series expansion of sine*, Filomat **38** (2024), no. 29, 10447–10462; available online at https://doi.org/10.2298/FIL2429447W.

[27] Z.-H. YANG, Y.-M. CHU, AND M.-K. WANG, *Monotonicity criterion for the quotient of power series with applications*, J. Math. Anal. Appl. **428** (2015), no. 1, 587–604; available online at https://doi.org/10.1016/j.jmaa.2015.03.043.

[28] Z.-H. YANG AND F. QI, *Monotonicity and inequalities for the ratios of two Bernoulli polynomials*, arXiv preprint (2024), available online at https://doi.org/10.48550/arxiv.2405.05280.

[29] H.-C. ZHANG, B.-N. GUO, AND W.-S. DU, *On Qi's normalized remainder of Maclaurin power series expansion of logarithm of secant function*, Axioms **13** (2024), no. 12, Art. 860, 11 pages; available online at https://doi.org/10.3390/axioms13120860.

[30] T. ZHANG AND F. QI, *Decreasing ratio between two normalized remainders of Maclaurin series expansion of exponential function*, AIMS Math. (2025), accepted.

[31] G.-Z. ZHANG AND F. QI, *On convexity and power series expansion for logarithm of normalized tail of power series expansion for square of tangent*, J. Math. Inequal. **18** (2024), no. 3, 937–952; available online at https://doi.org/10.7153/jmi-2024-18-51.

[32] G.-Z. ZHANG, Z.-H. YANG, AND F. QI, *On normalized tails of series expansion of generating function of Bernoulli numbers*, Proc. Amer. Math. Soc. **153** (2025), no. 1, 131–141; available online at https://doi.org/10.1090/proc/16877.

[33] T. ZHANG, Z.-H. YANG, F. QI, AND W.-S. DU, *Some properties of normalized tails of Maclaurin power series expansions of sine and cosine*, Fractal Fract. **8** (2024), no. 5, Art. 257, 17 pages; https://doi.org/10.3390/fractalfract8050257.

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Juan Zhang
School of Mathematics and Physics
Hulunbuir University
Hulunbuir 021008, Inner Mongolia, China
e-mail: zhangjuan@hlbec.edu.cn
https://orcid.org/0009-0007-6512-9963

Feng Qi
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo 454010, Henan, China
and
Retired researcher
17709 Sabal Court, Dallas, TX 75252-8024, USA
e-mail: qifeng618@gmail.com
https://orcid.org/0000-0001-6239-2968